Math Camp

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Unit 1

MSSM Program

Columbia University

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Math as a Language

- Speed reading is pointless:
  - Languages are learnt through practice
- Work through the problems
- Self-select

Why Master Math?

The role of math in sustainability management

- Scientific relationships and socio-economic behaviour is often complex.
- We need linear and nonlinear functions to model it.
- Math, especially calculus, simplifies the analysis of functions
- Math is often a tool of mystification designed to silence critics

Resources

- Foundations website being built
- MSSM math primer:
  - http://www.columbia.edu/~sgb2/Math_Primer_MSSM.pdf
- SIPA math primer:
  - http://www.columbia.edu/itc/sipa/math/

Math Camp Outline

- Unit 1: Functions, Equations, Slopes
- Unit 2: Derivatives, Integration, Statistics
- Unit 3: Capital Budgeting/Financial Statements

Unit 1 Outline

- Linear Functions
- Systems of Linear Equations
- Nonlinear Functions
- Slopes
- Univariate Calculus

Math Camp

Interlude

Functions

- **A function** is a **rule that assigns one number (output) to another number (input)**
  - For example, the function 2x takes an arbitrary number x as input and assigns "2 times the input" as the output
  - If x = 5, the function 2x = 2 × 5 = 10

A More Formal Definition

- A function f is a rule that assigns to each element x of a set an element $f(x)$ of a second set. The first set is the domain of f. The element $f(x)$ is the value of f at x.

$$\text{Domain} \xrightarrow{\quad f \quad} \text{Set of Function Values}$$

Another Definition

- A function is a set of ordered pairs whose first entries are all different.
  - The function that converts Celsius to Fahrenheit can be represented by the set of ordered pairs of the form $(C,(9/5)C+32)$. The number C is the number of degrees Celsius; the number $(9/5)C+32$ is the corresponding number of degrees Fahrenheit.
  - The function that computes the area of a circle from its radius can be thought of as the set of ordered pairs of the form $(r,\pi r^2)$.

Example

The government of Powellville has set up an assistance program for banks which make losses: any bank which says it faces losses of \$x billion from the pandemic will receive \$2x billion in low-interest loans.

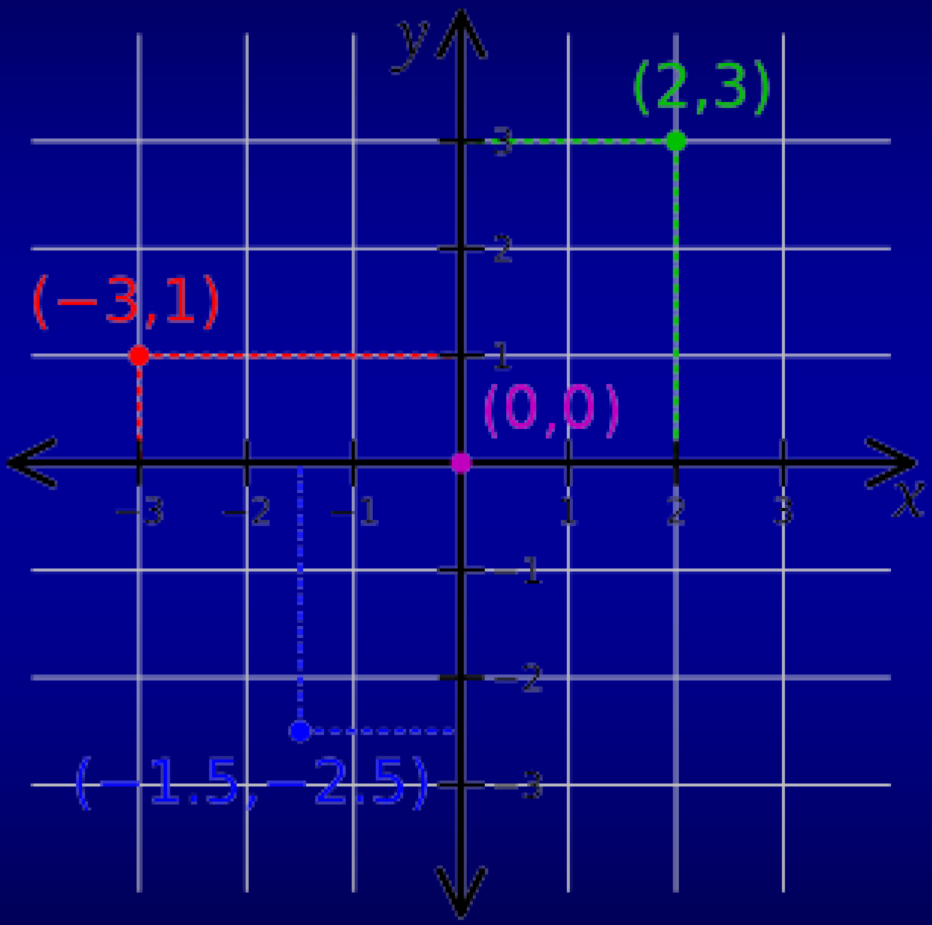
Tabular Representation

The government's assistance program is a rule (or function) which assigns a loan amount (the output) to a bank based on the bank's stated losses (the input).

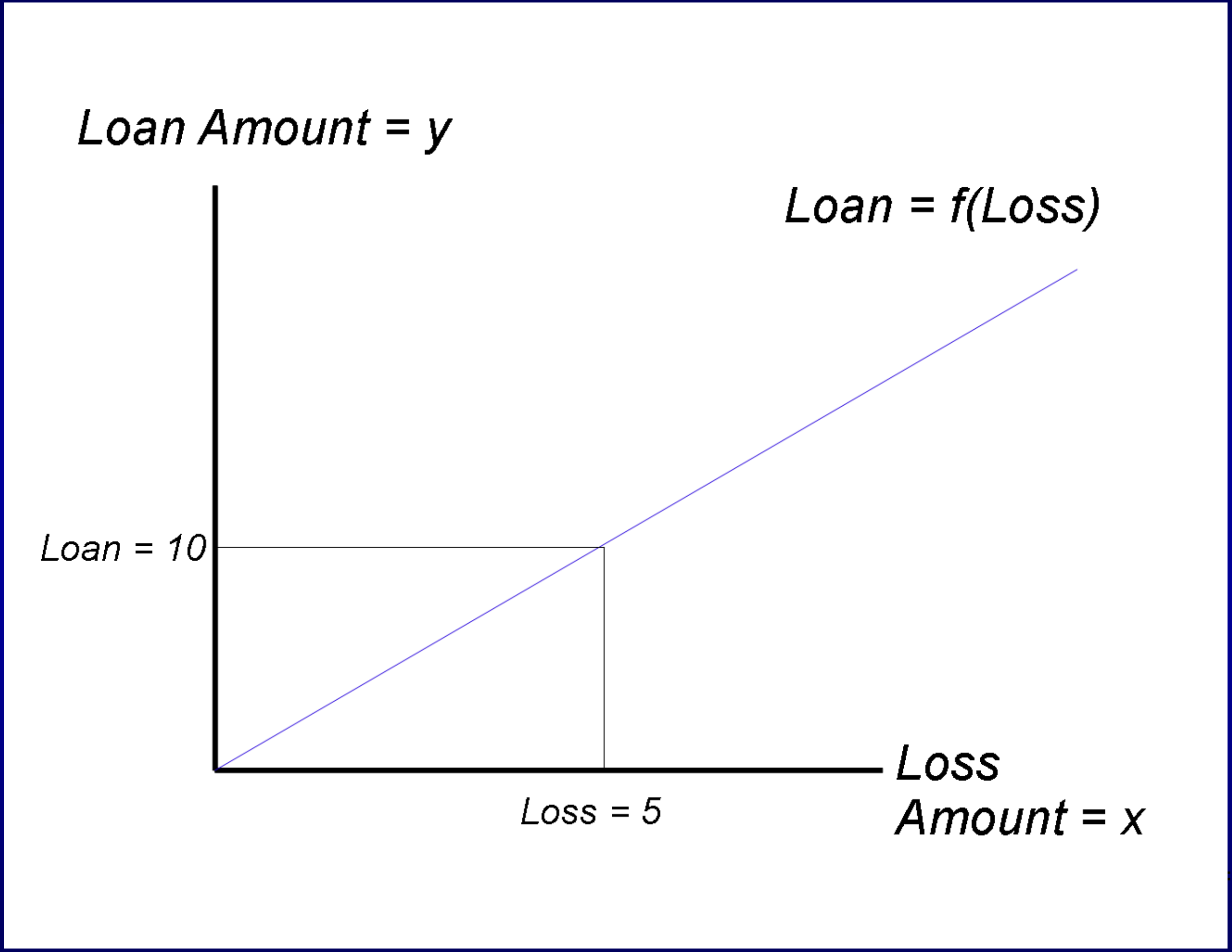
Amounts in $billions					
Loss amount (x)	0	5	10	15	20
Loan Amount (y)	0	10	20	30	40

Cartesian Plane

The Cartesian plane or coordinate system is a method of representing each point in the plane by a pair of numbers known as the coordinates. The coordinates represent distances from the vertical reference line (y-axis) and the horizontal reference line (x-axis)



Graphical Representation



All points on the blue line satisfy the equation y = f(x)

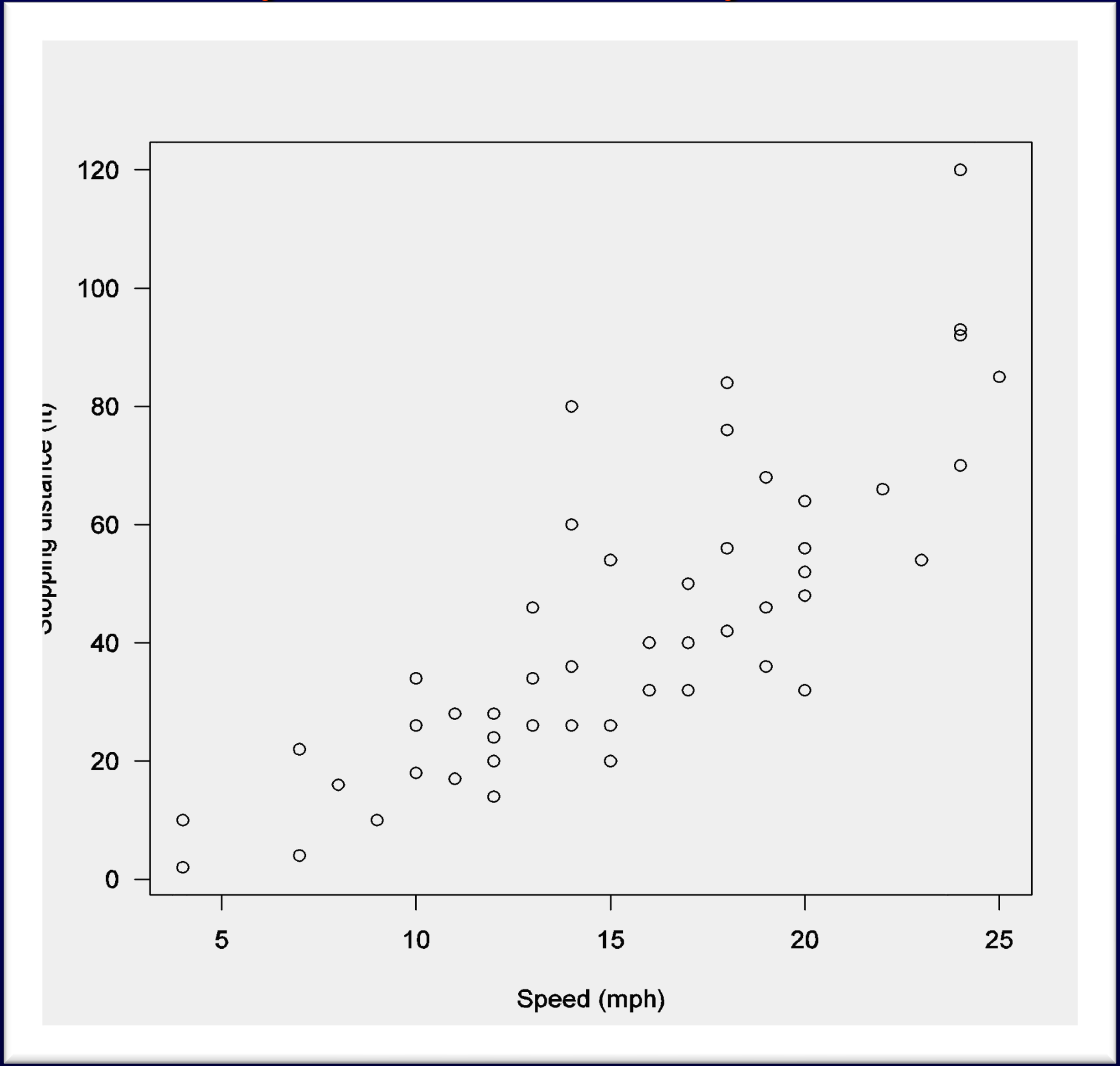
Graphical Representation

What is the domain of the above function?

Graphical Representation

What is the difference between the two representations?

Graphical Representation



A function whose form is not known with certainty.

Functional Notation

y = f(x) indicates that the variable y is a function of x

- **x**, the input variable, is called the "independent variable"
- **y**, the output variable, is called the "dependent variable"
- **f** is the functional relationship or rule that assigns particular values of y to particular values of x.
  - For example, y = f(x) = 2x
  - Loan amount = f(Loss Amount) = 2×Loss Amount

Example Functions

- Manufacturing cost is a function of quantity produced
  - $C = f(q)$ where C is manufacturing cost in \$ and q is quantity produced in units
- Desired residential square footage is a function of family size
  - **R = f(s)** where R is desired residential square footage and s is family size

Example

A firm spends x dollars on product development and y dollars on advertising. Its profit is described by the following relationship:

$F(x,y) = 36000 + 40x + 30y + (xy)/1000$

What is profit if the firm spends \$2000 on product development and \$5000 on advertising?

