A note on design of dynamic auctions

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Abstract

We study here an extension of the classic one period auction [1] to multiple periods and propose optimal revenue maximizing mechanisms over the entire horizon. The proposed mechanism also holds the properties of incentive compatibility (IC), individual rationality (IR) for each buyer in each time period as in the traditional setting. We highlight some key insights on how the seller can extract more revenue than the repeated application of static mechanism over multiple periods. The results in this work were developed independently of Vohra[], who has similar results. Both theories were based on charging the expected future payment as a way to maximize revenue under IC, IR constraints.

Problem Formulation

We consider the case of single seller, multiple buyers, sealed bid auction in each time period. We follow the same notation as in the [1] framework. At every time period t, signals are distributed i.i.d. for each bidder i with support $[a_i^t, b_i^t]$. The distribution function over the signal range (prior) at each step is assumed to be common knowledge (sellers and buyers). The valuation of bidder i when he receives signal t_i^t is $v^t(t_i) = t_i^t$. The evolution of the bidder's valuation is markovian over the horizon. The signal range and the distribution over the signals at time step t depends only on the allocation (a^t) made at time $t-1$. Given the structure of the sealed bid auction, this is a reasonable description and is also tractable. Vohra [] develop results for a much general set of evolutions but their technique forces assumptions on the hazard rate of the distribution. In this work, no hazard rate assumptions were made on the valuation function. Bidders observe their private signals at each time step, t_i^t, bid a value b_i^t. An allocation (a^t) is made by the mechanism, bidders valuations evolve and auction moves on to the next period

The aim of the seller is to construct a sequence of mechanisms (and announce them at time $t = 0$) to be implemented at every time period, that is revenue maximizing over the entire horizon, enforces incentive compatibility at each auction period, and is also individually rational for the buyers to participate in each time period.

Note: the results below are derived for the case of only 2 buyers and 2 period game. Extensions thereof to multiple buyers and time periods, to other simple generalizations of value function evolution should follow.

Proposed Mechanism

Mechanisms in period t is denoted by M^t

- BEGIN
- Bidders observe private signals t_i^1 and bid b_i^1 at $t = 1$
- An allocation is made and payments are collected according to mechanism M_1
- Bidders observe the allocations alone, valuations of the bidders evolve, they observe the private signal t_i^2 and bid b_i^2 at $t = 2$
- An allocation is made and payments are collected according to mechanism M_2
- STOP

M_2 is the optimal mechanism (in terms of payment and allocation) corresponding to the static single period auction. The idea is similar to the backward induction principle of dynamic programming. When there is 1 more stage to go, the dynamics of the game is essentially that of a single period static auction. Hence, Myerson's auction construction is optimal. The expressions below follow directly from the computations in [1].

$$x_i^2(t_{-i}^2, t_i^2) = p_i^2(t_{-i}^2, t_i^2) v_i^2(t_{-i}^2, t_i^2) - \int_{a_i}^{t_i} p_i^2(t_{-i}^2, s_i^2) ds_i, \forall i, t_i^2 \tag{1}$$

The design of mechanism in stage 1 is little more involved. The bidders might utilize the knowledge of the future horizon, evolution of the valuation functions to deviate from the truthful strategy. The mechanism should possibly overcharge or penalize any deviating behavior. Such a extra payment might also result in violating the individual rationality constraint. We show below that by charging the expected future utility (in excess) there is no deviation of bidders from truthful strategy and also individually rational. Importantly, we conclude such a payment rule is also revenue maximizing when the allocations are made according to a suitable *ironing* procedure.

Payment function for bidder i as specified by mechanism M_1: ($x_i^1 = x_i^{11} + x_i^{12}$) where
x_i^{11} is equal in value to the optimal payment for a single period auction and
x_i^{12} is the expected utility derived from period 2 i.e. $E[U_i^2 | t_{-i}^1, t_i^1]$

$$x_i^1(t_{-i}^1, t_i^1) = x_i^{11}(t_{-i}^1, t_i^1) + x_i^{12}(t_{-i}^1, t_i^1)$$
$$x_i^{11}(t_{-i}^1, t_i^1) = p_i^1(t_{-i}^1, t_i^1)v_i^1(t_{-i}^1, t_i^1) - \int_{a_i}^{t_i} p_i^1(t_{-i}^1, s_i^1)ds_i, \forall i, t_i^1$$
$$x_i^{12}(t_{-i}^1, t_i^1) = E[U_i^2|t_{-i}^1, t_i^1] = p_i^1(t_{-i}^1, t_i^1)E[U_i^2|t_{-i}^1, t_i^1, a^1 = 1] + p_{-i}^1(t_{-i}^1, t_i^1)E[U_i^2|t_{-i}^1, t_i^1, a^1 = 2] \tag{2}$$

The optimal allocation rule in period 1 follows a suitable *ironing* procedure that is described later.

