**IEOR 4106, Final Exam, Columbia University**
**Prof. S. Kou, Dec 21, 2004, from 4:10pm to 7pm.**

**1.** (10 pts). Suppose cars arrive at a gas service station according to a Poisson process with a rate $\lambda$ cars per hour.

(a) Let $Y$ be the number of arrivals within $t$ hours. What is the distribution of $Y$? What is $E(Y)$?

(b) Let $X$ be the time between two consecutive arrivals. What is the distribution of $X$? What is $E(X)$?

**2.** (10 pts). (M/M/1 queue). Suppose that customers arrive at a single server service station in accordance with a Poisson process with rate $\lambda$. Upon arrival each customer goes directly into service if the server is free; if not, then the customer joins the queue(that is, he/she waits in line). When the server finishes serving a customer, the customer leaves the system and the next customer in line, if there are any waiting, enters the service. The successive service time are assumed to be independent exponential random variables with mean $1/\mu$. This is a special case of the birth and death processes. Assume $\lambda < \mu$. Compute $P_n$, $n = 0, 1, ..., k$, where $P_n$ being the long-run proportion of time that the system has $n$ customers.

**3.** (10 pts). A job shop consists of three machines and two repairmen. The amount of time a machine works before breaking down is exponentially distributed with mean 12. The amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 10.

(a) What is the average number of machines not in use?

(b) What proportion of time are both repairmen busy?

**4.** (10 pts). Consider the random walk $S_n = \sum_{j=1}^{n} X_j$, $X_j \geq 0$, where $X_1$, $X_2$, ... are independent identitically distributed nonnegative random variables. Introduce the first passage time $\tau(t) = \min\{n \geq 0 : S_n > t\}$, and the overshoot $\gamma_t = S_{\tau(t)} - t$. Derive a renewal equation for $E(\gamma_t)$.

**5.** (10 pts). Let $B(t)$ be a standard Brownian motion. For $0 < s < t < \infty$, compute $\mathsf{E}[B(t)B(s)]$.

**6.** (10 pts). Let $B(t)$ be a standard Brownian motion. Introduce $M(T) = \max_{0 \leq s \leq T} B(s)$. What is $P\{M(T) \geq m, B(T) \leq x\}$, for $m \geq x$, $m \geq 0$?

**7.** (10 pts). Let $B(t)$ be a standard Brownian motion. Let $\tau = \min\{t \geq 0 : B_t = 2 - 4t\}$. What is $E[\tau]$?

**8.** (10 pts). Consider a simple trinomial tree:

| | | Stock prices at time T | Option payoffs |
|---|---|---|---|
| | $\nearrow$ | $uS_0$ | $C_u$ |
| $S_0$ | $\rightarrow$ | $S_0$ | $C_0$ |
| | $\searrow$ | $dS_0$ | $C_d$ |

Show that unless $\frac{C_u - C_d}{C_0 - C_d} = \frac{u-d}{1-d}$, it is impossible to get a riskless portfolio involving the option and the stock.