Formula Sheet, E3106, Introduction to Stochastic Models
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Ch. 4

1. Chapman-Kolmogorov equation. Consider a discrete Markov Chain X_n. Let $P_{ij}^{(n)} = P(X_{n+m} = j | X_m = i)$. Then $P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}$. Hence $\mathbf{P}^{(n)} = \mathbf{P}^n$, where $\mathbf{P}^{(n)} = (P_{ij}^{(n)})$ and $\mathbf{P} = (P_{ij})$.

2. Classification of states. Accessible, $i \to j$: if $P_{ij}^{(n)} > 0$ for some $n \geq 0$. Communicate, $i \longleftrightarrow j$: if $i \to j$ and $j \to i$. Class: all states that communicate. Irreducible: if there is only one class.

3. Recurrent and transient. Let $f_i = P(\text{ever come back to } i \mid \text{starts at } i)$. Transient: if $f_i < 1$. Recurrent: if $f_i = 1$.

(1) Suppose state i is transient. Then the probability that the MC will be in state i for exactly n time period is $f_i^{n-1}(1 - f_i)$; and $E[X] = \frac{1}{1-f_i}$, where X is the number of time period the MC will be in state i.

(2) State i is recurrent if $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$, and transient if $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$.

(3) If i is recurrent and $i \longleftrightarrow j$, then j is also recurrent. Therefore, recurrence and transience are class properties.

(4) Positive recurrent: if $E(T) < \infty$, where T is the time until the process returns to state i. Null recurrent if $E(T) = \infty$.

(5) Positive recurrent aperiodic states are call ergodic states.

4. π_i is defined to be the solution of $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij},\ j \geq 0,\ \sum_{i=0}^{\infty} \pi_i = 1$. Four interpretations of π_{ij}.

(1) "Limiting probabilities". For a irreducible ergodic MC, $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$.

(2) "Stationary probabilities". If $P(X_0 = j) = \pi_j$, then $P(X_n = j) = \pi_j$.

(3) "Long-run average frequencies". Let $a_j(N)$ be the number of periods a irreducible MC spends in state j during time periods $1, 2, ..., N$. Then as $N \to \infty$, $\frac{a_j(N)}{N} \to \pi_j$.

(4) $m_{jj} = 1/\pi_j$, where m_{jj} is the expectation of the number of transitions until the MC returns back to j, starting at j.

5. Gambler's ruin problem.

(1) It is a MC with $P_{00} = P_{NN} = 1$, and $P_{i,i+1} = p = 1 - P_{i,i-1},\ i = 1, ..., N-1$.

(2) Let P_i be the probability of reaching N before 0, starting with \$$i$. Then $P_i = pP_{i+1} + qP_{i-1}$. Furthermore,

$$P_i = \frac{1 - (q/p)^i}{1 - (q/p)^N}, \text{ if } p \neq \frac{1}{2};\ P_i = \frac{i}{N}, \text{ if } p = \frac{1}{2}.$$

(3) Let $\mathbf{P}_T$ be the transition matrix for transient states only. Then $\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1}$, where $\mathbf{S} = (s_{ij})$ and s_{ij} is the expected time periods that the MC is in state j, starting at i, for transient states i and j.

(4) Suppose i and j are transient states. Let f_{ij} be the probability of ever making a transition into state j, starting at i. Then $f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

Ch. 5

1. Exponential distribution: basic properties.

(a) density: $f(x) = \lambda e^{-\lambda x}$, if $x \geq 0$.

(b) distribution function: $F(x) = P(X \leq x) = 1 - e^{-\lambda x}$, if $x \geq 0$. $P(X \geq x) = e^{-\lambda x}$, $x \geq 0$.

(c) $E[X] = 1/\lambda$ and $Var[X] = \frac{1}{\lambda^2}$.

2. Exponential distribution: more properties.

(a) Memoryless: $P(X > t + s | X > t) = P(X > s)$, $E[X | X \geq t] = t + E[X]$.

(b) Sum: $X_1 + \cdots + X_n = \Gamma(n, \lambda)$, where $\Gamma(n, \lambda)$ is the gamma distribution with parameters n and λ whose density is $\lambda e^{-\lambda t} (\lambda t)^{n-1}/(n-1)!$.

