Supplement to “Dynamic optimal taxation with private information”.

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1 Proofs from Section 2

We state first several basic results from the component planner’s problem. Recall that this problem is:

$$B_t(w_t) = \inf_{\substack{\varphi:\Theta\to\mathcal{U},\varsigma:\Theta\to\mathcal{V}\\ \omega:\Theta\to\mathcal{W}_{t+1}}} \int_\Theta [C(\varphi(\theta)) - Y(\varsigma(\theta)) + q_t B_{t+1}(\omega(\theta))]d\pi \tag{S1}$$

subject to the temporary incentive-compatibility constraint:

$$\forall \theta,\theta',\quad \varphi(\theta) + \theta\varsigma(\theta) + \beta\omega(\theta) \geq \varphi(\theta') + \theta\varsigma(\theta') + \beta\omega(\theta'), \tag{S2}$$

and the promise-keeping constraint:

$$w_t = \int_\Theta [\varphi(\theta) + \theta\varsigma(\theta) + \beta\omega(\theta)]d\pi. \tag{S3}$$

In the terminal period T, the component planner solves:

$$B_T(w_T) = \inf_{\substack{\varphi:\Theta\to\mathcal{U}\\ \varsigma:\Theta\to\mathcal{V}}} \int_\Theta [C(\varphi(\theta)) - Y(\varsigma(\theta))]d\pi \tag{S4}$$

subject to the temporary incentive-compatibility constraint:

$$\forall \theta,\theta',\quad \varphi(\theta) + \theta\varsigma(\theta) \geq \varphi(\theta') + \theta\varsigma(\theta'), \tag{S5}$$

and the promise-keeping constraint:

$$w_T = \int_\Theta [\varphi(\theta) + \theta\varsigma(\theta)]d\pi. \tag{S6}$$

Since $\mathcal{Y}$ is compact and $v : \mathcal{Y} \to \mathbb{R}$ is continuous, $\mathcal{V}$ is compact. We assume in the remainder that $\mathcal{U}$, Θ and π are such that solutions exist to problems (S1) and (S4) and that each B_t is continuous. We give two sets of sufficient conditions. The first is that Θ is discrete and $\mathcal{U}$ is compact. In this case, standard arguments can be applied to establish that a solution to these problems exists and that each B_t is continuous. An argument of Kahn (1993, p. 128-9) shows that the following conditions are sufficient for a solution to problem (S4) to exist: 1) $\Theta = [\underline{\theta}, \overline{\theta}]$, 2) π admits a continuous density and 3) $\mathcal{U}$ is unbounded. Kahn's formulation coupled with the Theorem of the Maximum can be used to establish that B_T is continuous. These last arguments can then be applied iteratively to show that a solution to each problem (S1) exists and that each $\{B_t\}$ is continuous.

Once existence of a solution is established a routine dynamic programming argument can be used to establish the strict convexity of each B_t.

Lemma S1 *Each function* $\varphi^*_t(w_t, \cdot)$, $\varsigma^*_t(w_t, \cdot)$, $\omega^*_{t+1}(w_t, \cdot)$ *is monotone.*

Proof: By a standard argument, if (φ', ς') satisfies the period T incentive-compatibility (S5), then φ' and ς' are monotone in θ. Hence, $(\varphi^*_T(w_T, \cdot), \varsigma^*_T(w_T, \cdot))$ are monotone. Similarly, if $(\varphi', \varsigma', \omega')$ satisfies the incentive-compatibility condition (S2), then ς' and $d' = \varphi' + \beta\omega'$ are monotone. Hence, for $t < T$, $(d^*_t(w_t, \cdot), \varsigma^*_t(w_t, \cdot))$ are monotone, where $d^*_t(w_t, \cdot) = \varphi^*_t(w_t, \cdot) + \beta\omega^*_{t+1}(w_t, \cdot)$. Let $(\varphi'(d), \omega'(d))$ denote the solution to

$$\sup_{(\varphi', \omega'): d = \varphi' + \beta\omega'} [C(\varphi') + q_t B_{t+1}(\omega')].$$

It is easy to check that $(\varphi^*_t(w_t, \theta), \omega^*_{t+1}(w_t, \theta)) = (\varphi'(d^*_t(w_t, \theta)), \omega'(d^*_t(w_t, \theta)))$. The strict convexity of C and B_{t+1} imply that $(\varphi'(d), \omega'(d))$ are strictly increasing. Since d^*_t is monotone, it then follows that φ^*_t and ω^*_{t+1} are monotone as well. ■

Lemma S2 *Each* B_t, $t \in \{0, \dots, T\}$ *is strictly increasing.*

Proof: We first show B_T is strictly increasing. Let w_T and w'_T lie in $\mathcal{W}_T$ with $w_T > w'_T$. Suppose that $B_T(w_T) < B_T(w'_T)$. Let $\{\varphi(w_T, \cdot), \varsigma(w_T, \cdot)\}$ be a solution to (S4) at w_T. The incentive-compatibility constraints imply that $\varsigma(w_T, \cdot)$ is monotone increasing (see Lemma S1). If $\varsigma(w_T, \overline{\theta}) = \underline{v} = \min_{y \in \mathcal{Y}} v(y)$, it follows that the optimal $\{\varphi(w_T, \cdot), \varsigma(w_T, \cdot)\}$ are constant functions. Hence construct a new allocation $\{\varphi', \varsigma'\}$ by setting $\varsigma'(\theta) = \varsigma(w_T, \theta) = \underline{v}$ and $\varphi' = w'_T - E[\theta]\underline{v}$. This is feasible and incurs a lower cost than the original allocation. This contradicts $B_T(w_T) < B_T(w'_T)$.