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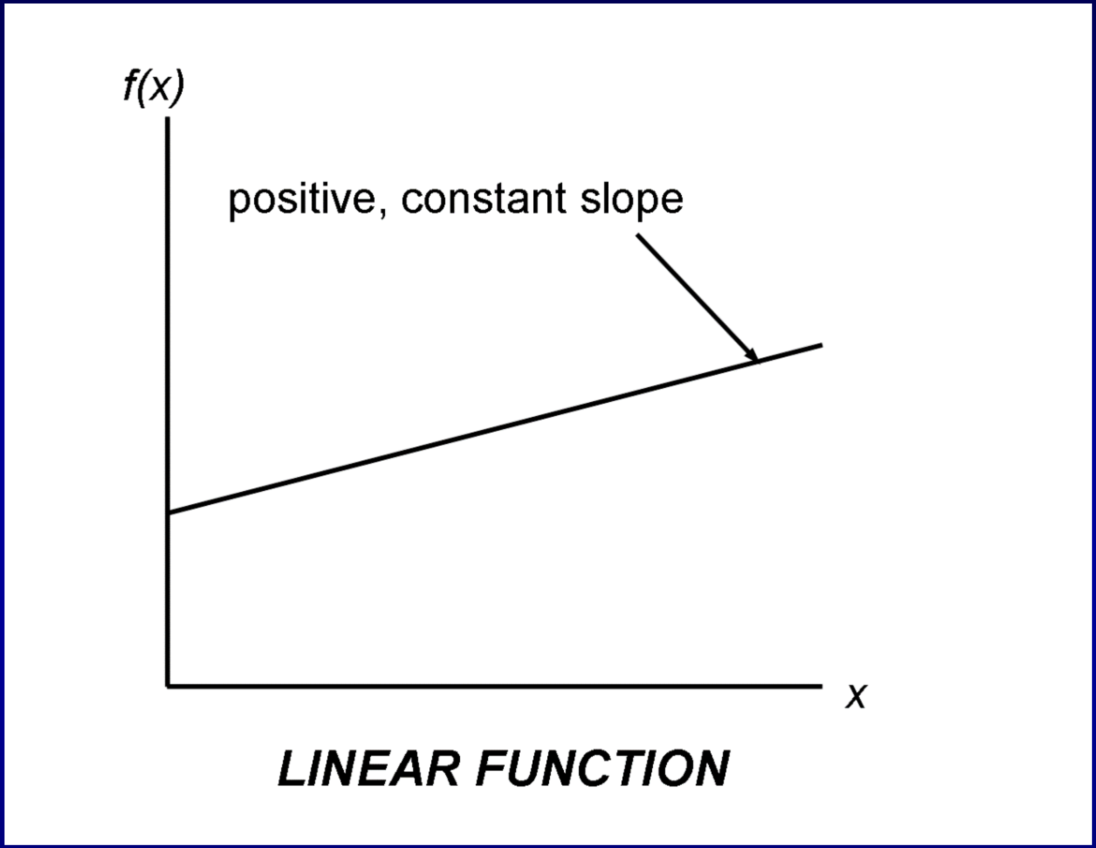
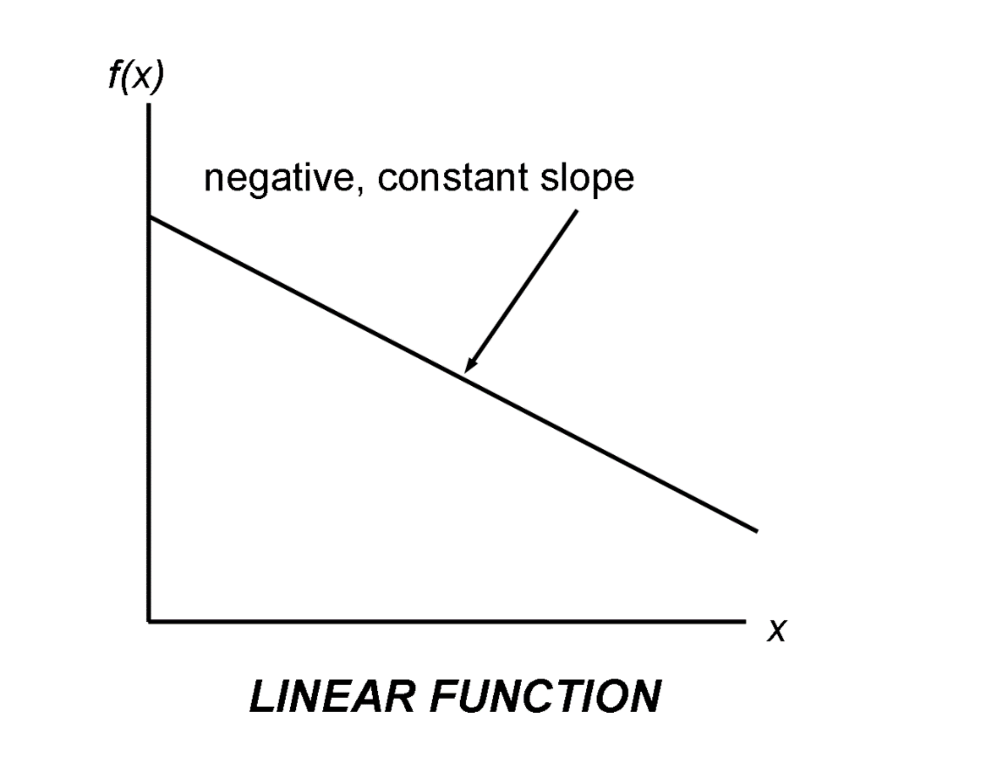
Interlude

Increasing Functions



- A function is increasing if its graph moves upward from left to right. More formally, f(x) is increasing if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

Decreasing Functions



- A function is decreasing if its graph moves downward from left to right. More formally, f(x) is decreasing if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$

Linear Functions

- Linear functions are those where the graph of the function is a straight line.
- A linear function has the following form:

  $y = f(x) = b + mx$

where

b is a constant term representing the y-intercept

m is the slope of the line

Intercepts

- The y-intercept is the point where the line crosses the y-axis.
- The x-intercept is the point where the line crosses the x-axis. What is the x-intercept of the function (in terms of y and b)?

$0 = b + mx$

$\Rightarrow -b = mx$

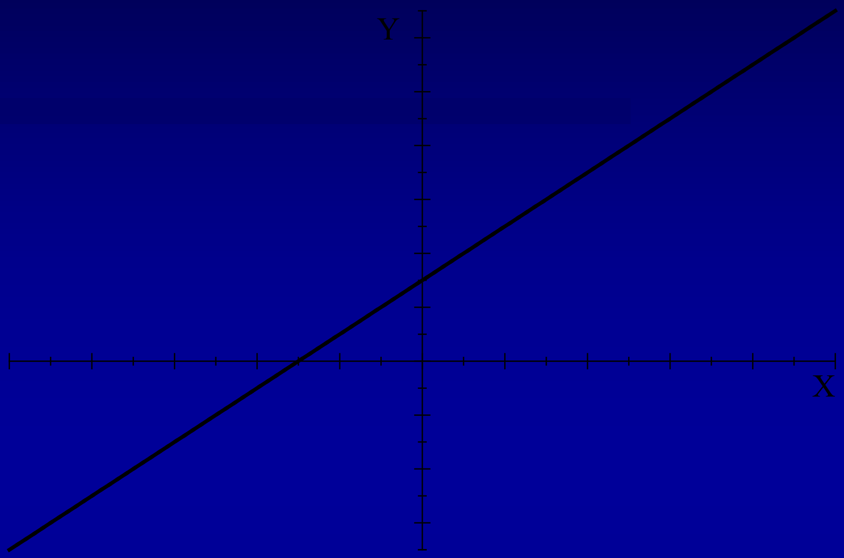
$\Rightarrow x = -b/m$

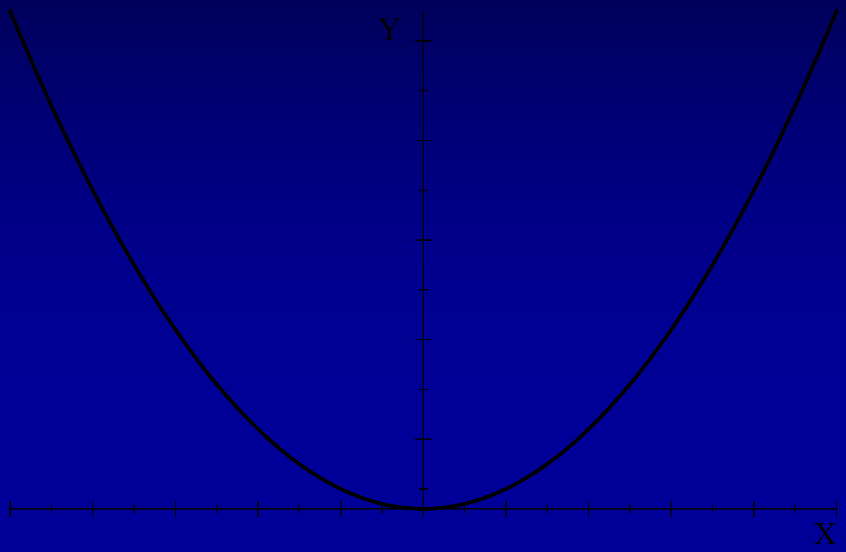
Slope

- The slope of a function is the change in y (the "rise") divided by the change in x (the "run").
- For linear functions:

  slope = coefficient of x = m
- The slope indicates
  - Steepness (magnitude of m)
  - Direction (sign of m)

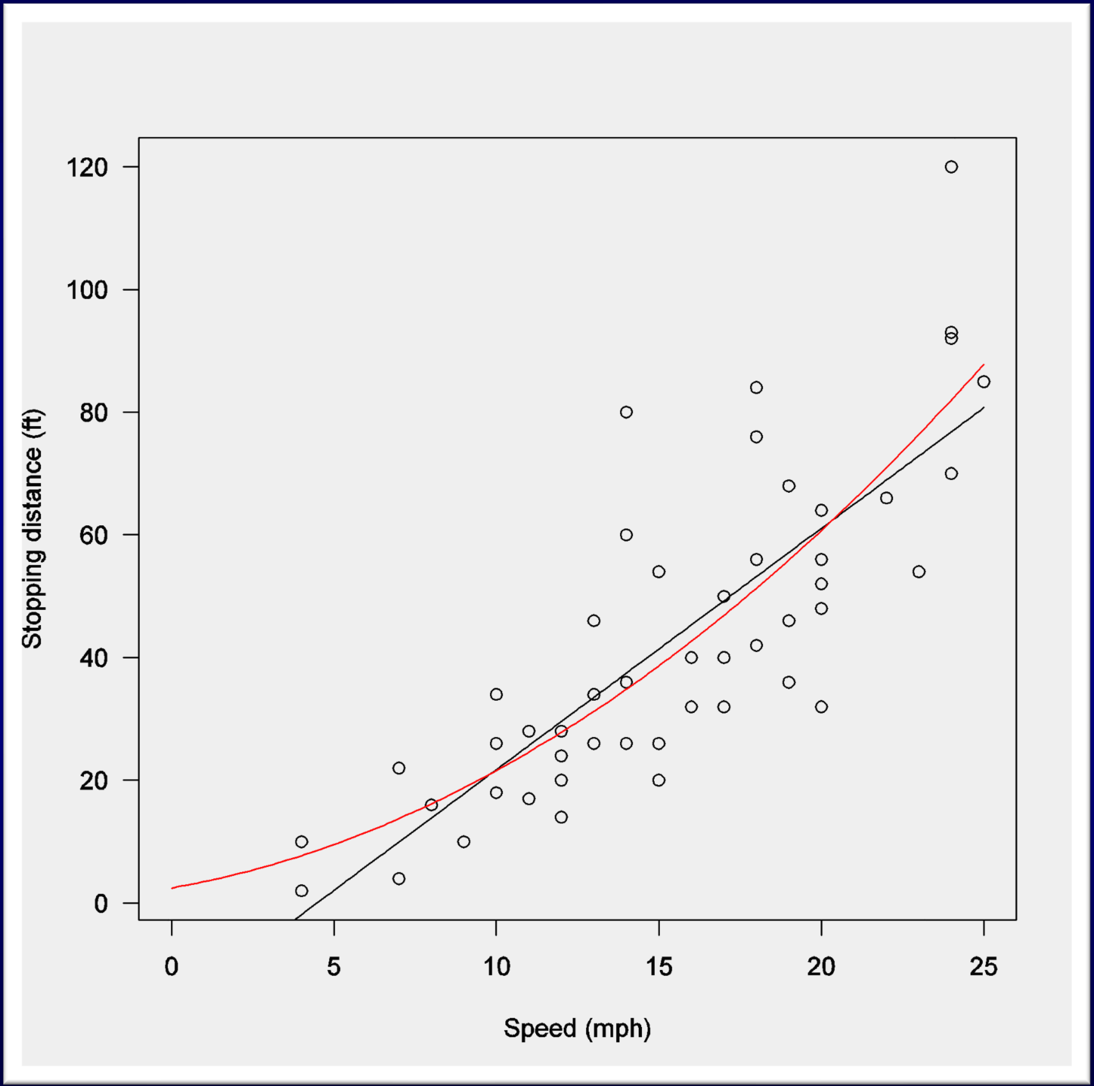
Linear vs Nonlinear Functions





- A **linear function** is a function whose graph is a straight line.
- A **nonlinear function** by definition has a different slope at every point along the curve, unlike a **linear function** whose slope stays constant.

Linear vs Nonlinear Functions



- An uncertain relationship can be modeled as a **linear function** or a **nonlinear function** such as a quadratic function.

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Interlude

Equation of a Linear Function

- Can you derive the equation of a linear function if:
  - you only know the slope of a linear function?

    NO
  - you only know the y-intercept of a linear function?

    NO

Deriving the Equation of a Linear Function

- In order to derive the equation:
  - EITHER, you must know the slope and the y-intercept
  - OR, you must know **two** points on the line

Problem 1

- Find the equation of the line which has a slope of 4 and a set of coordinates (3,-2).

Problem 1 Solution

- Find the equation of the line which has a slope of 4 and a set of coordinates (3,-2).

  $y = mx + b = 4x + ?$

  We need to find the y-intercept.

  Let $x_1 = 3$ and $y_1 = -2$. (x_1,y_1) is on the line.

  Let the y-intercept be (x_2,y_2)

  We know rise/run = 4

  Hence, $(y_1 - y_2)/(x_1 - x_2) = 4$

  $x_2 = 0$ (by definition)

  Solving $(-2 - y_2)/(3 - 0) = 4$ gives $y_2 = -14$

  Hence, $y = 4x - 14$

Problem 2

Find the equation of the line which passes through the points (-2,3) and (3,8).

Problem 2 Solution

- Find the equation of the line which passes through the points (-2,3) and (3,8).

  $y = mx + b = ?x + ?$

  We need to find the slope m

  $m = (y_1 - y_2)/(x_1 - x_2)$

  $m = (3 - 8)/(-2 - 3) = -5/-5 = 1$

  We need to find the y-intercept.

  Let the y-intercept be $(0, y_3)$

  Hence, $(y_1 - y_3)/(x_1 - 0) = 1$

  Solving $(3 - y_3)/(-2 - 0) = 1$ gives $y_3 = 5$

  Hence, $y = x + 5$

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Interlude

Systems of Linear Equations

- The analysis of many economic problems involves system of equations
- A system of equations is a set of equations in the same variables that are solved simultaneously
- Example:
  - Equation 1: $y = 4x - 14$
  - Equation 2: $y = x + 5$
- Solving means finding the values of x and y such that both equations are valid.

