1. Chapman-Kolmogorov equation. Consider a discrete Markov Chain $X_n$. Let $P^{(n)}_{ij} = P(X_{n+m} = j | X_m = i)$. Then $P^{(n+m)}_{ij} = \sum_{k=0}^{\infty} P^{(n)}_{ik} P^{(m)}_{kj}$. Hence $P^{(n)} = P^n$, where $P^{(n)} = (P_{ij}^{(n)})$ and $P = (P_{ij})$.

2. Classification of states. Accessible, $i \to j$: if $P^{(n)}_{ij} > 0$ for some $n \geq 0$. Communicate, $i \leftrightarrow j$: if $i \to j$ and $j \to i$. Class: all states that communicate. Irreducible: if there is only one class.

3. Recurrent and transient. Let $f_i = P(\text{ever come back to } i \mid \text{starts at } i)$. Transient: if $f_i < 1$. Recurrent: if $f_i = 1$.
   
   (1) Suppose state $i$ is transient. Then the probability that the MC will be in state $i$ for exactly $n$ time period is $f_i^{n-1}(1 - f_i)$; and $E[X] = \frac{1}{1 - f_i}$, where $X$ is the number of time periods the MC will be in state $i$.
   
   (2) State $i$ is recurrent if $\sum_{n=1}^{\infty} P^{(n)}_{ii} = \infty$, and transient if $\sum_{n=1}^{\infty} P^{(n)}_{ii} < \infty$.
   
   (3) If $i$ is recurrent and $i \leftrightarrow j$, then $j$ is also recurrent. Therefore, recurrence and transience are class properties.

4. Positive recurrent: if $E(T) < \infty$, where $T$ is the time until the process returns to state $i$. Null recurrent if $E(T) = \infty$.

5. Positive recurrent aperiodic states are called ergodic states.

4. $\pi_i$ is defined to be the solution of $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$, $j \geq 0$, $\sum_{i=0}^{\infty} \pi_i = 1$. Four interpretations of $\pi_{ij}$.
   
   (1) "Limiting probabilities". For an irreducible ergodic MC, $\lim_{n \to \infty} P^{(n)}_{ij} = \pi_j$.

   (2) "Stationary probabilities". If $P(X_0 = j) = \pi_j$, then $P(X_n = j) = \pi_j$.

   (3) "Long-run average frequencies". Let $a_j(N)$ be the number of periods a reducible MC spends in state $j$ during time periods $1, 2, ..., N$. Then as $N \to \infty$, $\frac{a_j(N)}{N} \to \pi_j$.

   (4) $m_{jj} = 1/\pi_j$, where $m_{jj}$ is the expectation of the number of transitions until the MC returns back to $j$, starting at $j$.

5. Gambler’s ruin problem.
   
   (1) It is a MC with $P_{00} = P_{NN} = 1$, and $P_{i,i+1} = p = 1 - P_{i,i-1}$, $i = 1, ..., N - 1$.

   (2) Let $P_t$ be the probability of reaching $N$ before 0, starting with $\$i$. Then $P_t = pP_{t+1} + qP_{t-1}$. Furthermore,

   $$P_t = \frac{1 - (q/p)^i}{1 - (q/p)^N}, \text{ if } p \neq \frac{1}{2}; \quad P_t = \frac{i}{N}, \text{ if } p = \frac{1}{2}.$$ 

   (3) Let $P_T$ be the transition matrix for transient states only. Then $S = (I - P_T)^{-1}$, where $S = (s_{ij})$ and $s_{ij}$ is the expected time periods that the MC is in state $j$, starting at $i$, for transient states $i$ and $j$.

   (4) Suppose $i$ and $j$ are transient states. Let $f_{ij}$ be the probability of ever making a transition into state $j$, starting at $i$. Then $f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.}

Ch. 5

1. Exponential distribution: basic properties.
Examples: Yule process, linear growth model, M/M/1/∞ queues.

2. The second definition of continuous-time Markov chain. (i) The amount of time stayed in state $i$ is an exponential random variable with rate $\nu_i$. (ii) Let $P_{ij} = P\{\text{next enters state } j|\text{the process leaves state } i\}$. Then $P_{ii} = 0$, and $\sum_j P_{ij} = 1$.

3. Birth-death process: birth rate $\lambda_i$, death rate $\mu_i$. It is a continuous time Markov chain with $\nu_0 = \lambda_0$, $\nu_i = \lambda_i + \mu_i$ (since the min of two exponential random variable is still an exponential random variable with the rate the sum of the two rates), and

$$P_{0,1} = 1, P_{i,i+1} = \lambda_i/(\lambda_i + \mu_i), P_{i,i-1} = \mu_i/(\lambda_i + \mu_i).$$

Examples: Yule process, linear growth model, M/M/1/∞ and M/M/s/∞ queues.
4. Transient probability \(P_{ij}(t) = P\{X(t+s) = j | X(s) = i\}\). Note \(P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s)\). Transition rate \(q_{ij} = v_i P_{ij}\). Also note the difference between the notations \(P_{ij}\) and \(P_{ij}(t)\).