Suppose then that $\varsigma(w_T, \overline{\theta}) > \underline{v} = \min_{y \in \mathcal{Y}} v(y)$. Let $\theta(\gamma) = \sup_{\varsigma(w_T, \theta) \leq \gamma} \theta$ and define

$$F(\gamma) = \int_{\underline{\theta}}^{\theta(\gamma)} [\varphi(w_T, \theta) + \varphi(w_T, \overline{\theta}) - \varphi(w_T, \theta(\gamma)) + \theta\varsigma(w_T, \theta)]d\pi + \int_{\theta(\gamma)}^{\overline{\theta}} [\varphi(w_T, \overline{\theta}) + \theta\varsigma(w_T, \theta(\gamma))]d\pi. \tag{S7}$$

Thus, $F : \mathcal{V} \to \mathbb{R}$ is increasing. Suppose $F(\varsigma(w_T, \underline{\theta})) \geq w'_T$. Then construct a new allocation by setting $\varsigma'(\theta) = \widetilde{\varsigma} = \varsigma(w_T, \underline{\theta})$ and $\varphi'(\theta)$ equal to a constant $\widetilde{\varphi}$. Reduce $\widetilde{\varphi}$ below $\varphi(w_T, \overline{\theta})$ until either $w'_T = \widetilde{\varphi} + E[\theta]\widetilde{\varsigma}$ or $\widetilde{\varphi} = \min_{u \in \mathcal{U}} u$. In the second case, reduce $\widetilde{\varsigma}$ until $w'_T = \widetilde{\varphi} + E[\theta]\widetilde{\varsigma}$. The new allocation is clearly incentive-compatible and has a cost strictly less than the original one. This contradicts $B_T(w_T) < B_T(w'_T)$.

Suppose next that $F(\varsigma(w_T, \underline{\theta})) < w'_T$. Then find γ' such that $\lim_{\gamma \uparrow \gamma'} F(\gamma) \leq w'_T \leq \lim_{\gamma \downarrow \gamma'} F(\gamma)$. Without loss of generality, we assume that $\{\varphi(w_T, \cdot), \varsigma(w_T, \cdot)\}$ are continuous from the right. Let $\Theta(\gamma') = \Theta \cap [0, \theta(\gamma'))$ and $\widetilde{\varsigma} = \lim_{\theta \uparrow \theta(\gamma'): \theta \in \Theta(\gamma')} \varsigma(w_T, \theta)$ and $\widetilde{\varphi} = \lim_{\theta \uparrow \theta(\gamma'): \theta \in \Theta(\gamma')} \varphi(w_T, \theta)$. Let $\varsigma^\lambda = (1 - \lambda)\widetilde{\varsigma} + \lambda\varsigma(w_T, \theta(\gamma'))$ and $\varphi^\lambda = (1 - \lambda)\widetilde{\varphi} + \lambda\varphi(w_T, \theta(\gamma'))$. Define

$$D(\lambda) = \int_{\underline{\theta}}^{\theta(\gamma')} [\varphi(w_T, \theta) + \varphi(w_T, \overline{\theta}) - \varphi^\lambda + \theta\varsigma(w_T, \theta)]d\pi + \int_{\theta(\gamma')}^{\overline{\theta}} [\varphi(w_T, \overline{\theta}) + \theta\varsigma^\lambda]d\pi. \tag{S8}$$

This function is continuous and increasing in λ. Additionally, $D(0) = \lim_{\gamma \uparrow \gamma'} F(\gamma) \leq w'_T \leq \lim_{\gamma \downarrow \gamma'} F(\gamma) \leq D(1)$. Thus, there exists a $\lambda' \in [0,1]$ such that $D(\lambda') = w'_T$. Consider the new utility allocation

$$\varphi'(\theta) = \begin{cases} \varphi(w_T, \theta) + \varphi(w_T, \overline{\theta}) - \varphi^{\lambda'} & \theta \in \Theta \cap [0, \theta(\gamma')) \\ \varphi(w_T, \overline{\theta}) & \theta \in \Theta \cap [\theta(\gamma'), \infty) \end{cases}$$

$$\varsigma'(\theta) = \begin{cases} \varsigma(w_T, \theta) & \theta \in \Theta \cap [0, \theta(\gamma')) \\ \varsigma^{\lambda'} & \theta \in \Theta \cap [\theta(\gamma'), \infty) \end{cases}$$