Solutions to Systems of Linear Equations

There are 3 possibilities:

1. There is a unique solution (the two lines cross in exactly one point)
2. There is no solution (the lines do not cross)
3. There are infinitely many solutions (the two lines coincide)

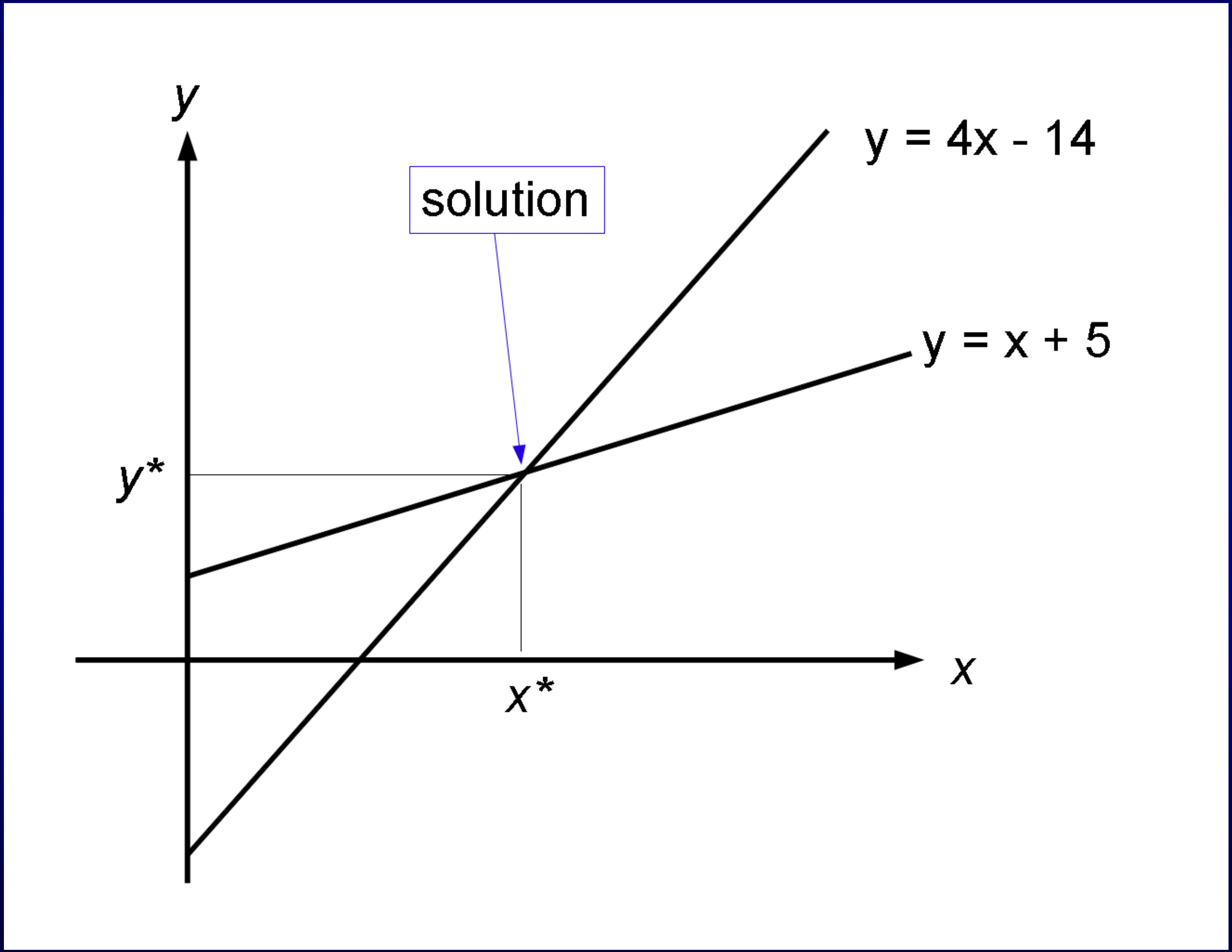
A system with a unique solution must have at least as many equations as unknowns.

Graphical Method

This method is convenient for systems with just 2 unknowns.

1. Graph the functions representing both equations.
2. The values of x and y at the point of intersection represent the solution to the system.

Graphical Method



Substitution Method

Solve one of the equations for one of the two variables and substitute into the other equation.

Substitution eliminates one variable so that you end up with one equation with one unknown that is easily solvable.

Substitution Method

Express y as a function of x in equation 1:

$y = 4x - 14$

Substitute this expression for y in equation 2:

$4x^* - 14 = x^* + 5$

$4x^* - x^* = 5 + 14$

$3x^* = 19$

$x^* = 19/3 = 6.33$

Substitute x* in either equation 1 or 2 to find y*:

$y^* = 19/3 + 5 = 11.33$

Application: Supply & Demand

The interaction of supply and demand in the market leads to the market equilibrium. The equilibrium price is the one at which demand equals supply. The equilibrium quantity is the quantity exchanged. The equilibrium is computed as a system of two equations and two unknowns: Price and Quantity

$Q_d = 300 - 6P$

$Q_s = \frac{1}{4} P$

Application: Supply & Demand

$Q_d = 300 - 6P$

$Q_s = \frac{1}{4} P$

What are the slopes of demand and supply? What are the Q and P intercepts? What is equilibrium price P* and quantity exchanged q*?

Application: Supply & Demand

Market equilibrium:

$300 - 6P = \frac{1}{4} P \Rightarrow 300 = 25/4\ P$

$P^* = (300 \times 4)/25 = 48$

$Q^* = \frac{1}{4} P^* = 48/4 = 12$

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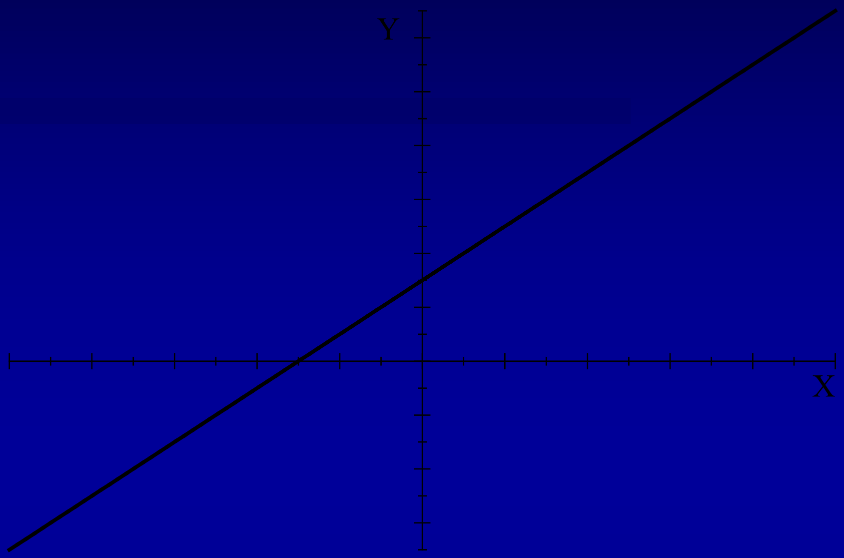
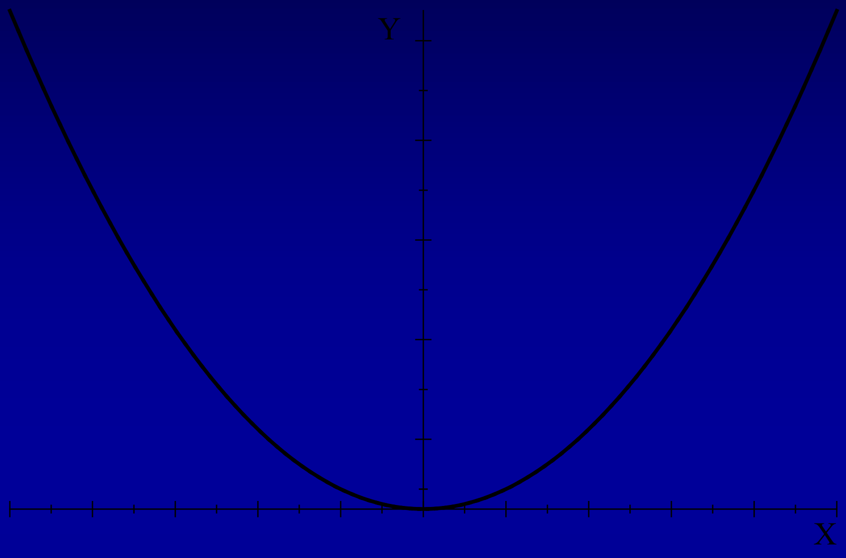
Interlude

Nonlinear Functions



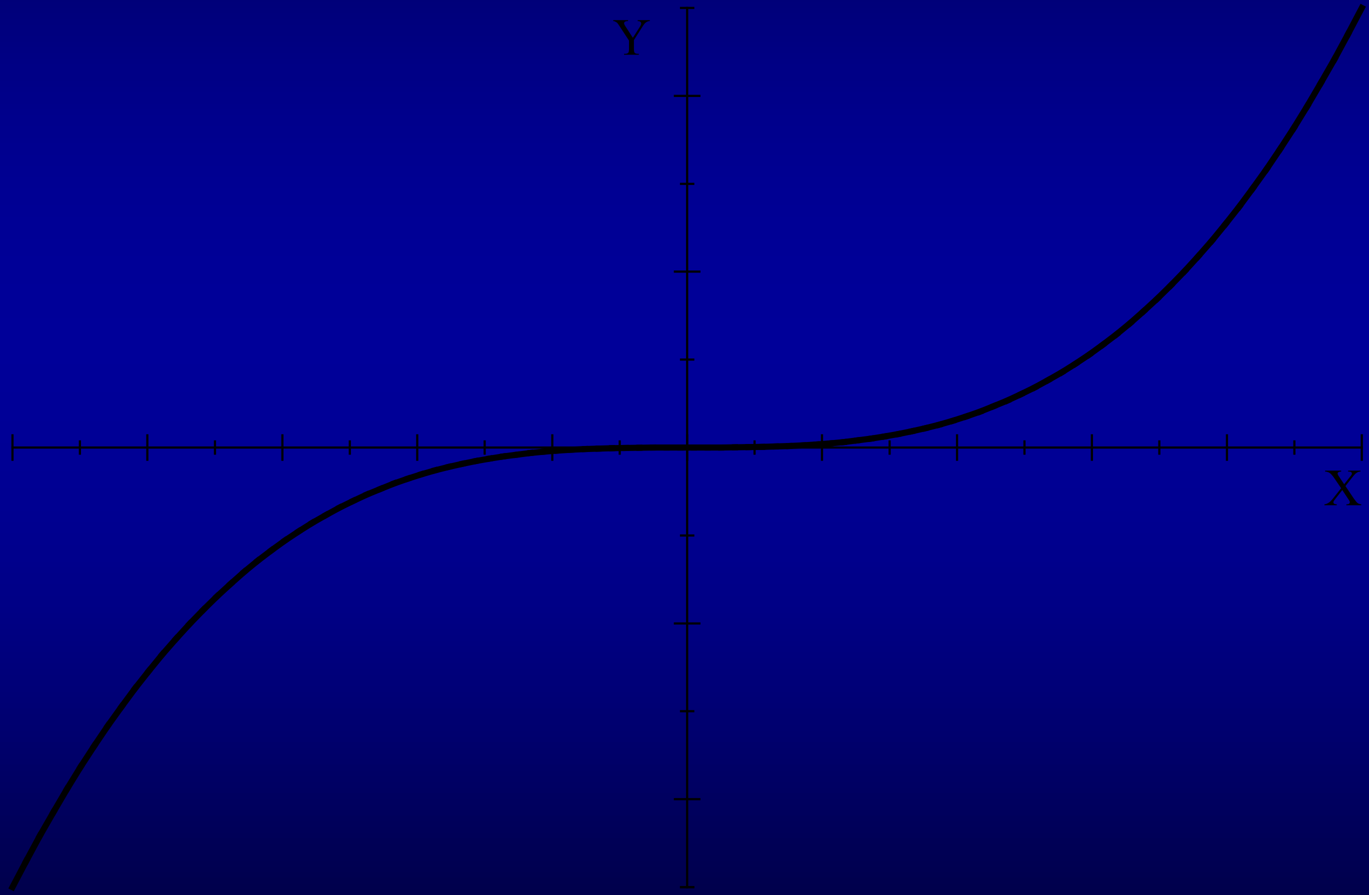


- A **nonlinear function** by definition has a different slope at every point along the curve, unlike a **linear function** whose slope stays constant.
- In order to calculate a constantly changing slope, we will use calculus techniques.

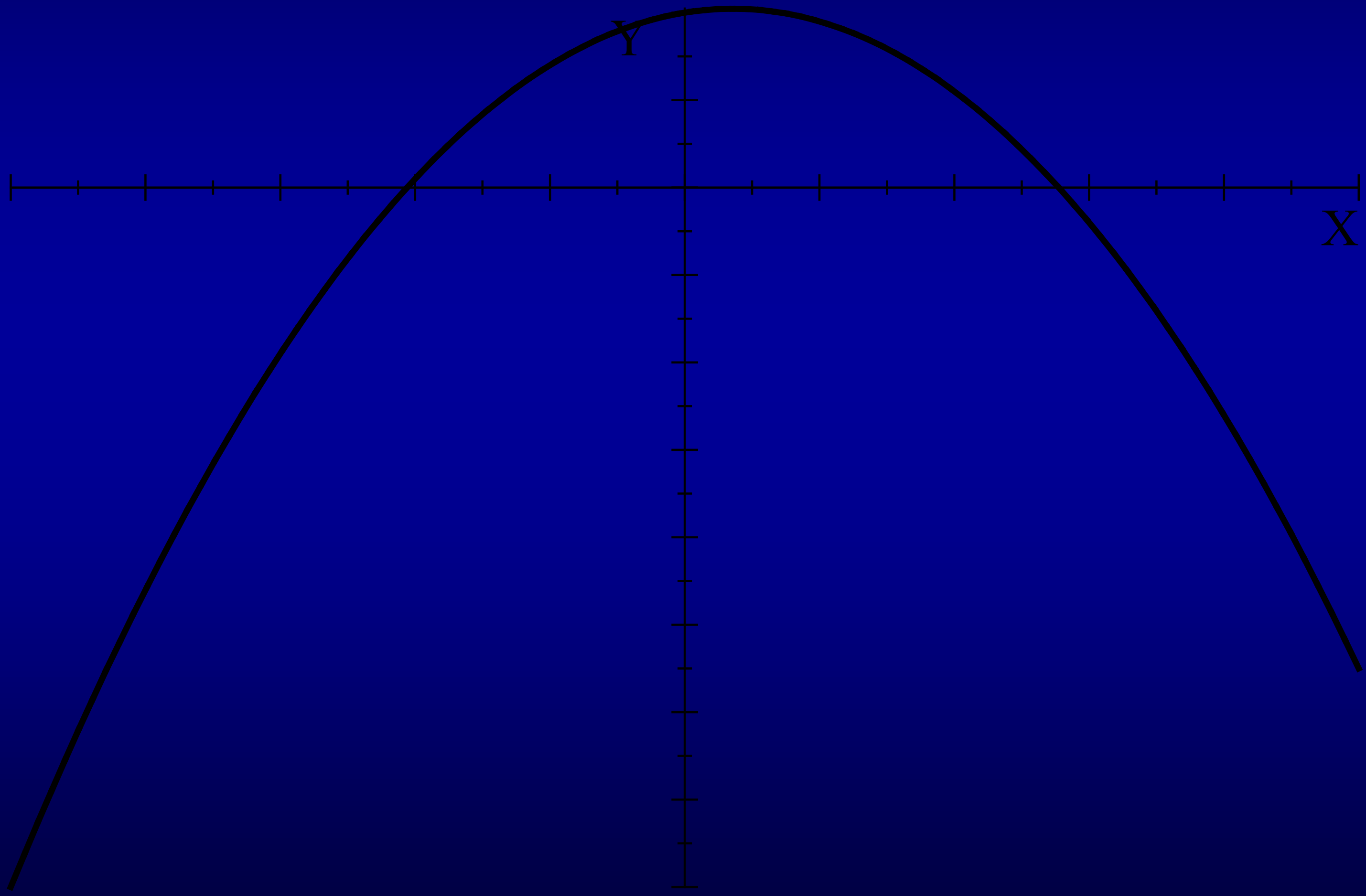
Examples of nonlinear functions

- power function
  - $y = f(x) = ax^c$
- polynomial function
  - $y = f(x) = ax^n + bx^{n-1} + \ldots + c$ (n integer)
- exponential function
  - $y = f(x) = e^x$
- logarithmic function
  - $y = f(x) = \ln x$
- piecewise linear function *e.g.* absolute value function
  - $y = f(x) = |x|$