(c) $P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$, where $X_1 = Ex(\lambda_1)$ and $X_2 = Ex(\lambda_2)$ are two independent exponential random variables.

(d) $P(\min\{X_1, ..., X_n\} > x) = \exp\{-x \sum_{i=1}^n \lambda_i\}$.

3. Hazard rate: $r(t) = \frac{f(t)}{1 - F(t)}$, where f and F are density and distribution functions, respectively. For an exponential random variable, $r(t) = \lambda$.

4. Poisson processes: three definitions.

(1) A counting process with independent increments such that $N(0) = 0$ and

$$P(N(t + s) - N(s) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

$$E[N(t) - N(s)] = \lambda(t - s),\ Var[N(t) - N(s)] = \lambda(t - s).$$

(2) A counting process with independent increments such that $N(0) = 0$ and $P(N(h) = 1) = \lambda h + o(h)$, $P(N(h) \geq 2) = o(h)$, as $h \to 0$.

(3) Let X_i be i.i.d. with the distribution $Ex(\lambda)$. Let $S_n = \sum_{i=1}^n X_i$ and $S_0 = 0$. Then $N(t) = \max\{n \geq 0 : S_n \leq t\}$. Here $\{X_i\}$ are called interarrival times, and S_n is called the nth arrival times.

(4) Splitting property. Given a Poisson process $N(t)$ with parameter λ, if an type-I event happens with probability p and type-II event happens with probability $1 - p$, then both $N_1(t)$ and $N_2(t)$ are independent Poisson processes with parameter λp and $\lambda(1 - p)$, respectively.

Ch. 6.

1. The first definition of continuous-time Markov chain.

$$P[X(t + s) = j | X(s) = i, \mathcal{F}_s] = P[X(t + s) = j | X(s) = i].$$

2. The second definition of continuous-time Markov chain. (i) The amount of time stayed in state i is an exponential random variable with rate v_i. (ii) Let P_{ij}=$P\{$next enters state $j |$the process leaves state $i\}$. Then $P_{ii} = 0$, and $\sum_j P_{ij} = 1$.

3. Birth-death process: birth rate λ_i, death rate μ_i. It is a continuous time Markov chain with $v_0 = \lambda_0$, $v_i = \lambda_i + \mu_i$ (since the min of two exponential random variable is still an exponential random variable with the rate the sum of the two rates), and

$$P_{0,1} = 1, P_{i,i+1} = \lambda_i/(\lambda_i + \mu_i),\ P_{i,i-1} = \mu_i/(\lambda_i + \mu_i).$$

Examples: Yule process, linear growth model, M/M/1/∞ and M/M/s/∞ queues.

4. Transient probability $P_{ij}(t) = P\{X(t+s) = j|X(s) = i\}$. Note $P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s)$. Transition rate $q_{ij} = v_i P_{ij}$. Also note the difference between the notations P_{ij} and $P_{ij}(t)$.

5. Kolmogorov 's backward equations:

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

Kolmogorov 's forward equations:

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

6. Limiting probabilities and the balance equations. Let $P_j = \lim_{t\to\infty} P_{ij}(t)$. Then $v_j P_j = \sum_{k\neq j} q_{kj} P_k$ for every j. Note that in the balance equations $v_j P_j$ is the rate at which the process leaves state j, and $\sum_{k\neq j} q_{kj} P_k$ is the rate at which the process enters state j.

7. Limiting probabilities for a birth-death process.

$$\lambda_0 P_0 = \mu_1 P_1, \lambda_1 P_1 = \mu_2 P_2, \lambda_2 P_2 = \mu_3 P_3, ..., \lambda_n P_n = \mu_{n+1} P_{n+1},\ n \geq 0.$$

Suppose $\mu_n \geq 0$ for all $n \geq 1$. Solving them yields

$$P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} P_0,\ P_0 = 1/\{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}\}.$$

Ch. 10.

1. Definition of the standard Brownian motion. (1) $B(0) = 0$. (2) $\{B(t),\ t \geq 0\}$ has stationary and independent increments. (3) $B(t)$ has a normal distribution with mean 0 and variance t.