This new allocation is incentive-compatible. For $\theta < \theta(\gamma')$, it is immediate from the incentive compatibility of the original allocation that for $\theta' < \theta(\gamma')$,

$$\varphi(w_T, \theta) + \varphi(w_T, \overline{\theta}) - \varphi^{\lambda'} + \theta\varsigma(w_T, \theta) \geq \varphi(w_T, \theta') + \varphi(w_T, \overline{\theta}) - \varphi^{\lambda'} + \theta\varsigma(w_T, \theta').$$

For $\theta' \geq \theta(\gamma')$, the associated allocation is $\{\varphi(w_T, \overline{\theta}), \varsigma^{\lambda'}\}$ and

$$\begin{aligned} \varphi(w_T, \theta) + \varphi(w_T, \overline{\theta}) - \varphi^{\lambda} + \theta\varsigma(w_T, \theta) &= \varphi(w_T, \theta) + \theta\varsigma(w_T, \theta) + \varphi(w_T, \overline{\theta}) - \varphi^{\lambda'} \\ &\geq (1-\lambda')\{\widetilde{\varphi} + \theta\widetilde{\varsigma}\} + \lambda'\{\varphi(w_T, \theta(\gamma')) + \theta\varsigma(w_T, \theta(\gamma'))\} \\ &\quad + \varphi(w_T, \overline{\theta}) - \varphi^{\lambda'} \\ &= \varphi(w_T, \overline{\theta}) + \theta\varsigma^{\lambda'}. \end{aligned}$$

Conversely, for $\theta \geq \theta(\gamma') > \theta'$,

$$\begin{aligned} \varphi(w_T, \overline{\theta}) + \theta\varsigma^{\lambda'} &= \varphi(w_T, \overline{\theta}) - \varphi^{\lambda} + (1-\lambda')\{\widetilde{\varphi} + \theta\widetilde{\varsigma}\} + \lambda'\{\varphi(w_T, \theta(\gamma')) + \theta\varsigma(w_T, \theta(\gamma'))\} \\ &\geq \varphi(w_T, \overline{\theta}) - \varphi^{\lambda} + (1-\lambda')\{\varphi(w_T, \theta') + \theta\varsigma(w_T, \theta')\} + \lambda'\{\varphi(w_T, \theta') + \theta\varsigma(w_T, \theta')\} \\ &\quad + (1-\lambda')(\theta - \widetilde{\theta})\widetilde{\varsigma} + \lambda'(\theta - \theta(\gamma'))\varsigma(w_T, \theta(\gamma')) \\ &= \varphi(w_T, \theta') + \varphi(w_T, \overline{\theta}) - \varphi^{\lambda} + \theta\varsigma(w_T, \theta'). \end{aligned}$$

Where we use the fact that if $\widehat{\varphi}(\widehat{\theta}) + \widehat{\theta}\varsigma(\widehat{\theta}) \geq \varphi(\theta) + \widehat{\theta}\varsigma(\theta)$ and $\varsigma(\widehat{\theta}) \geq \varsigma(\theta)$ then $\widehat{\varphi}(\widehat{\theta}) + \theta'\varsigma(\widehat{\theta}) \geq \varphi(\theta) + \theta'\varsigma(\theta)$ for $\theta' > \widehat{\theta}$. This new allocation delivers a cost strictly less than that implied by the original allocation. This contradicts $B_T(w_T) < B_T(w'_T)$ and we conclude that B_T is strictly increasing. A similar argument can be applied iteratively to establish that each B_t is strictly increasing. ■

2 Implementation for $T = \infty$

For $T = \infty$, we assume that $\mathcal{U}$ is a compact set. By our previous assumptions $\mathcal{Y}$ is compact and v continuous, hence, $\mathcal{V}$ is a compact set. In this infinite period setting, this assumption guarantees that a utility allocation satisfying the temporary incentive-compatibility constraints (6) also satisfies the incentive-compatibility condition (5). The definition of an equilibrium in a component planning economy is almost identical to that given in Definition 1 in the paper. We simply require that $\sum_{t=0}^{\infty} q_t < \infty$ and that Conditions 1 and 2 in that definition are replaced by:

$1'$ *For all t,* B_t *and* B_{t+1} *satisfy (S1). Additionally, each* $B_t \geq -\overline{y}(1 + \sum_{t=0}^{\infty} q_t)$.[1]

$2'$ *For all t,* $\{\varphi_t^*, \varsigma_t^*, \omega_{t+1}^*\}$ *attain the infima in the problems (S1).*

Similarly, the definition of a competitive equilibrium in a market economy is the same as Definition 2 except for the requirement that $\sum_{t=0}^{\infty} \widehat{q}_t < \infty$ and the replacement of Conditions 1 and 2 in that definition with:

$1'$ *For all t,* V_t *and* V_{t+1} *satisfy (17). Additionally, each* $V_t : B_t \to \mathcal{W}_t$.

$2'$ *For all t,* $\{\widehat{c}_t, \widehat{y}_t, \widehat{b}_{t+1}\}$ *attain the infima in the problems (17).*

The definition of implementation is identical to that in the finite period case. We now have the following proposition.

