E3106, Solutions to Homework 1

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Exercise 3. According to the conditions given by the question and the notation used in the above solution, we have $p_0 = 0.8, p_1 = 0.6, p_2 = 0.6, p_3 = 0.6, p_4 = 0.4, p_5 = 0.4, p_6 = 0.4, p_7 = 0.2$. Thus, the transition probability matrix P of the Markov chain is

$$P = \begin{bmatrix} 0.8 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 \end{bmatrix}.$$

Exercise 7. As in Exercise 2, we use (RR),(DR),(RD),(DD) to represent the 4 states $0, 1, 2, 3$ in Example 4.4 respectively. And we suppose that today is in the n-th state of the Markov chain. The required probability is equal to

$$P(X_{n+1} = (DR)|X_{n-1} = (DD)) + P(X_{n+1} = (RR)|X_{n-1} = (DD)).$$

Solution 1. By direct calculation.

$$\begin{aligned} & P(X_{n+1} = (DR)|X_{n-1} = (DD)) + P(X_{n+1} = (RR)|X_{n-1} = (DD)) \\ =& P(X_{n+1} = (DR), X_n = (DD)|X_{n-1} = (DD)) \\ & + P(X_{n+1} = (RR), X_n = (DR)|X_{n-1} = (DD)) \\ =& P(X_{n+1} = (DR)|X_n = (DD), X_{n-1} = (DD))P(X_n = (DD)|X_{n-1} = (DD)) \\ & + P(X_{n+1} = (RR)|X_n = (DR), X_{n-1} = (DD))P(X_n = (DR)|X_{n-1} = (DD)) \\ =& P(X_{n+1} = (DR)|X_n = (DD))P(X_n = (DD)|X_{n-1} = (DD)) \\ & + P(X_{n+1} = (RR)|X_n = (DR))P(X_n = (DR)|X_{n-1} = (DD)) \\ =& (.2)(.8) + (.5)(.2) \\ =& 0.26. \end{aligned}$$

Solution 2. By using Chapman-Kolmogorov equations. The required probability is $P_{30}^{(2)} + P_{31}^{(2)}$. Since

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix},$$

we have

$$\mathbf{P}^{(2)} = \mathbf{P}^2 = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}^2 = \begin{bmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.2 & 0.15 & 0.3 \\ 0.2 & 0.12 & 0.2 & 0.48 \\ 0.1 & 0.16 & 0.1 & 0.64 \end{bmatrix}.$$

Thus,

$$P_{30}^{(2)} + P_{31}^{(2)} = 0.1 + 0.16 = 0.26.$$

Exercise 8. Set

$$X_n = \begin{cases} 0 & \text{if coin 1 is flipped on the } n\text{-th day} \\ 1 & \text{if coin 2 is flipped on the } n\text{-th day} \end{cases}$$

then $\{X_n, n \geq 0\}$ is a Markov chain with probability transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

and initial distribution $P(X_0 = 0) = P(X_0 = 1) = 0.5$. And so,

$$\mathbf{P}^2 = \begin{bmatrix} 0.67 & 0.33 \\ 0.66 & 0.34 \end{bmatrix}$$

and

$$\mathbf{P}^3 = \begin{bmatrix} 0.667 & 0.333 \\ 0.666 & 0.334 \end{bmatrix}.$$

Hence,

$$\begin{aligned} P(X_3 &= 0) = P(X_0 = 0)P_{11}^{(3)} + P(X_0 = 1)P_{21}^{(3)} \\ &= .5(.667 + .666) = .6665. \end{aligned}$$

Exercise 9 Set

$$X_n = \begin{cases} 0 & \text{if coin 1 is flipped on the } n\text{-th day,} \\ 1 & \text{if coin 2 is flipped on the } n\text{-th day.} \end{cases}$$

Then $\{X_n, n \geq 0\}$ is a Markov chain with probability transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}.$$

Since Friday is 4 days after Monday and state 0 corresponds to flipping heads, the quantity we are interested in is P^4_{00}. We can compute $\mathbf{P}$ by squaring twice:

$$\mathbf{P}^2 = \begin{bmatrix} 0.67 & 0.33 \\ 0.66 & 0.34 \end{bmatrix}$$
$$\mathbf{P}^4 = (\mathbf{P}^2)^2 = \begin{bmatrix} 0.6667 & 0.3333 \\ 0.6666 & 0.3334 \end{bmatrix},$$

giving us $P^4_{00} = 0.6667$.
Exercise 10. Let $\{X_n, n \geq 0\}$ be the Markov chain defined in Example 4.3, whose transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}.$$

By transforming the state 2 into an absorbing state, we obtain a new transition probability matrix

$$\mathbf{Q} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Because the original Markov chain and the new Markov chain follows identical transition probabilities until the state 2 is entered, it follows that, starting in state 0 or 1 at time n, the probability the original Markov chain never enters state 2 by time $n+k$ is equal to the probability the new Markov chain is not in state 2 at time $n+k$. Therefore, the desired probability is equal to

$$P(X_{n+3} \neq 2 | X_n = 0) = 1 - P(X_{n+3} = 2 | X_n = 0) = 1 - Q^{(3)}_{02}.$$

Since

$$\mathbf{Q}^{(3)} = \mathbf{Q}^3 = \begin{bmatrix} 0.293 & 0.292 & 0.415 \\ 0.219 & 0.22 & 0.561 \\ 0 & 0 & 1 \end{bmatrix},$$

the desired probability is $1 - 0.415 = 0.585$.
Exercise 11. Let $\{X_n, n \geq 0\}$ and P be defined the same as in the above solution. By transforming the state 0 into an absorbing state, we obtaina a new transition probability matrix

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}.$$

Because the original Markov chain and the new Markov chain follows identical transition probabilities until the state 0 is entered, it follows that, starting in state 1 or 2 at time $n-k$, the probability the original Markov chain never enters state 0 by time $n-1$ is equal to the probability the transformed Markov chain is not in state 0 at time $n-1$. Therefore, the desired probability is equal to

$$\begin{aligned}
&P(X_n = 2|X_{n-4} = 2, X_{n-k} \neq 0, k = 1, 2, \ldots, 7) \\
=&P(X_n = 2|X_{n-1} \neq 0, X_{n-4} = 2) \\
=&\frac{P(X_n = 2, X_{n-1} \neq 0|X_{n-4} = 2)}{P(X_{n-1} \neq 0|X_{n-4} = 2)} \\
=&\frac{P(X_n = 2|X_{n-4} = 2)}{P(X_{n-1} \neq 0|X_{n-4} = 2)} \\
=&\frac{Q_{22}^4}{1 - Q_{20}^3}.
\end{aligned}$$

Since

$$\mathbf{Q}^{(3)} = \mathbf{Q}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.609 & 0.181 & 0.21 \\ 0.539 & 0.21 & 0.251 \end{bmatrix}$$

and

$$\mathbf{Q}^{(4)} = \mathbf{Q}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0.7053 & 0.1354 & 0.1593 \\ 0.6522 & 0.1593 & 0.1885 \end{bmatrix},$$

the desired probability is

$$\frac{0.1885}{1 - 0.539} = 0.409.$$