2. Brownian motion with drift: $W(t) = \mu t + \sigma B(t)$. Geometric Brownian motion: $e^{W(t)}$.

3. Martingale: $X(s) = E[X(t)|\mathcal{F}_s]$ for all $t \geq s$. Note that $E[X(s)] = E[X(t)]$ for all $t \geq s$. For example, $B(t)$ is a martingale, and $\exp\{aB(t) - a^2 t/2\}$ is also a martingale.

4. Martingale Stopping Theorem: If $X(t)$ is a martingale and $E[\tau] < \infty$, then $E[X(\tau)] = E[X(0)]$, where τ is a stopping time.

5. Geometric Brownian motion, $X(t) = \exp\{\sigma B(t) + \mu t\}$. Using (1) the moment generating function of a normal random variable W is given by $E[e^{aW}] = e^{aE[W] + a^2 Var(W)/2}$, and (2) the independent increments of Brownian motion, we can show that

$$E[X(t)|X(u),\ 0 \leq u \leq s] = X(0)e^{(t-s)(\mu+\sigma^2/2)}.$$

6. Reflection principle and the first passage time for standard Brownian motion. Let $\tau(a) = \min\{t : B(t) = a\}$, $a > 0$. For any $b \leq a$,

$$P(B(t) \leq b, \tau(a) \leq t) = P(B(t) \geq a + (a-b), \tau(a) \leq t) = P(B(t) \geq 2a - b) = \Phi(-\frac{2a-b}{\sqrt{t}}).$$

Therefore,

$$P(\tau(a) \leq t) = P(B(t) \leq a, \tau(a) \leq t) + P(B(t) > a, \tau(a) \leq t) = 2P(B(t) \geq a) = 2\Phi(-\frac{a}{\sqrt{t}}).$$

Note that $\{\tau(a) \leq t\}$=$\{\max_{0\leq s\leq t} B(s) \geq a\}$.

7. Option pricing: binomial tree. No arbitrage argument.

8. Black-Scholes formula for the European call option. The price of the European call option price u_0 is $u_0 = \mathsf{E}^*\left(e^{-rT}(S(T)-K)^+\right)$, where under the risk-neutral probability P^*

$$S(t) = S(0)\exp\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\}.$$

Evaluating the expectation yields the following formula for the price of the call option:

$$u_0 = S(0)\cdot\Phi(\mu_+) - Ke^{-rT}\Phi(\mu_-),$$

where

$$\mu_\pm := \frac{1}{\sigma\sqrt{T}}[\log(S(0)/K) + (r \pm \left(\sigma^2/2\right))T] \text{ and } \Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}du.$$

Ch. 7 Renewal Theory

Let $S_n = \sum_{i=1}^{n} X_i$, where $X's$ are i.i.d. random variables.

1. Renewal Process: $N(t) = \max\{n : S_n \leq t\}$. The first passage time $\tau(t) = \min\{n : S_n > t\}$. The overshoot $\gamma(t) = S_{\tau(t)} - t$. Note that if $X_i \geq 0$, then

$$N(t) \geq n \Leftrightarrow S_n \leq t, \quad \text{and} \quad \tau(t) = N(t) + 1.$$

2. Wald's equation: Suppose T is a stopping time. If $E[T] < \infty$, then we have $E[\sum_{i=1}^{T} X_i] = E[T]E[X]$.

3. Renewal Equation. Suppose $X_i \geq 0$. Then an equation of the form

$$A(t) = a(t) + \int_0^t A(t-x)dF(x),$$

where $F(x) = P(X \leq x)$, is call the renewal equation. Explicit solutions of the renewal equation are available only for some special cases of $F(x)$, such as uniform distribution.

4. Elementary renewal theorem: As $t \to \infty$,

$$\tau(t)/t \to \frac{1}{\mu},\ N(t)/t \to \frac{1}{\mu},\ E[\tau(t)/t] \to \frac{1}{\mu},\ E[N(t)/t] \to \frac{1}{\mu},$$

where $\mu = E[X]$.