Proofs of Incentive compatibility & Individual Rationality

$U_i^1(p^1, x^1, t_i^1)$ is the utility to go function at $t = 1$ if player i has value t_i^1 and bids t_i^1.
$U_i^1(p^1, x^1, (t_i^1, s_i^1))$ is the utility to go function at $t = 1$ if player i has value t_i^1 and bids s_i^1
$U_i^{1s}(p, x, (t_i^1, s_i^1))$ is the utility obtained in period $t = 1$ alone if player i has value t_i^1 and bids s_i^1

$$\begin{aligned} U_i^1(p^1, x^1, t_i^1) &= \int [v_i^1(t_{-i}^1, t_i^1)p_i^1(t_{-i}^1, t_i^1) - x_i^1(t_{-i}^1, t_i^1) + E[U_i^2|t_{-i}^1, t_i^1]]f_{t_{-i}^1}dt_{-i}^1 \\ &= [v_i^1(t_{-i}^1, t_i^1)p_i^1(t_{-i}^1, t_i^1) - x_i^{11}(t_{-i}^1, t_i^1) - x_i^{12}(t_{-i}^1, t_i^1) + E[U_i^2|t_{-i}^1, t_i^1]]f_{t_{-i}^1}dt_{-i}^1 \\ &= \int [v_i^1(t_{-i}^1, t_i^1)p_i^1(t_{-i}^1, t_i^1) - x_i^{11}(t_{-i}^1, t_i^1)]f_{t_{-i}^1}dt_{-i}^1 \\ &= \int f_{t_{-i}^1}dt_{-i}^1 \int_{a_i^1}^{t_i^1} p_i^1(t_{-i}^1, r_i^1)dr_i^1 \end{aligned}$$
$$[From Equation (2)] \tag{3}$$

Next,

$$\begin{aligned} U_i^1(p^1, x^1, (t_i^1, s_i^1)) &= \int [v_i^1(t_{-i}^1, t_i^1)p_i^1(t_{-i}^1, s_i^1) - x_i^{11}(t_{-i}^1, s_i^1)]f_{t_{-i}^1}dt_{-i}^1 \\ &= \int [p_i^1(t_i^1, s_i^1)(t_i^1 - s_i^1) + \int_{a_i}^{s_i} p_i^1(t_i^1, r_i^1)dr_i^1]f_{t_{-i}^1}dt_{-i}^1 \end{aligned}$$

(Using structure of valuation function and Equation (2))

The expressions above are of the same form as a single period auction and hence incentive compatibility follows from the arguments of [1]. IC conditions for period 2 are clear from the definition of the mechanism. Hence,
$U_i^1(p^1, x^1, t_i^1) \geq U_i^1(p^1, x^1, (t_i^1, s_i^1))$ and $U_i^2(p^2, x^2, t_i^2) \geq U_i^2(p^2, x^2, (t_i^2, s_i^2))$

The intuition of the above reduction is to map a multi period game into the dynamics of a static game using suitable payments and allocations and then we use the optimal mechanism for a single period problem.

From equation (3) derived above, $U_i^1(p^1, x^1, t_i^1) \geq 0 \forall t_i^1$ and $U_i^1(p^1, x^1, a_i^1) = 0$ which establishes individual rationality in period 1. IR in period 2 follows from the definition of the mechanism.