Proposition 1 *Assume* $\mathcal{U}$ *and* $\mathcal{V}$ *are compact. Let* $\xi^{CP} = \{\{q_t\}_{t=0}^{\infty}, \{\varphi_t^*, \varsigma_t^*, \omega_{t+1}^*\}_{t=0}^{\infty}, \{B_t\}_{t=0}^{\infty}, \{\Psi_{t+1}\}_{t=0}^{\infty}\}$ *be an* ***equilibrium*** *of a infinite-period component planner economy* $\mathcal{E}^{CP}(\{\underline{U}_{t+1}\}_{t=0}^{\infty}, \{G_t\}_{t=0}^{\infty}, \Psi_0)$. *Then, the associated constrained efficient utility allocation can be* ***implemented*** *by a competitive equilibrium in a market economy with taxes and borrowing limits.*

Proof: Truncate the infinite-period market economy at $T < \infty$. Set the agent's period $T+1$ value function in the truncated economy to $\widehat{V}_{T+1} = B_{T+1}^{-1}$. Set the prices in the market economy to $\{q_t\}_{t=0}^{\infty}$. Subdivide each period $t \in \{0, \ldots, T\}$ into two sub-periods. In the first the agent exerts labour supply y_t and obtains some after-tax resources $x_t = b_t + y_t - T_t(b_t, y_t)$. In the second, the agent allocates x_t between consumption c_t and savings b_{t+1}. Each period of the component planner's problem can be similarly sub-divided. In the first sub-period, the agent makes a report and the component planner allocates utility from labour supply and an interim utility promise; in the second, the planner allocates the interim utility promise between utility from consumption and a continuation utility promise. By repeated application of the argument in the proof of Proposition 1, beginning in period T, a sequence of tax functions $\{T_t\}_{t=0}^{T}$ and borrowing limits $\{\underline{b}_{t+1}\}_{t=0}^{T-1}$ can be constructed such that confronted with these sequences, it is optimal for an agent with initial wealth $B_0(w_0)$ to choose the same period T-truncated allocation, $\{c_t, y_t\}_{t=0}^{T}$ as is awarded to an agent with a w_0-utility promise by the component planner.

By successively increasing T, a sequence of tax functions $\{T_t\}_{t=0}^{\infty}$ and borrowing limits $\{\underline{b}_{t+1}\}_{t=0}^{\infty}$ can be constructed which, along with $\{G_t\}_{t=0}^{\infty}$ and Λ_0 define a market economy. Set $f = B_0$. If an agent with an initial quantity of claims $B_0(w_0)$ selects $z^*(w_0)$, the allocation obtained by a w_0-promise agent in the component economy, then she too receives a payoff of w_0. Suppose there is some alternative allocation available to the agent in the constructed market economy that gives a payoff of $\widetilde{U}^{\infty} > w_0 + \varepsilon > w_0$. Let $\widetilde{U}^T$ denote the payoff earned from this allocation over the initial T periods, and let $\widetilde{b}_{T+1}$ denote the agent's savings at date T under this allocation. Similarly, let U^T be the agent's payoff from $z^*(w_0)$ over the initial T periods and let b_{T+1} be the agent's savings at date T under this allocation. Now since agents choose to select the component planner allocation in the truncated economy: $U^T(w_0) + \beta^T E\widehat{V}_{T+1}(b_{T+1}) \geq \widetilde{U}^T + \beta^T E\widehat{V}_{T+1}(\widetilde{b}_{T+1})$ Hence, since the agent's utility functions are bounded, $w_0 = \lim_{T\to\infty}[U^T(w_0) + \beta^T E\widehat{V}_{T+1}(b_{T+1})] \geq \lim_{T\to\infty}[\widetilde{U}^T + \beta^T E\widehat{V}_{T+1}(\widetilde{b}_{T+1})] = \widetilde{U}^{\infty} > w_0 + \varepsilon$. This is a contradiction. It then follows that *in the untruncated economy*, the agent with initial wealth $B_0(w_0)$ selects the allocation obtained by an agent with an initial utility promise of w_0. Condition 2 in the implementation definition is satisfied. ■

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[1] This boundedness in conjunction with an argument similar to Theorem 4.14 in Stokey, Lucas and Prescott (1989) guarantees that the sequence of $\{B_t\}_{t=0}^{\infty}$ satisfies the Bellman equation (S1).