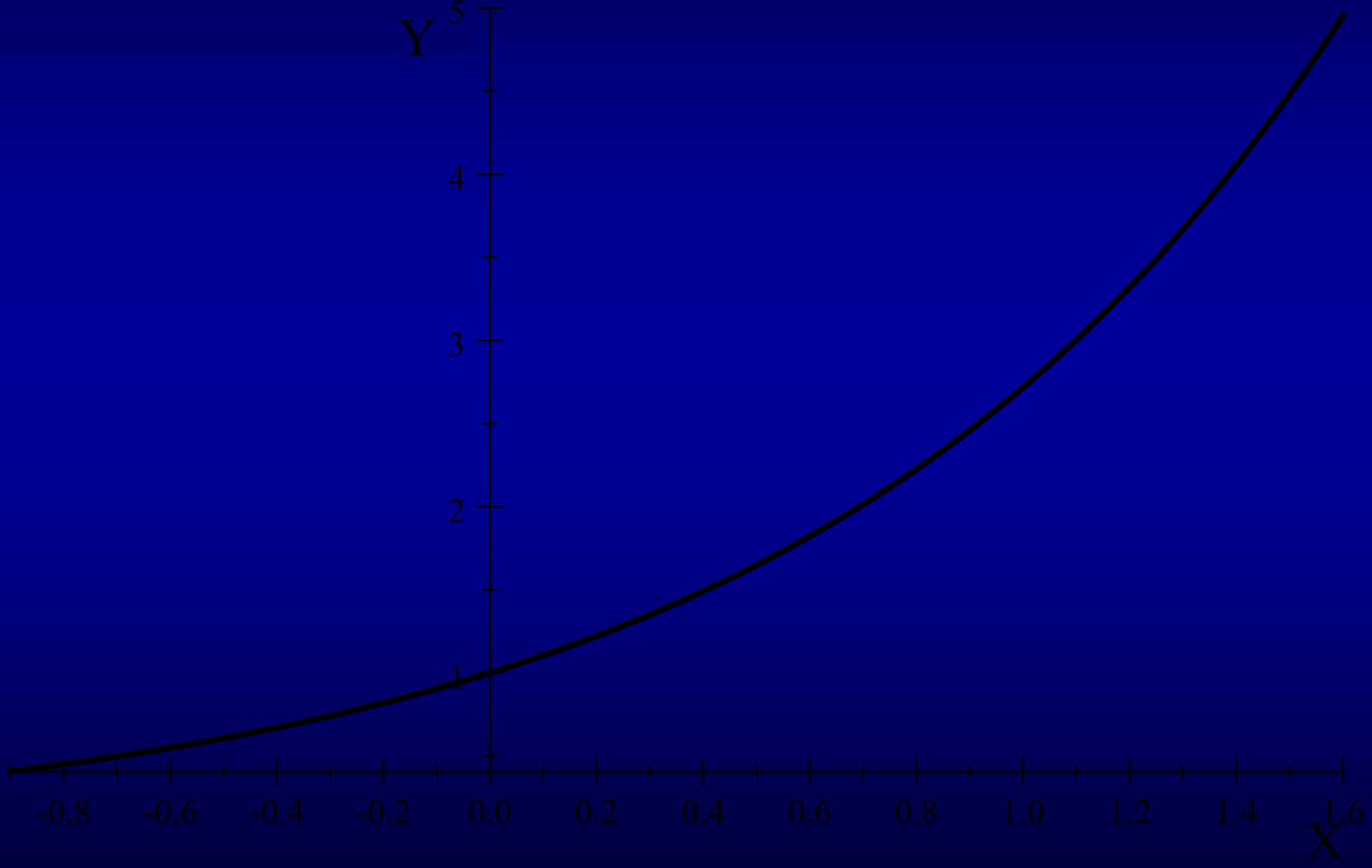
power function $y = 2x^3$



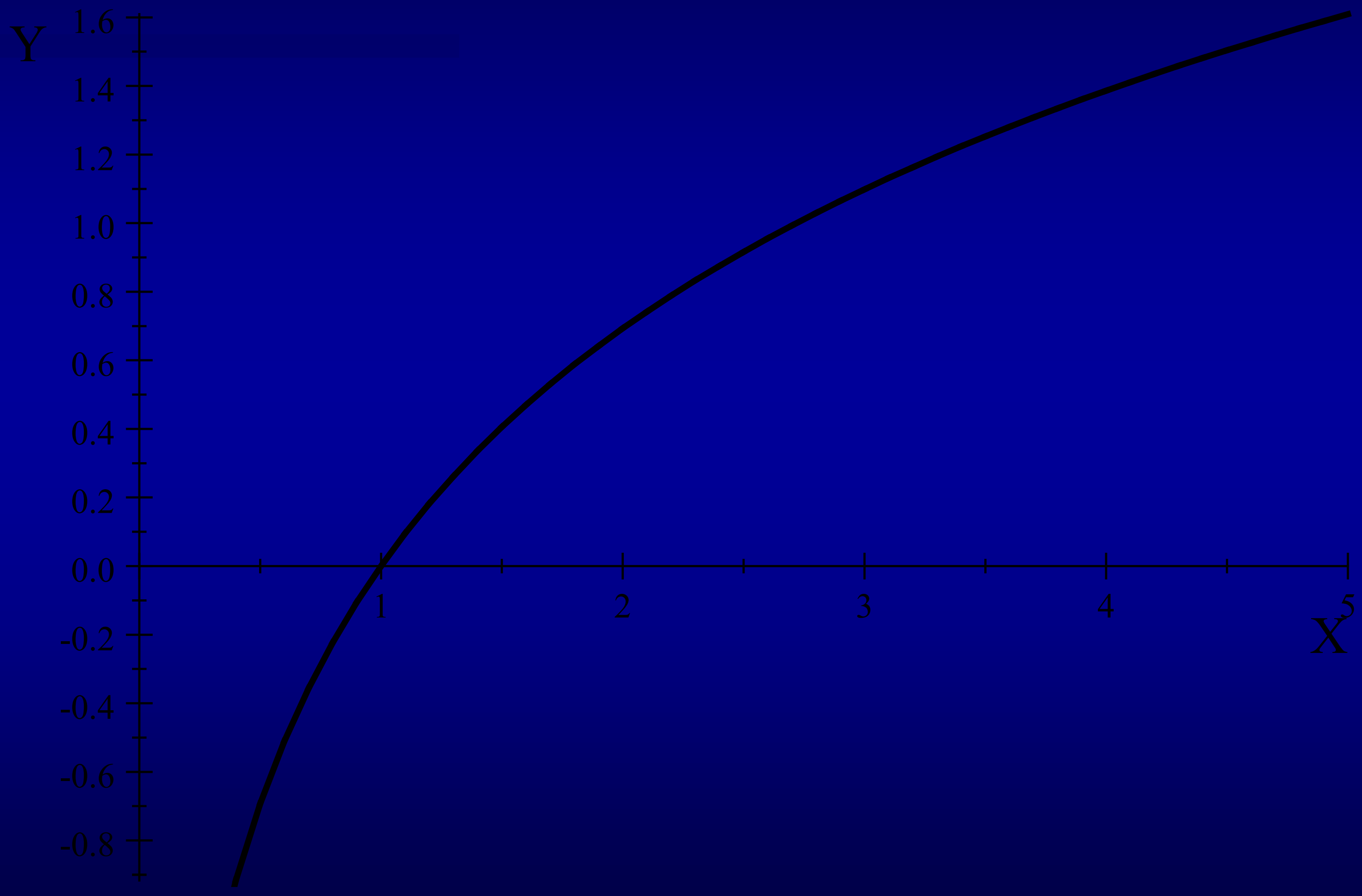
polynomial function $y = -7x^2 + 5x + 40$



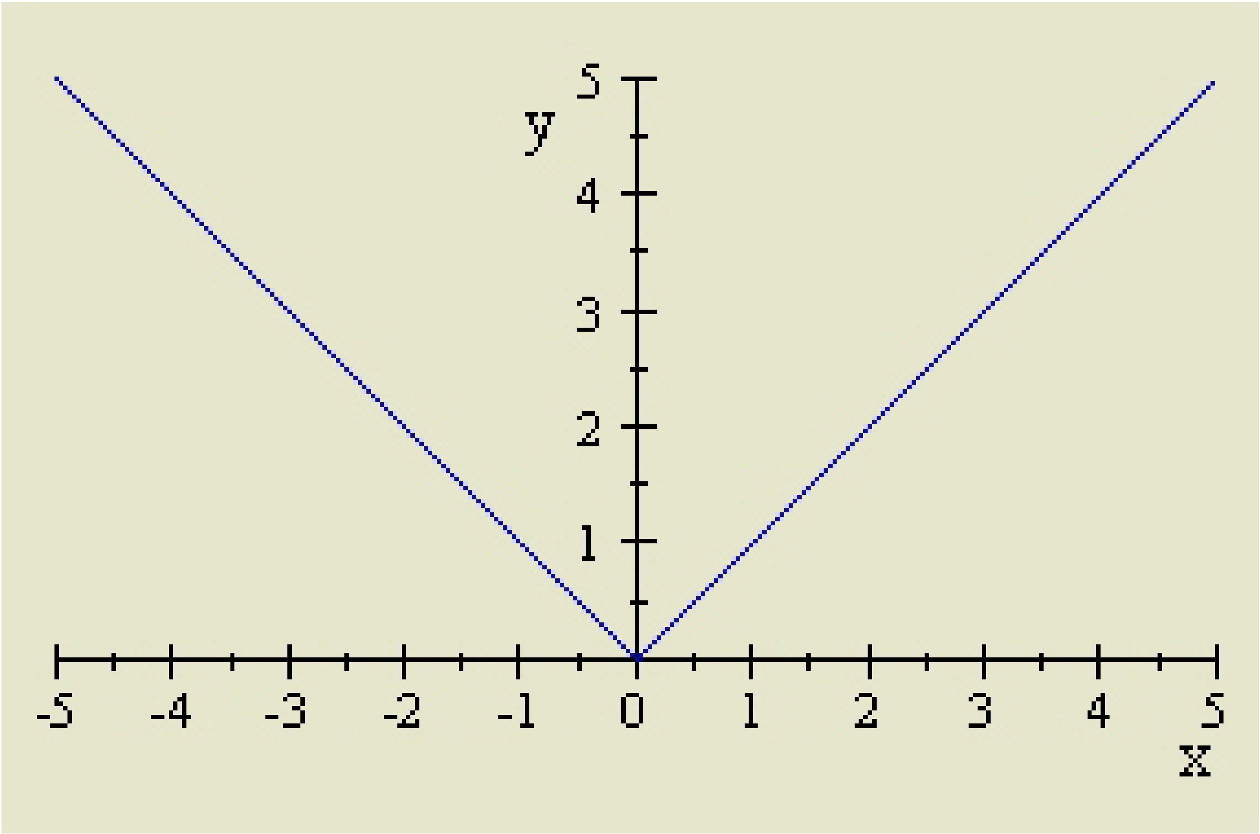
exponential function $y = e^x$



logarithmic function y = ln x



Absolute value function y = |x|



$f(x) = |x|$

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Interlude

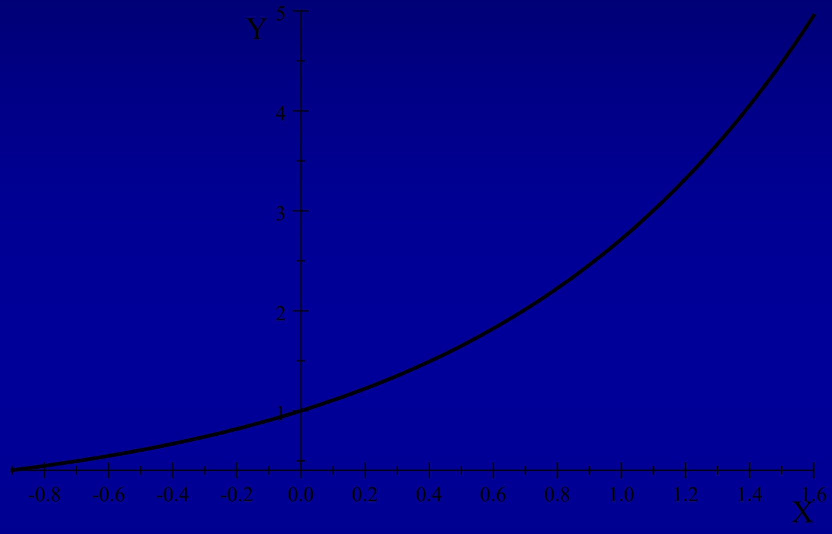
Inverse functions

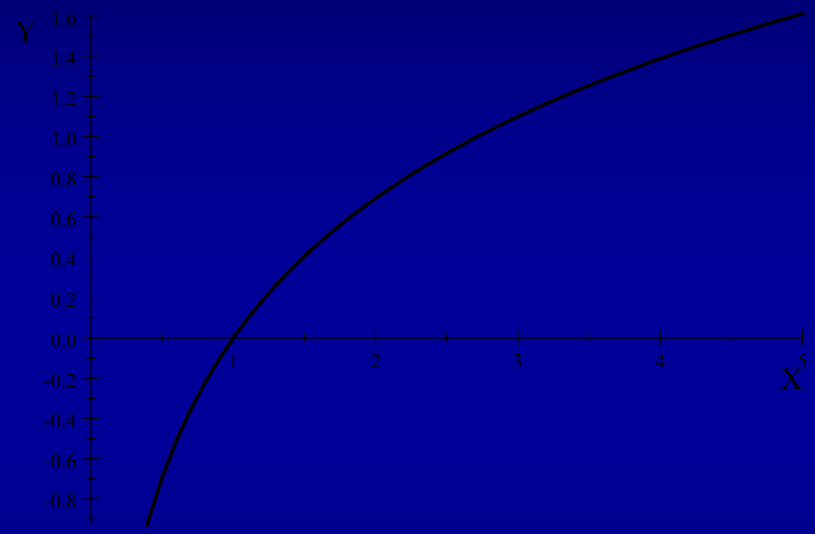
- A function assigns a **unique** value of y to each value of x.
  - What is the vertical line test for a function?
- A **monotonic** function is one that **always increases** or **always decreases**.
- $\therefore$ for monotonic functions, there is a unique value of x associated with each value of y. The rule that assigns a unique value of x to each value of y in this way is called the **inverse** function.

Inverse function Example

- If $y = 2x$, the inverse function is obtained by solving for x: $x = y/2$.
- Is there an inverse function for the function $y = x^2$?
- Is the function $y = x^2$ a monotonic function?

Inverse function Example





- If $y = e^x$, the inverse function is obtained by solving for x: $x = \ln y$.