5. Kolmogorov ’s backward equations:

\[
P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).
\]

Kolmogorov ’s forward equations:

\[
P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).
\]

6. Limiting probabilities and the balance equations. Let \(P_j = \lim_{t \to \infty} P_{ij}(t)\). Then \(v_j P_j = \sum_{k \neq j} q_{kj} P_k\) for every \(j\). Note that in the balance equations \(v_j P_j\) is the rate at which the process leaves state \(j\), and \(\sum_{k \neq j} q_{kj} P_k\) is the rate at which the process enters state \(j\).

7. Limiting probabilities for a birth-death process.

\[
\lambda_0 P_0 = \mu_1 P_1, \lambda_1 P_1 = \mu_2 P_2, \lambda_2 P_2 = \mu_3 P_3, \ldots, \lambda_n P_n = \mu_{n+1} P_{n+1}, n \geq 0.
\]

Suppose \(\mu_n \geq 0\) for all \(n \geq 1\). Solving them yields

\[
P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} P_0, P_0 = 1/\{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}\}.
\]

Ch. 10.

1. Definition of the standard Brownian motion. (1) \(B(0) = 0\). (2) \(\{B(t), t \geq 0\}\) has stationary and independent increments. (3) \(B(t)\) has a normal distribution with mean 0 and variance \(t\).

2. Brownian motion with drift: \(W(t) = \mu t + \sigma B(t)\). Geometric Brownian motion: \(e^{W(t)}\).

3. Martingale: \(X(s) = E[X(t) | \mathcal{F}_s]\) for all \(t \geq s\). Note that \(E[X(s)] = E[X(t)]\) for all \(t \geq s\). For example, \(B(t)\) is a martingale, and \(\exp\{aB(t) - a^2 t/2\}\) is also a martingale.

4. Martingale Stopping Theorem: If \(X(t)\) is a martingale and \(E[\tau] < \infty\), then \(E[X(\tau)] = E[X(0)],\) where \(\tau\) is a stopping time.

5. Geometric Brownian motion, \(X(t) = \exp\{\sigma B(t) + \mu t\}\). Using (1) the moment generating function of a normal random variable \(W\) is given by \(E[e^{aW}] = e^{aE[W] + \sigma^2 Var(W)/2}\), and (2) the independent increments of Brownian motion, we can show that

\[
E[X(t)|X(u), 0 \leq u \leq s] = X(0)e^{(t-s)(\mu + \sigma^2/2)}.
\]

6. Reflection principle and the first passage time for standard Brownian motion. Let \(\tau(a) = \min\{t : B(t) = a\}, a > 0\). For any \(b \leq a\),

\[
P(B(t) \leq b, \tau(a) \leq t) = P(B(t) \geq a + (a - b), \tau(a) \leq t) = P(B(t) \geq 2a - b) = \Phi\left(-\frac{2a - b}{\sqrt{t}}\right).
\]

Therefore,

\[
P(\tau(a) \leq t) = P(B(t) \leq a, \tau(a) \leq t) + P(B(t) > a, \tau(a) \leq t) = 2P(B(t) \geq a) = 2\Phi\left(-\frac{a}{\sqrt{t}}\right).
\]
Note that \( \{ \tau(a) \leq t \} = \{ \max_{0 \leq s \leq t} B(s) \geq a \} \).


8. Black-Scholes formula for the European call option. The price of the European call option price \( u_0 \) is
   \[
   u_0 = E^* \left( e^{-rT}(S(T) - K)^+ \right),
   \]
   where under the risk-neutral probability \( P^* \)
   \[
   S(t) = S(0) \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right\}.
   \]
Evaluating the expectation yields the following formula for the price of the call option:
   \[
   u_0 = S(0) \cdot \Phi(\mu_+) - Ke^{-rT} \Phi(\mu_-),
   \]
where
   \[
   \mu_\pm := \frac{1}{\sigma \sqrt{T}} \left[ \log \left( \frac{S(0)}{K} \right) + (r \pm \frac{\sigma^2}{2})T \right] \text{ and } \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du.
   \]

**Ch. 7 Renewal Theory**

Let \( S_n = \sum_{i=1}^{n} X_i \), where \( X \)'s are i.i.d. random variables.

1. Renewal Process: \( N(t) = \max \{ n : S_n \leq t \} \). The first passage time \( \tau(t) = \min \{ n : S_n > t \} \). The overshoot \( \gamma(t) = S_{\tau(t)} - t \). Note that if \( X_i \geq 0 \), then
   \[
   N(t) \geq n \iff S_n \leq t, \quad \text{and} \quad \tau(t) = N(t) + 1.
   \]

2. Wald’s equation: Suppose \( T \) is a stopping time. If \( E[T] < \infty \), then we have \( E[\sum_{i=1}^{T} X_i] = E[T]E[X] \).

3. Renewal Equation. Suppose \( X_i \geq 0 \). Then an equation of the form
   \[
   A(t) = a(t) + \int_{0}^{t} A(t-x) dF(x),
   \]
where \( F(x) = P(X \leq x) \), is called the renewal equation. Explicit solutions of the renewal equation are available only for some special cases of \( F(x) \), such as uniform distribution.

4. Elementary renewal theorem: As \( t \to \infty \),
   \[
   \frac{\tau(t)}{t} \to \frac{1}{\mu}, \quad \frac{N(t)}{t} \to \frac{1}{\mu}, \quad E[\tau(t)/t] \to \frac{1}{\mu}, \quad E[N(t)/t] \to \frac{1}{\mu},
   \]
where \( \mu = E[X] \).