3 Deriving Wedges

For expositional ease, we focus on the case $\Theta = [\underline{\theta}, \overline{\theta}]$ and assume that the functions that comprise the optimal utility allocation rule, $\zeta^* = \{\{\varphi_t^*, \varsigma_t^*, \omega_{t+1}^*\}_{t=0}^{T-1}, \varphi_T^*, \varsigma_T^*\}$, are piecewise differentiable in θ.[2] We assume that π admits a continuous density ρ. As a notational convention, if $x : \Theta \to \mathbb{R}$ is a piecewise differentiable function, then we denote its derivative at θ by $\dot{x}(\theta)$. Define the function $U : \Theta \to \mathbb{R}$ by:

$$U(\theta) \equiv \varphi(\theta) + \theta\varsigma(\theta) + \beta\omega(\theta). \tag{S9}$$

By standard arguments, for example Salanié (1997), $\{\varphi, \varsigma, \omega\}$ is incentive compatible if and only if:

$$\begin{aligned} \dot{U}(\theta) &= \varsigma(\theta) \qquad \text{a.e. } \theta, && \text{(S10)} \\ & \varsigma(\theta) \ \text{ non-decreasing in } \theta. && \text{(S11)} \end{aligned}$$

We formulate the period $t \in \{0, \ldots, T-1\}$ problem of a component planner with utility promise w_t as an optimal control problem.[3] To obtain the component planner's period t Hamiltonian, we use (S9) to replace φ with U and we drop the constraint that $\varsigma(\theta)$ is non-decreasing[4]:

$$\begin{aligned} H_t^{w_t}(U, \varsigma, \omega; \chi, \lambda, \phi)(\theta) = -\chi(U(\theta) - w_t)\rho(\theta) - \lambda(\theta)\varsigma(\theta) + \\ [C(U(\theta) - \theta\varsigma(\theta) - \beta\omega(\theta)) - Y(\varsigma(\theta)) + q_t B_{t+1}(\omega(\theta))]\,\rho(\theta) + \phi(\theta)(\omega(\theta) - \underline{U}_t)\rho(\theta). \end{aligned} \tag{S12}$$

Here χ is the multiplier on the promise-keeping constraint $w_t = \int_{\underline{\theta}}^{\overline{\theta}} U(\theta)\rho(\theta)d\theta$, λ is the costate variable associated with the incentive compatibility constraint (1) and $\phi(\theta)$ is the multiplier on the constraint $\omega(\theta) \geq \underline{U}_t$. The component planner's problem in period t can then be written as:

$$B_t(w_t) = \inf_{\{\chi,\lambda,\phi\}} \sup_{\{U,\varsigma,\omega\}} \int H_t^{w_t}(U, \varsigma, \omega; \chi, \lambda, \phi)(\theta)d\theta. \tag{S13}$$

Denote the optimal multipliers by $\{\chi_t^*, \lambda_t^*, \phi_t^*\}$. The first order conditions from (S13) imply a particular pattern of wedges. We describe this pattern in terms of the constrained efficient resource allocation rule.

THE INSURANCE WEDGE: The first order condition for $U(\theta)$ at t, re-expressed in terms of consumption is:

$$\dot{\lambda}_t^*(w_t, \theta) = -\left[\chi_t^*(w_t) - \frac{1}{u'(c_t^*(w_t, \theta))}\right]\rho(\theta), \tag{S14}$$

Thus, if $\dot{\lambda}_t^*(w_t, \cdot)/\rho(\cdot)$ is non-constant then an agent's marginal utility of consumption is not equated across states and insurance against preference shocks is imperfect. As observed in Lemma 1, φ_t^*, and, hence, c_t^* are monotone in θ. Strict monotonicity stems from the binding incentive-compatibility constraint.[5]

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[2] Kahn (1993) provides conditions for an optimal static mechanism to be absolutely continuous.

[3] The period T problem can be similarly formulated. For brevity we omit it.

[4] See Salanié (1997) for sufficient conditions to ensure that it is not binding.

[5] The transversality condition on U implies that $\lambda_t^*(w_t, \overline{\theta}) = \lambda_t^*(w_t, \underline{\theta}) = 0$. From Lemma S1, c_t^* is non-increasing. It follows that $\dot{\lambda}_t^*(w_t)$ is non-increasing and if $\lambda_t^*(w_0, \theta) \neq 0$ for some θ, then $\dot{\lambda}_t^*(w_0, \cdot)$ is non-constant.

THE EFFORT WEDGE: The first order conditions for $U(\theta)$ and $\varsigma(\theta)$ at t, re-expressed in terms of consumption and labour imply:

$$\frac{-\theta v'(y_t^*(w_t,\theta))}{u'(c_t^*(w_t,\theta))} = 1 + \frac{\lambda_t^*(w_t,\theta)\, v'(y_t^*(w_t,\theta))}{\rho(\theta)}, \tag{S15}$$

Since the social shadow price of labour is 1, $-\frac{\lambda_t^*(w_t,\theta)v'(y_t^*(w_t,\theta))}{\rho(\theta)}$ represents the wedge between this shadow price and the agent's private shadow price of labour. When $\lambda_t^*(w_t,\theta) > 0$, it is positive.