Inverse Function Example

Compute $F(C)=(9/5)C+32$ for $C=20$.

Write C as a function of F, i.e., determine a formula for the function $C(F)$.

Implicit functions

- A function is usually defined explicitly as $y = f(x)$.
- Sometimes all we know about a function y is that it satisfies an equation such as $2x^2y - 3xy^3 + 5x = 10$
- Something implied or understood, but not directly expressed, is said to be implicit. A function is defined implicitly by stating the equation that it satisfies.
- Here, y is defined implicitly as a function of x.

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Interlude

Application: Power Plant Choice

Consider the problem of choosing which generator can most cheaply serve an electricity load of a certain duration.

The choice will depend on fixed cost, variable cost and usage.

Suppose the overnight cost of building a coal power plant and a gas turbine are given below:

Coal Plant	$1,050/kW
Gas turbine	$350/kW

Application: Power Plant Choice

Overnight cost is the cost of a construction project if no interest was incurred during construction, as if the project was completed "overnight." It is equivalent to the present value cost that would have to be paid as a lump sum up front to completely pay for a construction project.

The overnight cost concept allows comparison of construction costs incurred over differing construction periods.

Application: Power Plant Choice

Overnight cost OC must be amortized (or 'levelized') over the useful life of the power plant, to determine the annual fixed cost FC in $/kWy

$$FC = \frac{r \times OC}{1 - e^{-rT}}$$

Where r is the annual interest rate and T the life of the plant

Application: Power Plant Choice

With r = 0.1 (or 10%) and T = 40 for coal plants and T = 20 for gas turbines, the levelized fixed costs per year are:

Coal Plant	$106.96/kWy
Gas turbine	$40.48/kWy

Suppose variable costs per year are:

Coal Plant	$87.60/kWy
Gas turbine	$306.60/kWy

Application: Power Plant Choice

If the power plant was run 100% of the time, the annual breakeven revenue would be the sum of the levelized FC and annual VC.

Since the power plant will not likely be run 100% of the time, we define cf as the proportion of time the plant is run, i.e. the capacity factor.

Application: Power Plant Choice

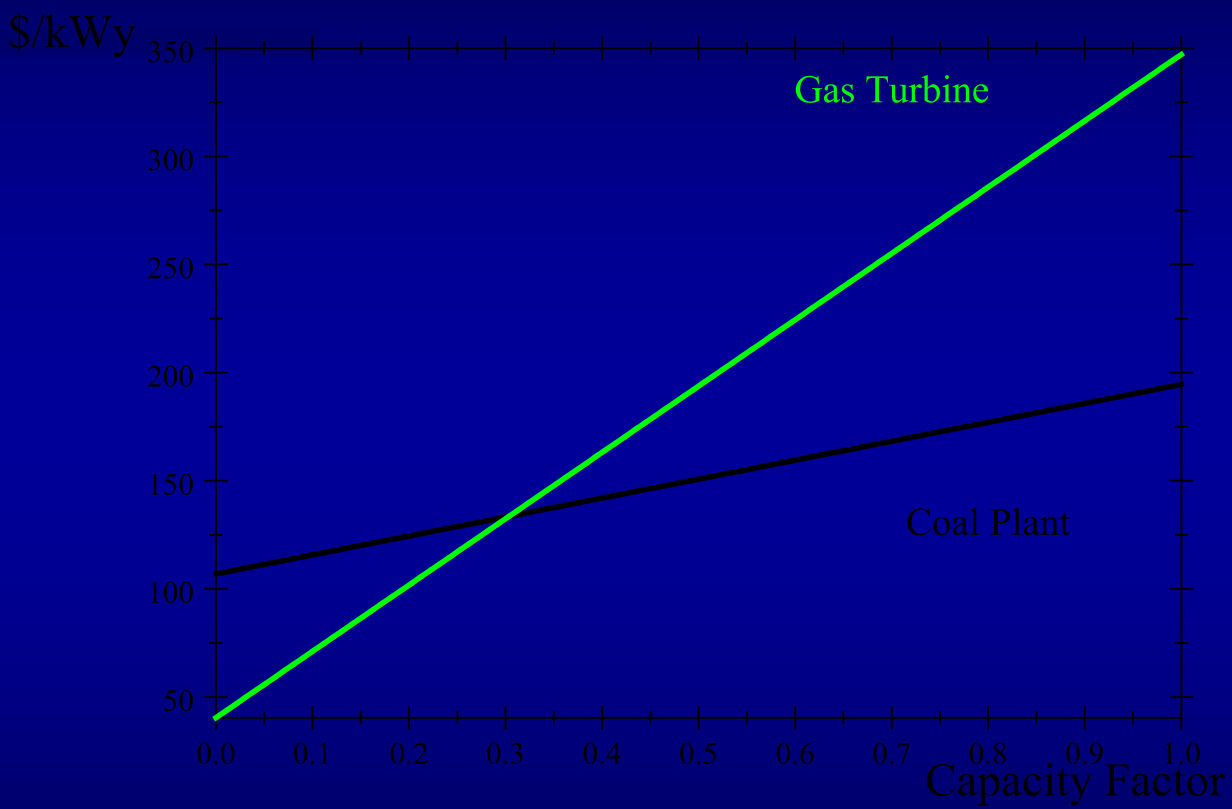
We define the screening curves for each type of plant:

Coal Plant $\$106.96 + \$87.60 \times cf$

Gas turbine $\$40.48 + \$306.60 \times cf$

At what capacity factors is the gas turbine the cheaper choice?

Application: Power Plant Choice



Application: Power Plant Choice

At what capacity factors is the gas turbine the cheaper choice?

The gas turbine is cheaper at capacity factors lower than 0.304, when the lower fixed cost of a gas turbine is not offset by the higher variable cost. At high capacity factors (usage), the coal plant is cheaper since it has a lower variable cost than the gas turbine.

Application: Supply & Demand

Consider the following demand & supply functions for the Toyota Highlander Hybrid & the Chevy Suburban. Assume that prices are always positive.

<u>Market for Highlanders</u>:

Demand: $Qd = 60000 - 2p$

Supply: $Qs = 4p - 90000$

<u>Market for Suburbans</u>:

Demand: $Qd = 45000 – p$

Supply: $Qs = 2p – 75000$

Application: Supply & Demand

1. What is the equilibrium price and quantity of Highlanders and Suburbans sold?
2. Compute the consumer surplus in both markets.

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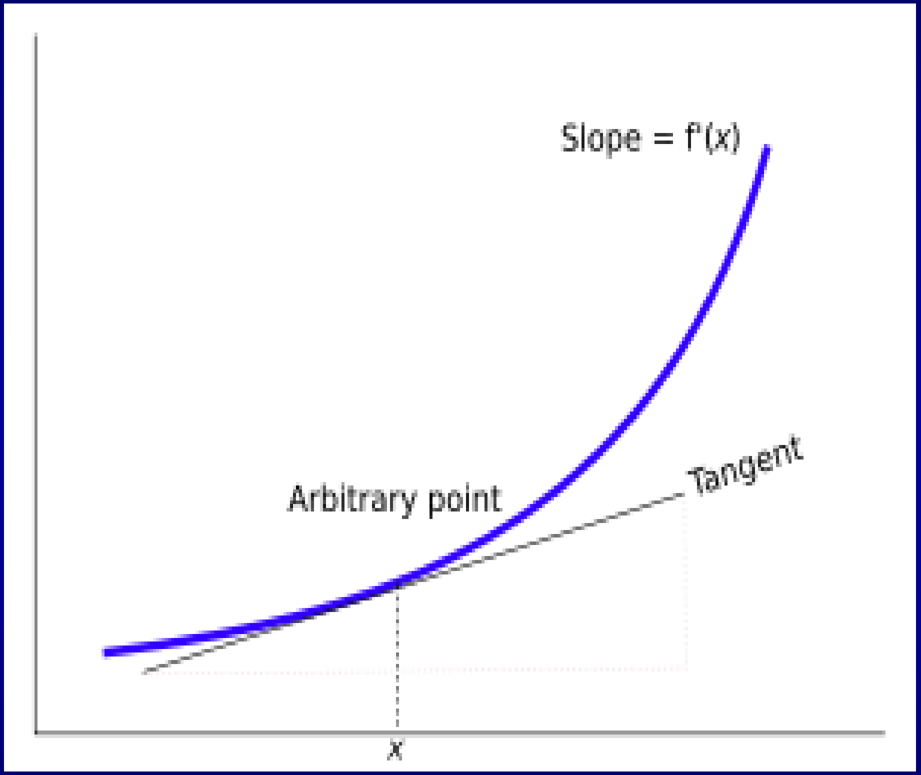
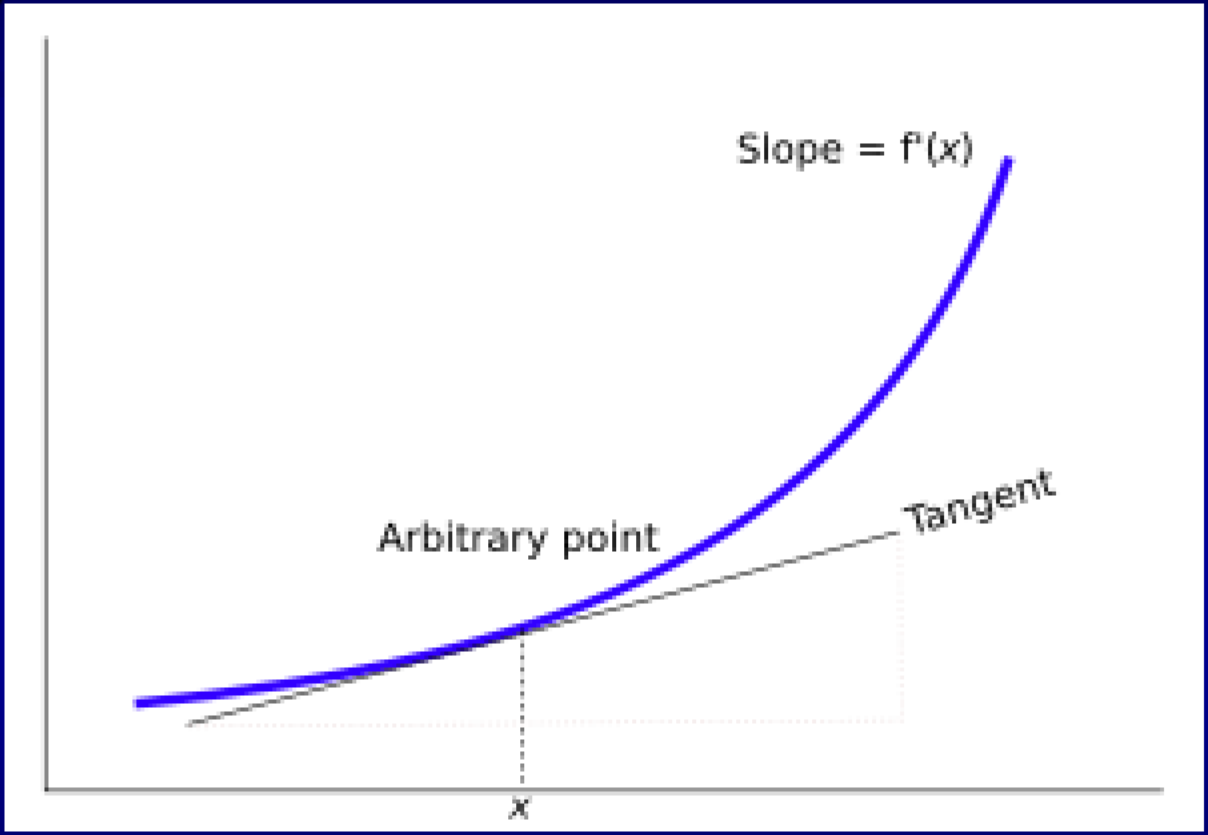
Interlude

Nonlinear Functions & Calculus

- Calculus allows the determination of the slope (rise/run) of a nonlinear function at a point.
- The slope of a nonlinear function at a point is the slope of the tangent line at that point.

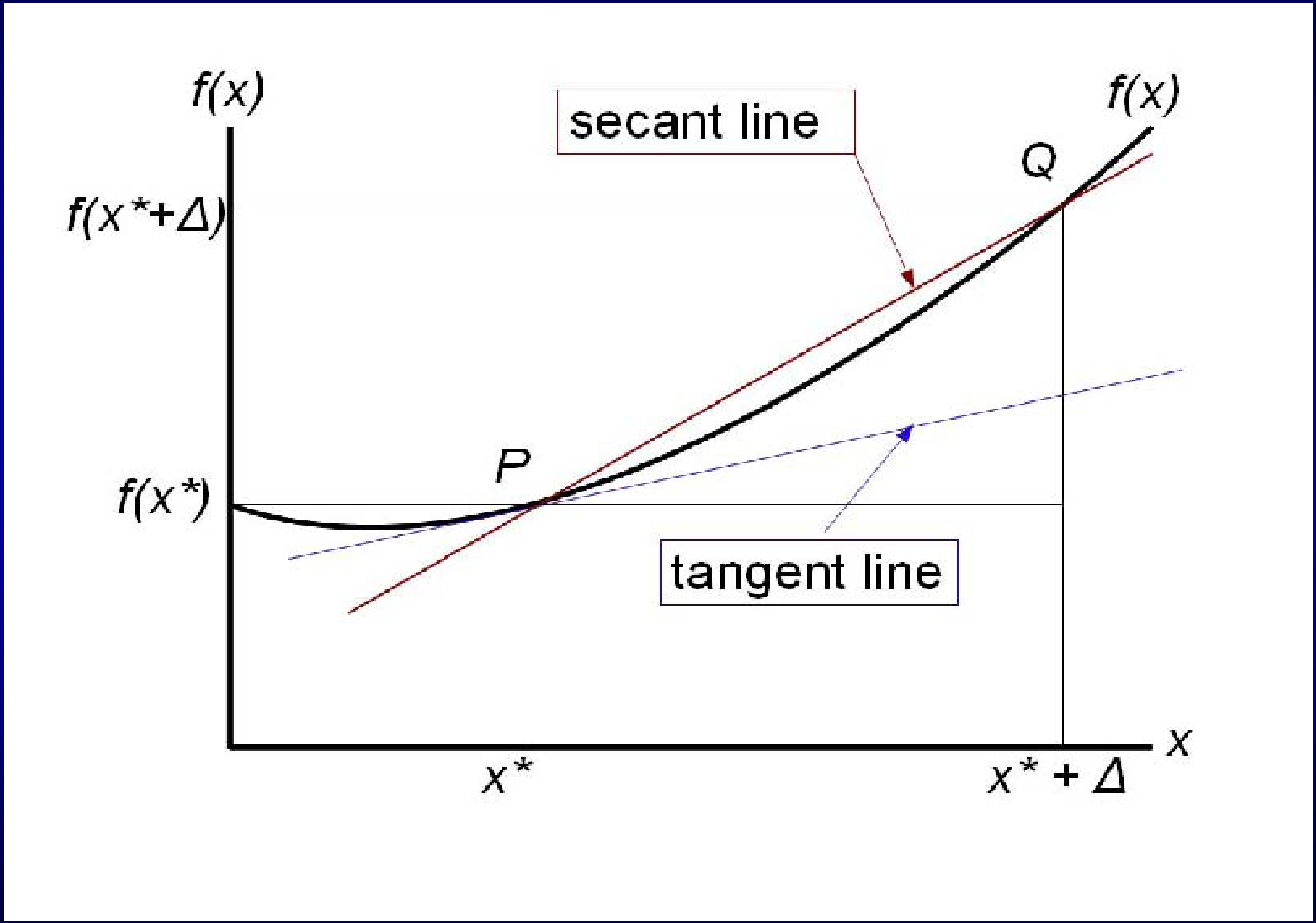


Slope of Non-linear Function



The tangent line passes through *(x, f(x))*. The derivative *f '(x)* of a curve at a point is the slope (rise over run) of the line tangent to that curve at that point.