Next, we derive an equivalent condition of incentive compatibility for a dynamic auction similar in structure of the one period auction [1].

$$\begin{aligned} U_i^1(p^1, x^1, t_i^1) &\geq U_i^1(p^1, x^1, (t_i^1, s_i^1)) \\ &= U_i^{1s}(p, x, (t_i^1, s_i^1)) + E[U_i^2 | t_{-i}^1, s_i^1] \\ &= U_i^{1s}(p^1, x^1, s_i^1) + (t_i^1 - s_i^1) Q_i^1(p_i^1, s_i^1) + E[U_i^2 | t_{-i}^1, s_i^1] \\ &= U_i^1(p^1, x^1, s_i^1) + (t_i^1 - s_i^1) Q_i^1(p_i^1, s_i^1) \end{aligned}$$

where $Q_i^1(p_i^1, s_i^1) = \int_{T_{-i}} p_i^1(t_{-i}^1, s_i^1) f_{-i}(t_{-i}^1) dt_{-i}^1$

From arguments of [1], it follows
$U_i^1(p^1, x^1, t_i^1) = U_i^1(p^1, x^1, a_i^1) + \int_{a_i^1}^{t_i^1} Q_i^1(p_i^1, s_i^1) ds_i^1$

Proof of revenue maximizing mechanism

$$\begin{aligned} U_0(p, x) &=^a \int_T [\Sigma_{i \in N}(x_j^1(t^1) + E[x_j^2(t^2)|t^1])] f(t) dt \\ &=^b \int_T \Sigma_{j \in N}(x_j^1(t^1) - p_j^1(t^1) v_j^1(t^1) + p_j^1(t^1) v_j^1(t^1) + E[x_j^2(t^2) - p_j^2(t^2) v_j^2(t^2)] + E[p_j^2(t^2) v_j^2(t^2)]) f(t) dt \\ &=^c \int_T \Sigma_{i \in N}[(t_i^1 - \frac{(1 - F_i^1(t_i))}{f_i^1(t_i^1)}) p_i^1(t_i^1) + E[p_j^2(t^2) v_j^2(t^2)]] f(t) dt - \Sigma_{i \in N} U_i^1(p, x, a_i) \\ &=^d \int_T \Sigma_{i \in N}[(t_i^1 + \Sigma_{j \in N} E[p_j^2 t_j^2 | a^1 = i] - \frac{1 - F_i^1(t_i)}{f_i^1(t_i^1)}) p_i^1(t_i^1)] f(t) dt - \Sigma_{i \in N} U_i^1(p, x, a_i) \end{aligned}$$

Equation (a) represents the expected to go utility of the seller (expected revenue obtained by the seller in the entire horizon). It is clear from (a) that Mechanism M_2 defined earlier is optimal for period 2. For any outcome of M_1, M_2 extracts optimal revenue in period 2.

Equation (c) follows from the aforementioned equivalent conditions and Myerson [1].

Equation (d) - The first 2 terms together represent the expected revenue to the seller and the third term represents the *information rent* that he needs to afford. From the payment construction and IR constraints, it is clear that $U_i^1(p, x, a_i) = 0 \ \forall i$ which is optimal. We also realize that the terms inside the integral are functions of t_i^1 alone and hence *ironing* this modified function to construct the allocation rule is optimal in period 1 [1].

Example

Model with 1 player, 1 seller with reserve price r. Players valuations are $U[0,1]$ in both periods. They don't evolve at all. The reserve price, $r = \frac{1}{2}$ (fixed). If the buyer bids more than $\frac{1}{2}$, he gets the object, otherwise the seller retains the object. The expected utility for the buyer in period 2 is $\frac{1}{8}$. Payment Mechanism in period 1: If you win pay: $\frac{1}{2} + \frac{1}{8}$; if you lose pay $\frac{1}{8}$ Clearly, this mechanism yields more profit than Myerson[1] mechanism implemented twice in both periods since the latter charges $\frac{1}{2}$ and 0 respectively.we charge a constant $\frac{1}{8}$ more uniformly.

The new mechanism is IR (clear) and IC. Intuitive proof for IC,
if the buyers value is less than $\frac{1}{2}$:

- If he bids above $\frac{1}{2}$ he loses more than what he would have if he had bid less than $\frac{1}{2}$.
- If he bids any other value less than $\frac{1}{2}$ still he pays the same $\frac{1}{8}$ without the object.

If the buyers value is greater than $\frac{1}{2}$:

- he bids less than $\frac{1}{2}$, then he does not get the object and his utility is -$\frac{1}{8}$
- If he bids something else above $\frac{1}{2}$, his utility still remains the same and he does not make more profit at all.

The above arguments can be formalized, implying truthful bidding is the best for the bidder given this scene.

Period 2 arguments are clear (Just 1 period static Auction)

References

[1] R.B. Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58, 1981.