THE INTERTEMPORAL WEDGE: Combining the first order conditions for $U(\theta)$, $\omega(\theta)$ and the envelope condition from the component planner's first period problem yields the following *inverted Euler equation*:

$$\frac{1}{u'(c_t^*(w_t,\theta))} = \frac{q_t}{\beta} E_{\theta'}\left[\frac{1}{u'(c_{t+1}^*(\omega_{t+1}^*(w_t,\theta),\theta'))}\right] + \eta_t^*(w_t,\theta),$$

Assuming that the lower bound on $\omega_{t+1}^*(w_t,\theta)$ does not bind and $\eta_t^*(w_t,\theta) = 0$, we have, by Jensen's inequality,

$$q_t \leq \beta E_t\left[\frac{u'(c_{t+1}^*(\omega_{t+1}^*(w_t,\theta),\theta'))}{u'(c_t^*(w_t,\theta))}\right]. \tag{S16}$$

This inequality is strict if $u'(c_{t+1}^*(\omega_{t+1}^*(w_t,\theta),\cdot))$ is non-constant, which is the case in this problem when the incentive-compatibility constraint binds. Thus, there is a wedge between the social shadow price of a claim q_t and the private shadow price $\beta E_{\theta'}[u'(c_{t+1}^*(\omega_{t+1}^*(w_t,\theta),\theta'))]/u'(c_t^*(w_t,\theta))$.

4 Proofs from Section 4

The following lemma characterises the value and policy functions from Revealing Examples 1 and 2.

Lemma 2: *Assume that* $\inf_{u\in\mathcal{U}} C(u+\underline{\theta}\Delta v) - C(u) < \Delta y < \sup_{u\in\mathcal{U}} C(u) - C(u-\underline{\theta}\Delta v)$.

1. *There exists a* $\underline{w}_1$ *such that for* $w_1 < \underline{w}_1$, $\varsigma_1^*(w_1,\overline{\theta}) = v(\overline{y})$ *and for* $w_1 > \underline{w}_1$, $\varsigma_1^*(w_1,\overline{\theta}) = v(\underline{y})$. *Similarly, there exists a* $\overline{w}_1 \geq \underline{w}_1$, *such that for* $w_1 < \overline{w}_1$, $\varsigma_1^*(w_1,\underline{\theta}) = v(\overline{y})$ *and for* $w_1 > \overline{w}_1$, $\varsigma_1^*(w_1,\underline{\theta}) = v(\underline{y})$.

2. *There exists a* $\underline{w}_0$ *such that for* $w_0 \leq \underline{w}_0$, $\omega^*(w_0) \leq \underline{w}_1$ *and for* $w_0 > \underline{w}_0$, $\omega^*(w_0) > \underline{w}_1$. *Similarly, there exists a* $\overline{w}_0 \geq \underline{w}_0$, *such that for* $w_0 < \overline{w}_0$, $\omega^*(w_0) < \underline{w}_1$ *and for* $w_0 \geq \overline{w}_0$, $\omega^*(w_0) \geq \overline{w}_1$.

PROOF, PART 1. The two incentive constraints imply that $\varsigma(\overline{\theta}) \geq \varsigma(\underline{\theta})$ and $\varphi(\underline{\theta}) \geq \varphi(\overline{\theta})$. Hence, the "upward" incentive constraint: $\varphi(\underline{\theta}) + \underline{\theta}\varsigma(\underline{\theta}) \geq \varphi(\overline{\theta}) + \underline{\theta}\varsigma(\overline{\theta})$ must hold with equality at the optimum. If not $\varphi(\underline{\theta}) > \varphi(\overline{\theta})$, but then the strict convexity of C implies that the planner can reduce her costs by lowering $\varphi(\underline{\theta})$ by ε and raising $\varphi(\overline{\theta})$ by $\varepsilon\pi(\underline{\theta})/\pi(\overline{\theta})$. The promise keeping and upward incentive constraint can then be used to eliminate the φ variables from the planner's problem. We drop the downward incentive constraint $\varphi(\overline{\theta}) + \overline{\theta}\varsigma(\overline{\theta}) \geq \varphi(\underline{\theta}) + \overline{\theta}\varsigma(\underline{\theta})$, and consider the following problem:

$$\sup_{\{\varsigma(\underline{\theta}),\varsigma(\overline{\theta})\}} \{C(w_1 - (\overline{\theta}-\underline{\theta})\varsigma(\overline{\theta})\pi(\overline{\theta}) - \underline{\theta}\varsigma(\underline{\theta})) - Y(\varsigma(\underline{\theta}))\}\pi(\underline{\theta}) + \{C(w_1 - E[\theta]\varsigma(\overline{\theta})) - Y(\varsigma(\overline{\theta}))\}\pi(\overline{\theta}). \tag{S17}$$

It follows from (S17), the convexity of C and $-Y$ and the assumption that there is a critical A such that if $A(w_1) \equiv w_1 - (\overline{\theta}-\underline{\theta})\varsigma_1^*(w_1,\overline{\theta})\pi(\overline{\theta}) > A$, then $\varsigma_1^*(w_1,\underline{\theta}) = v(\underline{y})$. If $A(w_1) < A$, $\varsigma_1^*(w_1,\underline{\theta}) = v(\overline{y})$. Suppose that $A(w_1) > A$, then