What is slope of f(x) at x*?



Limits

- What is the slope of the tangent line?
  - It is the limit of the slope of the secant line as Δ gets small.
- slope of secant line =

$(f(x+\Delta) - f(x))/\Delta$

- slope of tangent line =

$\lim_{\Delta \to 0} (f(x+\Delta) - f(x))/\Delta$

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Interlude

Derivatives and Differentiation

Differentiation is the process of finding a derivative

- The derivative $f'(x)$ of a function $f(x)$ is the slope of $f(x)$.
- The derivative depends upon x in some way i.e. the derivative is also a function of x
- The derivative is found by differentiating a function of the form $y = f(x)$
- When x is substituted into the derivative, the result is the slope of the original function $y = f(x)$.

Derivatives and Differentiation

- There are many different ways to indicate the operation of differentiation, also known as finding or taking the derivative
- $f'(x) = f'$
- $f'(x) = y'$
- $f'(x) = df(x)/dx$
- $f'(x) = dy/dx$
- $f'(x) = d/dx[f(x)]$

Interpretations of the Derivative

The derivative of f(x) can be interpreted as:

- the slope of f(x)
- the rate of change of f(x) for a unit change in x
- the marginal value of f(x)

Interpretations of the Derivative

Example: Cost functions

- Cost as fn of output q
  - $c(q) = q^3 - 10q^2 + 40q$
- Marginal Cost c'(q) = the slope of c(q) = the change in cost resulting from a unit increase in output q

The Second Derivative

The second derivative of a function is the derivative of the derivative of the function.

$d^2f(x)/dx^2$ or $f''(x)$

- $f''(x) > 0 \Rightarrow$ f is convex
- $f''(x) < 0 \Rightarrow$ f is concave

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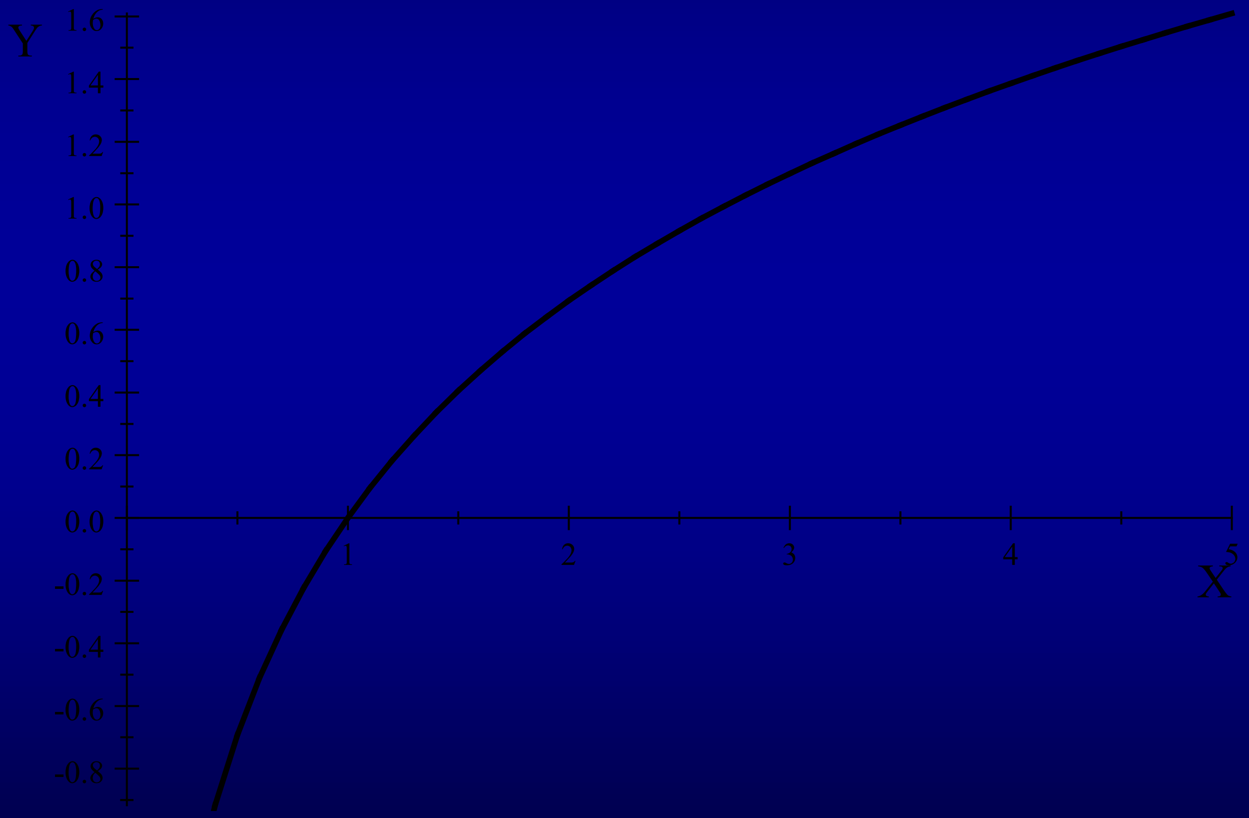
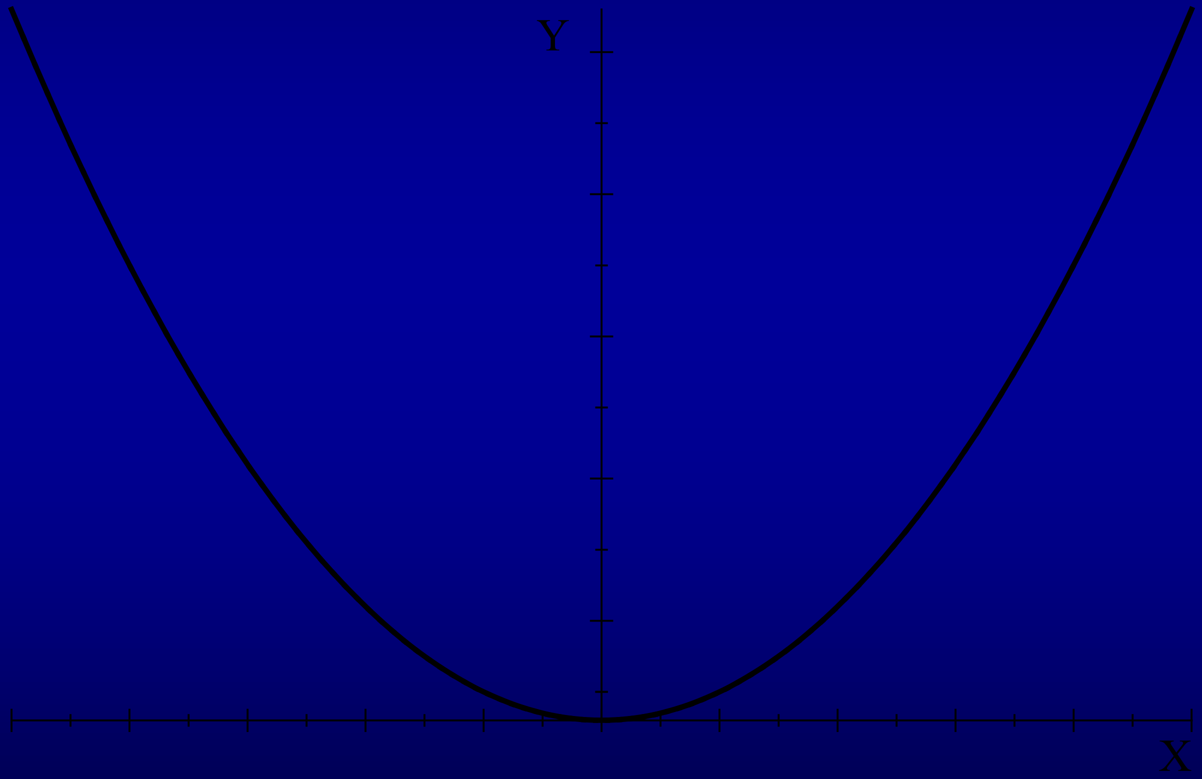
A Concave Function



Concavity

A differentiable function *f* is **concave** if its derivative function *f '* is monotonically decreasing: a concave function has a decreasing slope. ("*Decreasing*" here means "non-increasing", rather than "strictly decreasing", and thus allows zero slopes.)

A Convex Function



Convexity

A differentiable function *f* is **convex** if its derivative function *f* ' is monotonically increasing: a convex function has an increasing slope. ("*Increasing*" here means "non-decreasing", rather than "strictly increasing", and thus allows zero slopes.)

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Interlude

Differentiation Rules

- The rules are applied to each **term** within a function separately. Then the results from the differentiation of each term are added together, being careful to preserve signs.
- Rule 1: the derivative of a constant (e.g. $c = 15$) is zero

Differentiation Rules

- Rule 2: When *x* is raised to the power of 1, the slope is the coefficient on that *x*
  - What is the slope of $y = 2x + 15$?
- Rule 3: When *x* is raised to a power $n > 1$, then to get the slope, pull out the coefficient, multiply it by the power of *x*, then multiply that term by x, carried to the power *n-1*
  - What is the slope of $y = 5x^3 + 10$?
  - Derivative of $5x^3 = 5(3)(x)^{(3-1)} = 15x^2$

A Differentiation Example

$y = x^3 - 10x^2 + 40x$

The power rule combined with the coefficient rule is used as follows: pull out the coefficient, multiply it by the power of x, then multiply that term by x, carried to the power of n - 1.

$dy/dx = 3x^2 - 20x + 40$

A Differentiation Example

$dy/dx = 3x^2 - 20x + 40$

How do we interpret this?

- First, decide what part of the domain of the original function ($y = x^3 + 10x^2 + 40x$) you are interested in.
- For example, suppose you would like to know the slope of y when the variable x takes on a value of 3.
- Substitute x = 3 into the function of the slope and solve:
- $dy/dx = 3(3)^2 - 20(3) + 40 = 27 - 60 + 40 = 7$
- Therefore, we have found that when x = 3, the function y has a slope of + 7.

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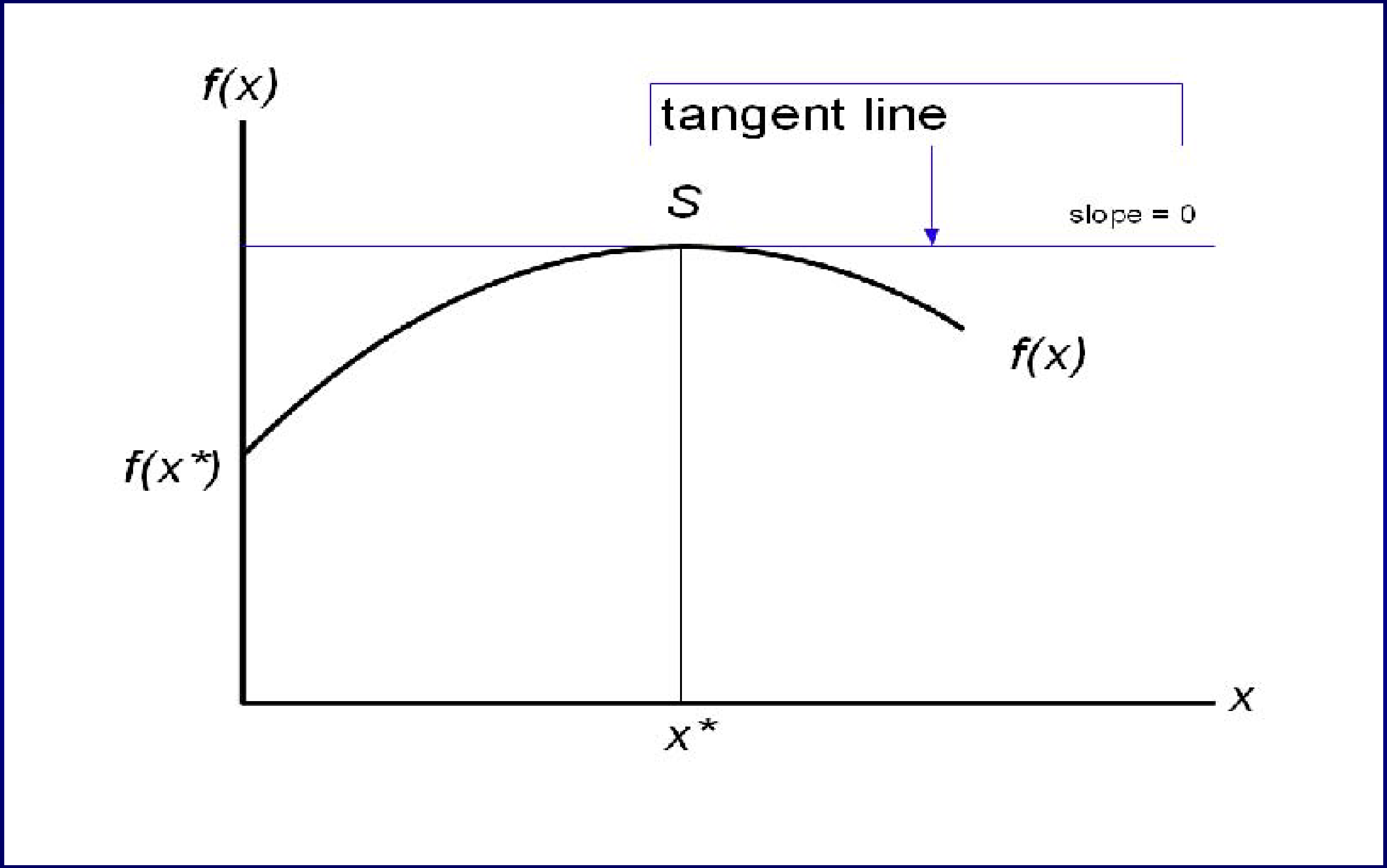
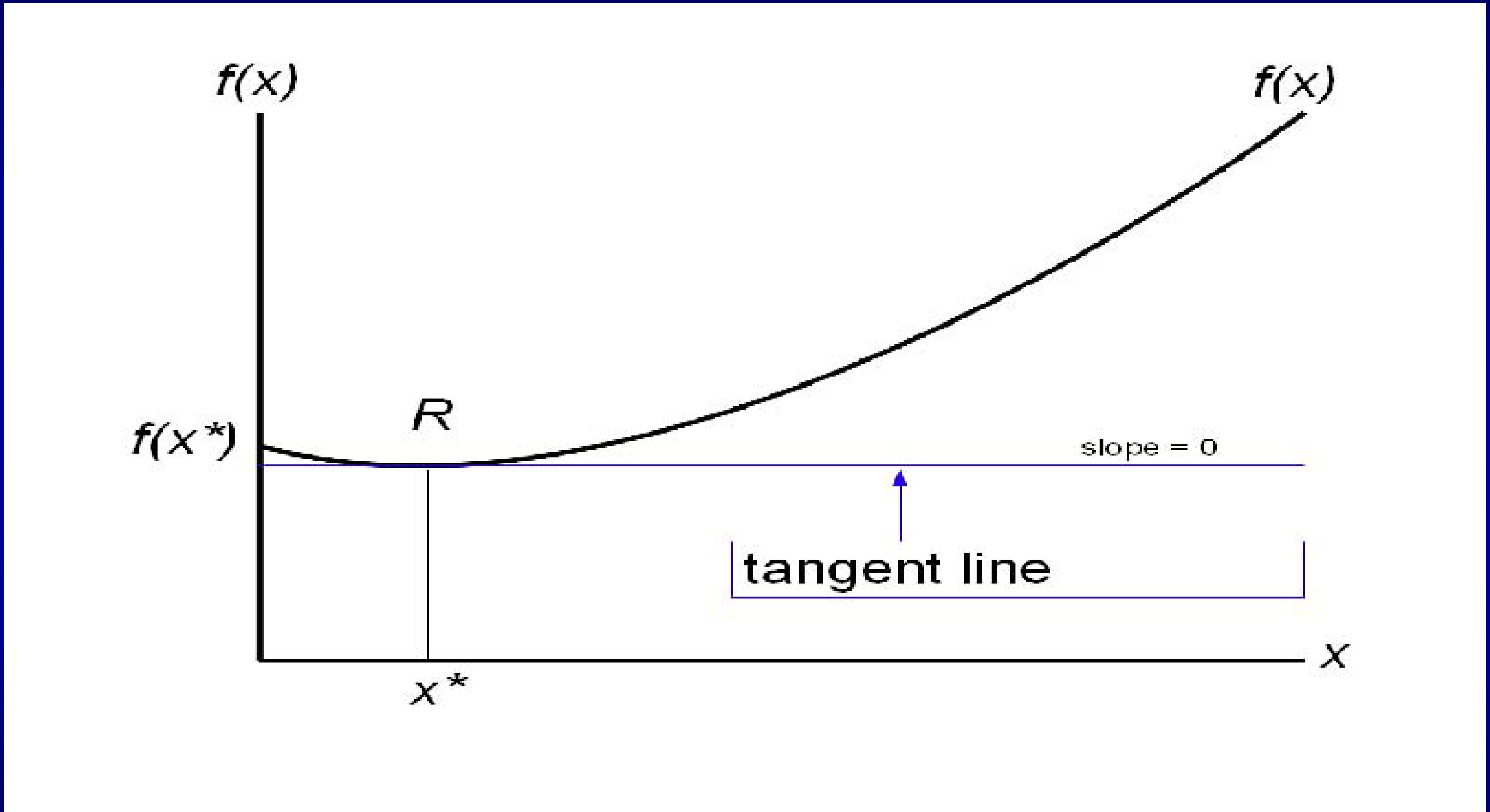
Optimization