$C(A(w_1) - \underline{\theta} v(\underline{y})) - \underline{y} < C(A(w_1) - \underline{\theta} v(\overline{y})) - \overline{y}$. Also, $w_1 = A(w_1) + (\overline{\theta} - \underline{\theta})\varsigma_1^*(w_1, \overline{\theta})\pi(\overline{\theta}) > A(w_1)$. So, by the convexity of C, $C(w_1 - \underline{\theta} v(\underline{y})) - \underline{y} < C(w_1 - \underline{\theta} v(\overline{y})) - \overline{y}$. Again by the strict convexity of C, $C(w_1 - E[\theta]v(\underline{y})) - \underline{y} < C(w_1 - E[\theta]v(\overline{y})) - \overline{y}$. Thus, if $A(w_1) > A$, it follows from (S17) that the cost in state $\overline{\theta}$ is lower if $\varsigma(\overline{\theta}) = v(\underline{y})$. The cost in state $\underline{\theta}$ is also lower if $\varsigma(\overline{\theta}) = v(\underline{y})$. So in fact $\varsigma_1^*(w_1, \overline{\theta}) = v(\underline{y})$. It follows that there exists a critical $\overline{w}_1 = A + (\overline{\theta} - \underline{\theta})v(\underline{y})\pi(\overline{\theta})$, such that for $w_1 > \overline{w}_1$, $\varsigma_1^*(w_1, \underline{\theta})$ and $\varsigma_1^*(w_1, \overline{\theta})$ equal $v(\underline{y})$, for $w_1 < \overline{w}_1$, $\varsigma_1^*(w_1, \underline{\theta})$ equals $v(\overline{y})$. For $w_1 < \overline{w}_1$, $\varsigma_1^*(w_1, \overline{\theta})$ solves

$$\sup_{\{\varsigma(\overline{\theta})\}} \{C(w_1 - (\overline{\theta} - \underline{\theta})\varsigma(\overline{\theta})\pi(\overline{\theta}) - \underline{\theta} v(\overline{y}))\}\pi(\underline{\theta}) + \{C(w_1 - E[\theta]\varsigma(\overline{\theta})) - Y(\varsigma(\overline{\theta}))\}\pi(\overline{\theta}).$$

It follows easily from the convexity of C and $-Y$ that there exists a critical $\underline{w}_1 \in \mathcal{W}_1$, such that for $w_1 > \underline{w}_1$, $\varsigma^*(w_1, \overline{\theta}) = v(\underline{y})$, and for $w_1 < \underline{w}_1$, $\varsigma_1^*(w_1, \overline{\theta}) = v(\overline{y})$. (The set $w_1 < \underline{w}_1$ may be empty). It follows from the above discussion that the solution to (S17) satisfies the upwards incentive constraint with equality and $\varsigma_1^*(w_1, \cdot)$ is non-decreasing. Hence, this solution satisfies the downwards incentive constraint and solves the original component planner's problem.

Part 2. It follows from Part 1 that:

$$B_1(w) = \begin{cases} B_{11}(w_1) = C(w_1 - E[\theta]v(\overline{y})) - \overline{y} & \text{for} \quad w_1 < \underline{w}_1 \\ B_{12}(w_1) = \{C(w_1 - (\overline{\theta} - \underline{\theta})v(\underline{y})\pi(\overline{\theta}) - \underline{\theta} v(\overline{y})) - \overline{y}\}\pi(\underline{\theta}) & \text{for} \quad w_1 \in (\underline{w}_1, \overline{w}_1) \\ \qquad + \{C(w_1 - E[\theta]v(\underline{y})) - \underline{y}\}\pi(\overline{\theta}) & \\ B_{13}(w_1) = C(w_1 - E[\theta]v(\underline{y})) - \underline{y} & \text{for} \quad w_1 > \overline{w}_1. \end{cases}$$

Let $B_{0i}(w_0) = \sup C(w_0 - \beta w_1) + q B_{1i}(w_1)$. Let $w_{1i}^*(w_0)$ denote the solutions to these problems, where $C'(w_0 - \beta w_{1i}^*(w_0)) = q/\beta B_{1i}'(w_{1i}^*(w_0))$. Clearly, $B_{11}' > B_{12}' > B_{13}'$, and so $w_{11}^*(w_0) < w_{12}^*(w_0) < w_{13}^*(w_0)$. Now for $i = 1, 2$, $B_{0i} - B_{0i+1}$ is continuous and differentiable. Also, $B_{0i}'(w_0) = C'(w_0 - \beta w_{1i}^*(w_0)) > C'(w_0 - \beta w_{1i+1}^*(w_0)) = B_{0i+1}'(w_0)$ and so $B_{0i} - B_{0i+1}$ is decreasing. It follows that there exist a pair of numbers $(\underline{w}_1, \overline{w}_1)$ each in $\mathcal{W}_1$ that satisfy the conditions of the Lemma. (Note: $\underline{w}_1$ may equal $\min \mathcal{W}_1$ or $\max \mathcal{W}_1$ and similarly for $\overline{w}_1$. It follows from the first order conditions $C'(w_0 - \beta w_{1i}^*(w_0)) = q/\beta B_{1i}'(w_{1i}^*(w_0))$ and the strict convexity of B_{1i} that each $w_{1i}^*(w_0)$ is increasing. Hence, ω^* is increasing. ■