- f(x) attains a maximum at point S

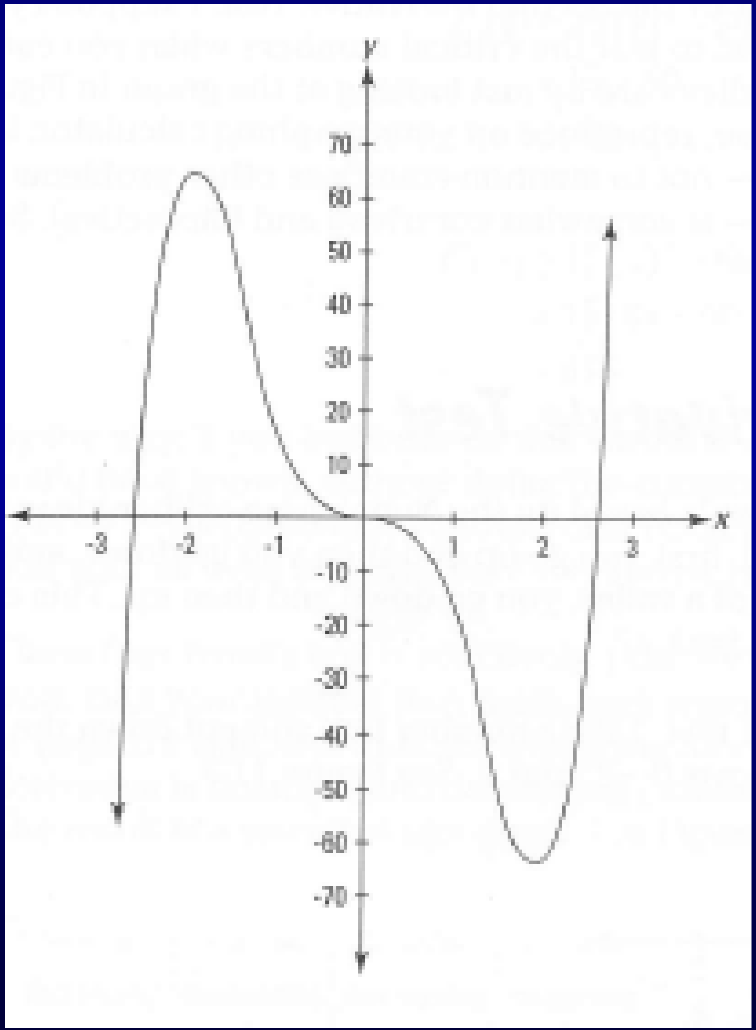
Optimization



- f(x) attains a minimum at point R

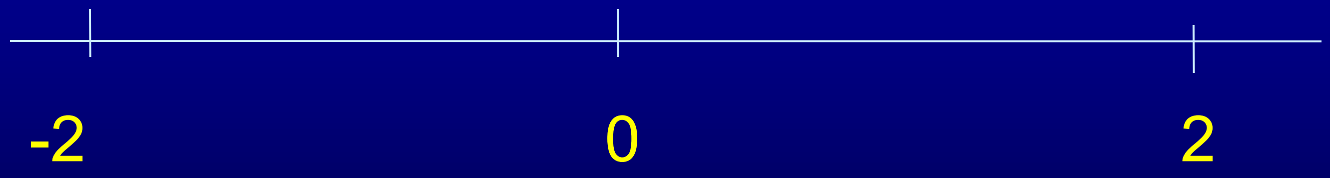
Finding the Maxima and Minima

- $f(x) = 3x^5 - 20x^3$
- Find the derivative of f using the power rule
- Set the derivative to 0 and solve for x
- Now determine whether peaks or valleys occur at those x values



First Order Condition

- x = 0, -2, or 2
- Choose values from within/between each region to plug into the first derivative



Critical numbers

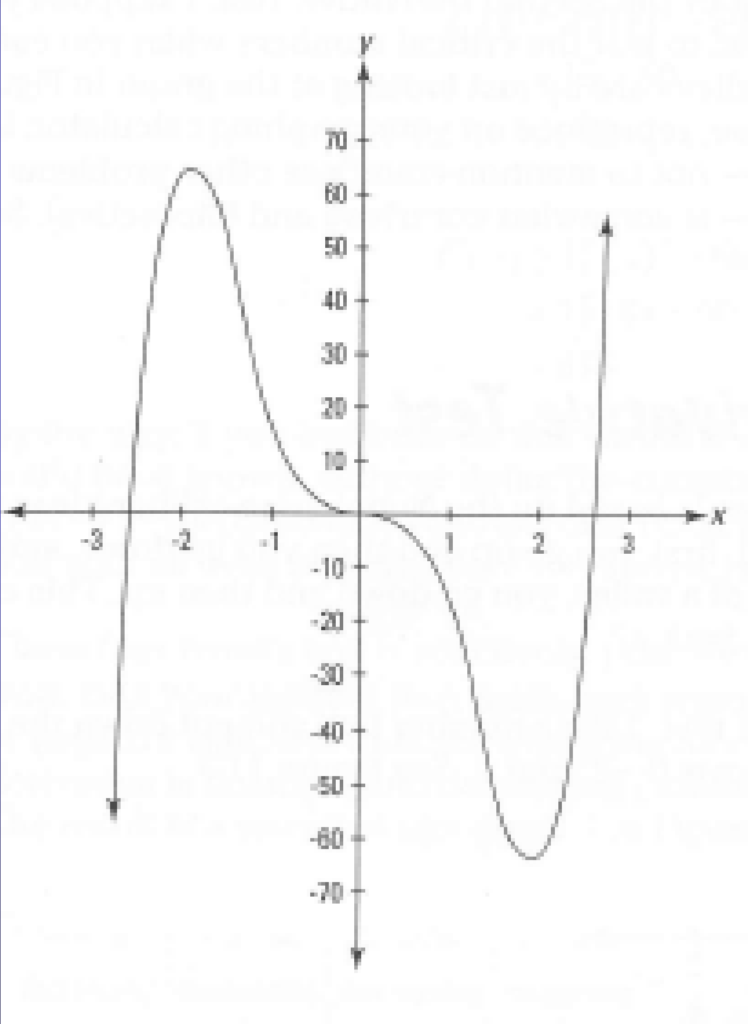
•Choose -3, -1, 1, 3

Slope near Critical Points

- $f'(x) = 15x^4 - 60x^2$ (x= -3,-1, 1, 3)
- $f'(-3) = 15(-3)^4 - 60(-3)^2 = 675$
- $f'(-1) = -45$
- $f'(1) = -45$
- $f'(3) = 675$
- These values tell us the slope of the curve at points around the critical points

Values at Critical Points

- Obtain the function values (or heights of the local extrema)
- Plug in the x values into the ORIGINAL function f(x)
- $f(x) = 3x^5 - 20x^3$
- $f(-2) = 3(-2)^5 - 20(-2)^3 = 64$
- $f(2) = -64$
- Thus, the local max is (-2, 64) and local min is (2,-64)



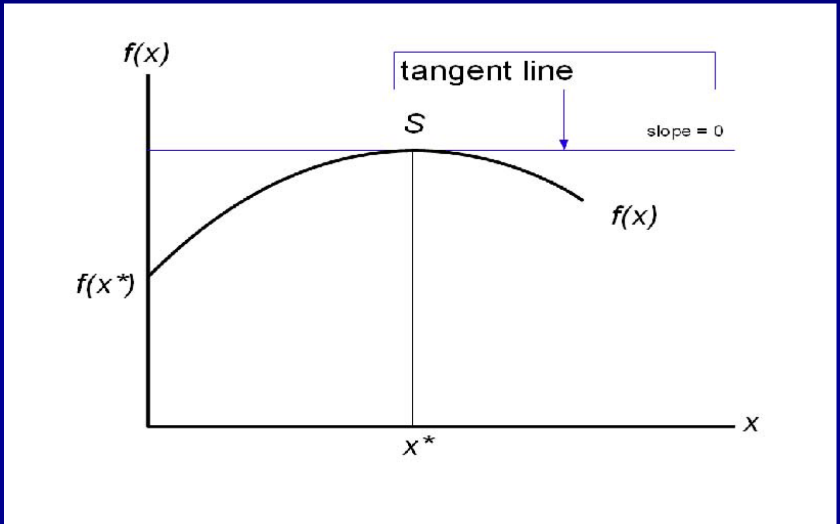
Maximization Conditions

If $f(x)$ is a smooth fn that attains a maximum at x^*, then

$f'(x^*) = 0$

and

$f''(x^*) \leq 0$.



Concave at x^*

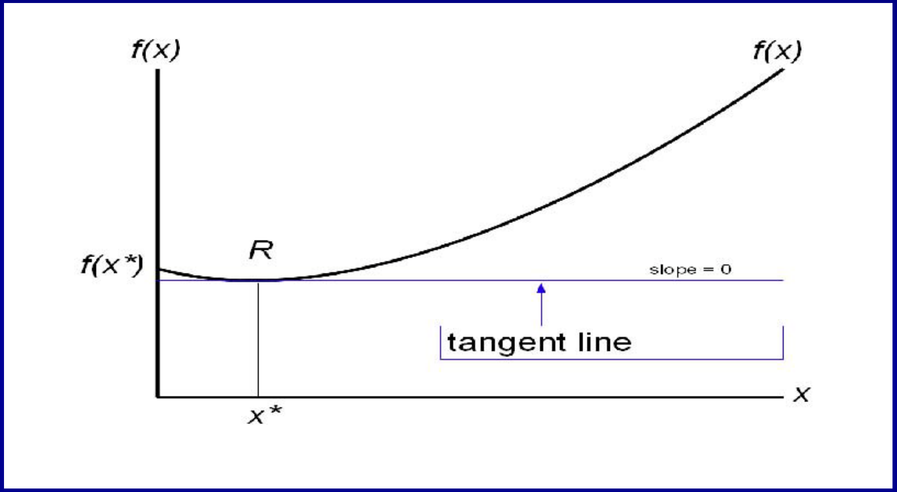
Minimization Conditions

If f(x) is a smooth fn that attains a minimum at x*, then

$f'(x^*) = 0$

and

$f''(x^*) \geq 0.$



Convex at x*

Math Camp

Interlude

Application: Revenue Maximization

Suppose that on a certain route, an airline carries 8,000 passengers per month, each paying \$100. The airline wants to increase the fare. However, the market research department estimates that for each \$1 increase in fare, the airline will lose 10 passengers. Determine the price that maximizes the airline's revenue.

Application: Revenue Maximization

Let x = price per ticket

n = number of passengers

Revenue = nx

We know that n is a function of x, as follows:

n = [original n] – [lost passengers due to fare increase]

$n = 8000 - (x - 100)\ 10$

$n = 8000 - (10x - 1000)$

$n = 9000 - 10x$

Application: Revenue Maximization

Revenue = nx

Revenue = (9000 – 10x)x = 9000x - $10x^2$

What is the slope of the revenue fn?

R' = 9000 - 20x

First Order Condition: Set R' = 0

9000 – 20x = 0

9000 = 20x

x = 450

The revenue-maximizing fare is $450.

Application: Inventory Control

Suppose that a supermarket wants to establish an optimal inventory policy for frozen orange juice that optimally balances refrigeration and delivery costs. It is estimated that a total of 1200 cases will be sold at a steady rate during the next year. The manager plans to place several orders of the same size equally spaced throughout the year. Use the following data to determine the economic order quantity, that is, the order size that minimizes the total ordering and carrying cost.

1. The order cost per delivery is $75.
2. It costs $8 in electricity to refrigerate one case of orange juice for one year.

Application: Inventory Control

Let x = order quantity

n = # of orders per year

Inventory cost = delivery cost + refrigeration cost

= 75n + 8 (average inventory)

= 75n + 8 (x/2)

$c = 75n + 4x$

We know that $nx = 1200$

Application: Inventory Control

$c = 75n + 4x$ and we know that $nx = 1200$

$n = 1200/x$

$c = 90000/x + 4x$

What is the slope of the cost fn?

$c' = -90000x^{-2} + 4$

First Order Condition: Set $c' = 0$

$-90000x^{-2} + 4 = 0$ or $4 = 90000/x^2$

$x = 150$

The economic order quantity is 150 cases.

Math Camp

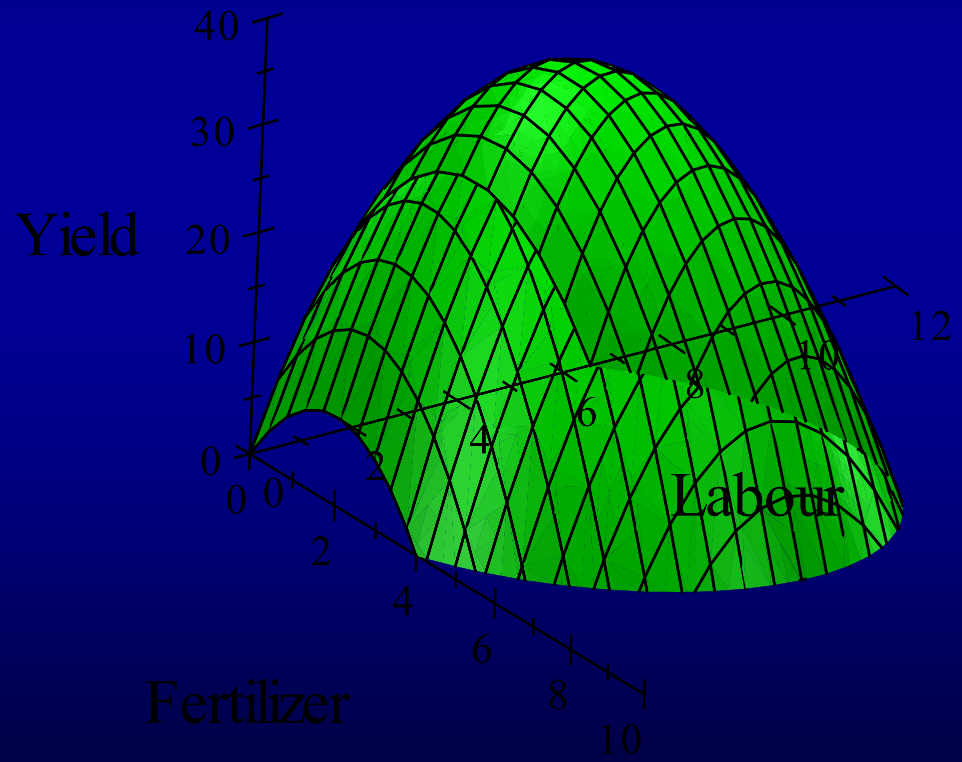
Interlude

Multivariate Functions

- So far, f(x) was a function of one variable x
- Suppose we have a multivariate function $f(x_1, x_2)$
- Example: Yield is a function of units of labour b and units of fertilizer z
  - $y(b,z) = 13b - 2b^2 + bz + 8z - 2z^2$

Multivariate Functions

$y(b,z) = 13b - 2b^2 + bz + 8z - 2z^2$

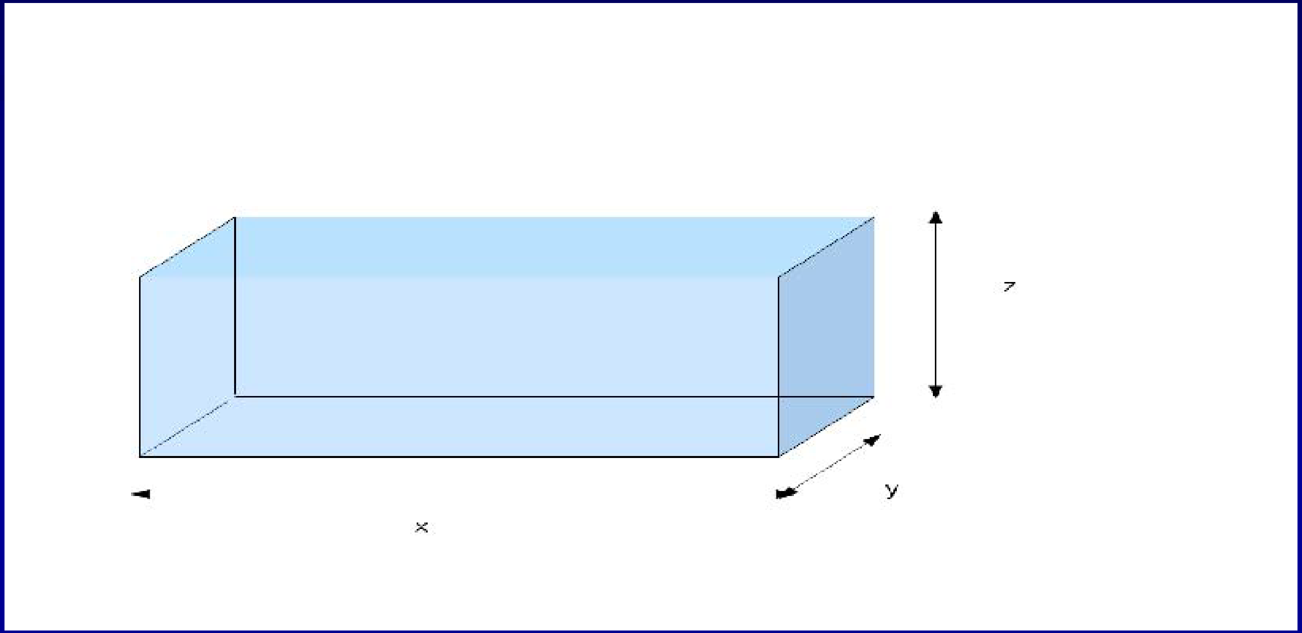


Multivariate Functions

$y(b,z) = 13b - 2b^2 + bz + 8z - 2z^2$

- Calculate $y(2,3)$
  - Answer: 30
- Calculate $y(2,z)$
  - Answer: $13 \times 2 - 2 \times 2^2 + 2z + 8z - 2z^2$
  - $= 18 + 10z - 2z^2$

Example: Heat Loss



	Roof	East side	West side	North side	Sou th side	Floor
Heat loss (per sq ft)	10	8	6	10	5	1
Area (per sq ft)	xy	yz	yz	xz	xz	xy

Example: Heat Loss

- Find a formula for the total heat loss as a function of x,y and z.
  - $f(x,y,z) = 10xy + 8yz + 6yz + 10xz + 5xz + 1xy$
  - $= 11xy + 14yz + 15xz$

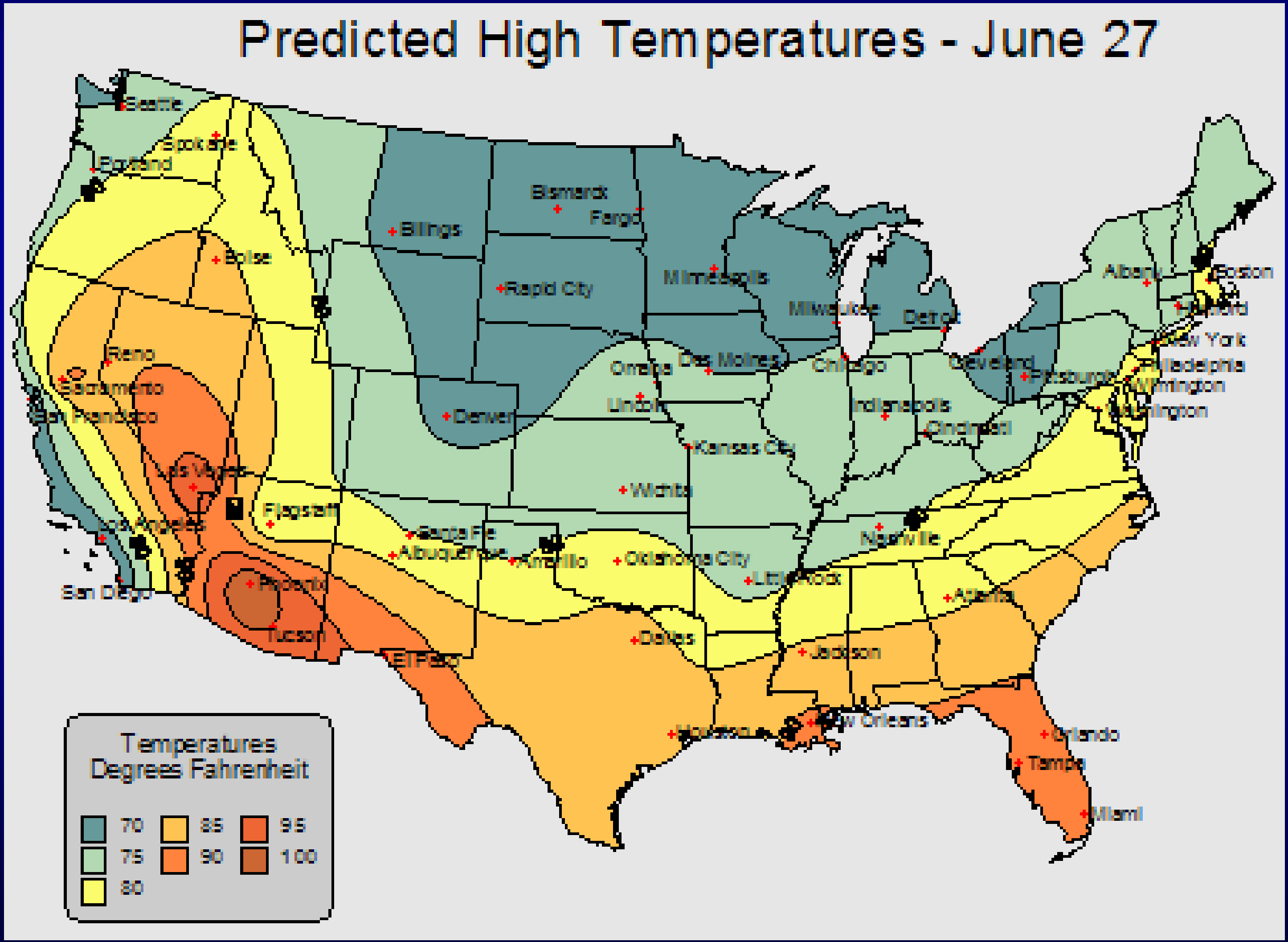
Example: Heat Loss

- What is the heat loss when $x = 100$, $y = 70$ and $z = 50$?
  - $f(100,70,50) = 11(100)(70) + 14(70)(50) + 15(100)(50)$
  - $= 77,000 + 49,000 + 75,000$
  - $= 201,000$

Level Curves

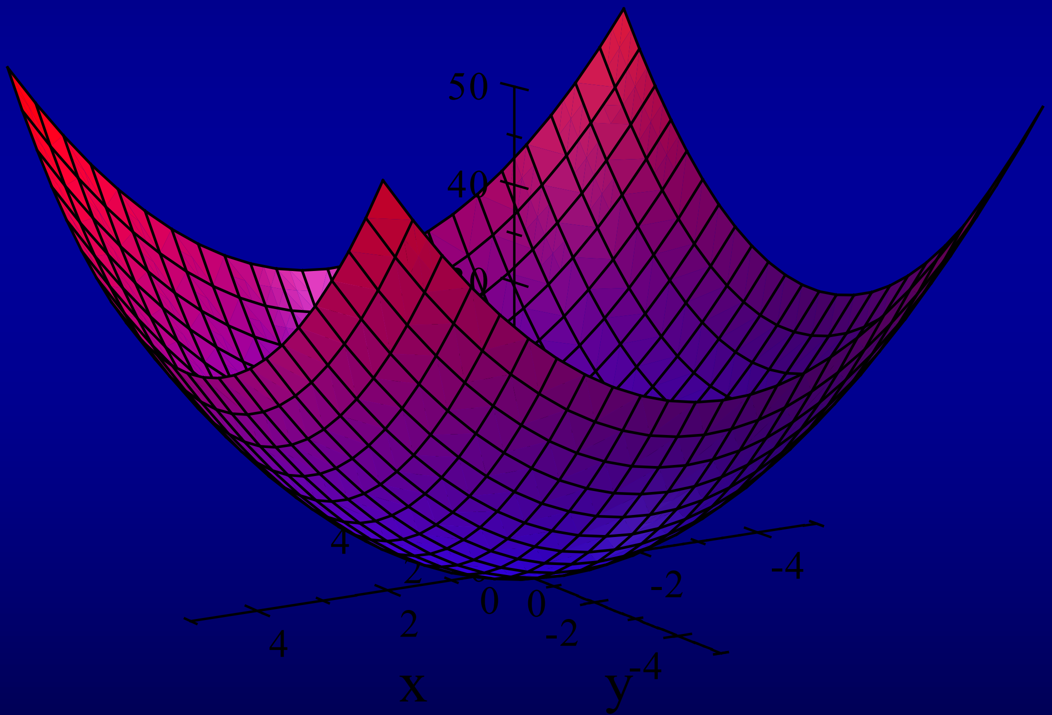
- For a fixed number c, the graph of the equation $f(x,y) = c$ is called the *level curve* of height c.

Example: Isotherms

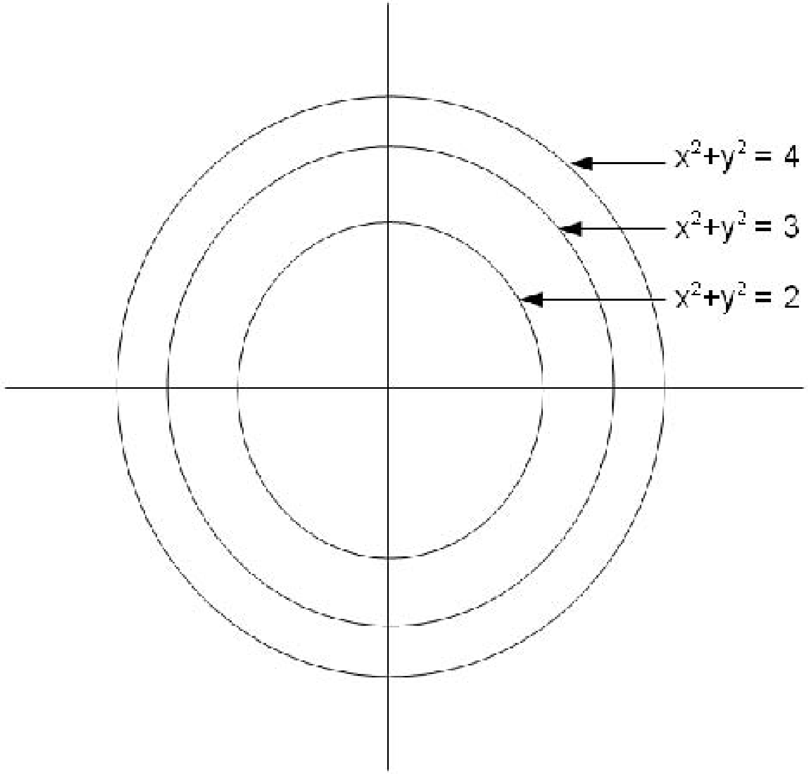


Level Curves

- $f(x,y) = x^2 + y^2$



Level Curves



Example: Isoquants

- Determine the level curve for the production function $f(L,K) = 60L^{0.75}K^{0.25}$ at height 600.
- The level curve is the graph of $f(L,K) = 600$, or
  - $60L^{0.75}K^{0.25}=600$
  - $K^{0.25} = 10/(L^{0.75})$
  - $K = 10{,}000/L^3$

Example: Isoquants

$K = 10{,}000/L^3$

or

$f(L,K) = 600$