Lemma S3. *B_1 is differentiable on $(\underline{w}_1, \overline{w}_1)$.*

PROOF: It follows from Lemma 2 that

$$B_1(w) = \begin{cases} B_{11}(w_1) = C(w_1 - E[\theta]v(\overline{y})) - \overline{y} & \text{for} \quad w_1 < \underline{w}_1 \\ B_{12}(w_1) = \{C(w_1 - (\overline{\theta} - \underline{\theta})v(\underline{y})\pi(\overline{\theta}) - \underline{\theta} v(\overline{y})) - \overline{y}\}\pi(\underline{\theta}) & \text{for} \quad w_1 \in (\underline{w}_1, \overline{w}_1) \\ \qquad + \{C(w_1 - E[\theta]v(\underline{y})) - \underline{y}\}\pi(\overline{\theta}) & \\ B_{13}(w_1) = C(w_1 - E[\theta]v(\underline{y})) - \underline{y} & \text{for} \quad w_1 > \overline{w}_1. \end{cases}$$

Each B_{1i} is strictly convex and differentiable. Hence, B_1 is piecewise strictly convex and differentiable except at $\underline{w}_1$ and $\overline{w}_1$. ■

Lemma 5: *If $C''' > 0$, then $\frac{\partial \widehat{\theta}_1^*}{\partial w_1} \leq 0$.*

PROOF: The component planner's problem can be written as:

$$\min_{\widehat{\theta} \in [\underline{\theta}, \overline{\theta}]} \{C(w_1 + K + \widehat{\theta}(1 - \Pi(\widehat{\theta}))\Delta v - \Delta v \int_{\widehat{\theta}}^{\overline{\theta}} \theta f(\theta) d\theta) - \overline{y}\}\Pi(\widehat{\theta})$$

$$+\{C(w_1 + K - \widehat{\theta}\Pi(\widehat{\theta})\Delta v - \Delta v \int_{\widehat{\theta}}^{\overline{\theta}} \theta f(\theta) d\theta) - \underline{y}\}(1 - \Pi(\widehat{\theta})).$$

Assuming the boundary constraints do not bind, the first order condition with respect to $\widehat{\theta}$ is:

$$\begin{aligned}0 &= C'(w_1 + K + \widehat{\theta}^*(1 - \Pi(\widehat{\theta}^*))\Delta v - \Delta v \int_{\widehat{\theta}^*}^{\overline{\theta}} \theta f(\theta) d\theta) \\ &\quad -C'(w_1 + K - \widehat{\theta}^* \Pi(\widehat{\theta}^*)\Delta v - \Delta v \int_{\widehat{\theta}^*}^{\overline{\theta}} \theta f(\theta) d\theta)\}\Pi(\widehat{\theta}^*)(1 - \Pi(\widehat{\theta}^*))\Delta v \\ &\quad +[\{C(w_1 + K + \widehat{\theta}^*(1 - \Pi(\widehat{\theta}^*))\Delta v - \Delta v \int_{\widehat{\theta}^*}^{\overline{\theta}} \theta f(\theta) d\theta) - \overline{y}\} \\ &\quad -\{C(w_1 + K - \widehat{\theta}^* \Pi(\widehat{\theta}^*)\Delta v - \Delta v \int_{\widehat{\theta}^*}^{\overline{\theta}} \theta f(\theta) d\theta) - \underline{y}\}]f(\widehat{\theta}^*).\end{aligned}$$

So we have:

$$\frac{\partial \widehat{\theta}^*}{\partial w_1} = -\frac{N}{D},$$

where

$$\begin{aligned}N &= \{C''(\overline{u}_1^*) - C''(\underline{u}_1^*)\}\Pi(\widehat{\theta}^*)(1 - \Pi(\widehat{\theta}^*)) \\ &\quad +[\{C'(\overline{u}_1^*) - \overline{y}\} - \{C'(\underline{u}_1^*) - \underline{y}\}]f(\widehat{\theta}^*)\end{aligned}$$

and

$$\begin{aligned}D &= \{C''(\overline{u}_1^*)(1 - \Pi(\widehat{\theta}^*)) + C''(\underline{u}_1)\Pi(\widehat{\theta}^*)\}\Pi(\widehat{\theta}^*)(1 - \Pi(\widehat{\theta}^*))\Delta v^2 \\ &\quad +\{C'(\overline{u}_1^*)(2 - 3\Pi(\widehat{\theta}^*)) - C'(\underline{u}_1^*)(1 - 3\Pi(\widehat{\theta}^*))\}f(\widehat{\theta}^*)\Delta v \\ &\quad +[\{C(\overline{u}_1^*) - \overline{y}\} - \{C(\underline{u}_1^*) - \underline{y}\}]f'(\widehat{\theta}^*)\end{aligned}$$

If $C''' > 0$, then $N > 0$. Since $\widehat{\theta}^*$ is a minimum, $D > 0$. Consequently, the wealth effect is negative. ■

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