Exercise 3. According to the conditions given by the question and the notation used in the above solution, we have $p_0 = 0.8, p_1 = 0.6, p_2 = 0.6, p_3 = 0.6, p_4 = 0.4, p_5 = 0.4, p_6 = 0.4, p_7 = 0.2$. Thus, the transition probability matrix $P$ of the Markov chain is

$$P = \begin{bmatrix}
0.8 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\
0.6 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\
0 & 0.6 & 0 & 0 & 0 & 0.4 & 0 & 0 \\
0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0 \\
0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 \\
0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0.8
\end{bmatrix}.$$ 

Exercise 7. As in Exercise 2, we use (RR),(DR),(RD),(DD) to represent the 4 states 0,1,2,3 in Example 4.4 respectively. And we suppose that today is in the $n$-th state of the Markov chain. The required probability is equal to

$$P(X_{n+1} = (DR)|X_{n-1} = (DD)) + P(X_{n+1} = (RR)|X_{n-1} = (DD)).$$

Solution 1. By direct calculation.

\begin{align*}
P(X_{n+1} = (DR)|X_{n-1} = (DD)) &+ P(X_{n+1} = (RR)|X_{n-1} = (DD)) \\
= P(X_{n+1} = (DR), X_n = (DD)|X_{n-1} = (DD)) &+ P(X_{n+1} = (RR), X_n = (DR)|X_{n-1} = (DD)) \\
= P(X_{n+1} = (DR)|X_n = (DD), X_{n-1} = (DD))P(X_n = (DD)|X_{n-1} = (DD)) &+ P(X_{n+1} = (RR)|X_n = (DR), X_{n-1} = (DD))P(X_n = (DR)|X_{n-1} = (DD)) \\
= P(X_{n+1} = (DR)|X_n = (DD)) &+ P(X_{n+1} = (RR)|X_n = (DR))P(X_n = (DR)|X_{n-1} = (DD)) \\
= (0.2)(0.8) &+ (0.5)(0.2) \\
= 0.26.
\end{align*}

Solution 2. By using Chapman-Kolmogorov equations. The required prob-
ability is \( P^{(2)}_{30} + P^{(2)}_{31} \). Since

\[
P = \begin{bmatrix}
0.7 & 0 & 0.3 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.4 & 0 & 0.6 \\
0 & 0.2 & 0 & 0.8
\end{bmatrix},
\]

we have

\[
P^{(2)} = P^2 = \begin{bmatrix}
0.7 & 0 & 0.3 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.4 & 0 & 0.6 \\
0 & 0.2 & 0 & 0.8
\end{bmatrix}^2 = \begin{bmatrix}
0.49 & 0.12 & 0.21 & 0.18 \\
0.35 & 0.2 & 0.15 & 0.3 \\
0.2 & 0.12 & 0.48 & \\
0.1 & 0.16 & 0.1 & 0.64
\end{bmatrix}.
\]

Thus,

\[
P^{(2)}_{30} + P^{(2)}_{31} = 0.1 + 0.16 = 0.26.
\]

**Exercise 8.** Set

\[
X_n = \begin{cases}
0 & \text{if coin 1 is flipped on the } n\text{-th day} \\
1 & \text{if coin 2 is flipped on the } n\text{-th day}
\end{cases}
\]

then \( \{X_n, n \geq 0\} \) is a Markov chain with probability transition matrix

\[
P = \begin{bmatrix}
0.7 & 0.3 \\
0.6 & 0.4
\end{bmatrix}
\]

and initial distribution \( P(X_0 = 0) = P(X_0 = 1) = 0.5 \). And so,

\[
P^2 = \begin{bmatrix}
0.67 & 0.33 \\
0.66 & 0.34
\end{bmatrix}
\]

and

\[
P^3 = \begin{bmatrix}
0.667 & 0.333 \\
0.666 & 0.334
\end{bmatrix}.
\]

Hence,

\[
P(X_3 = 0) = P(X_0 = 0)P^{(3)}_{11} + P(X_0 = 1)P^{(3)}_{21} = 0.5(0.667 + 0.666) = 0.6665.
\]

**Exercise 9** Set

\[
X_n = \begin{cases}
0 & \text{if coin 1 is flipped on the } n\text{-th day,} \\
1 & \text{if coin 2 is flipped on the } n\text{-th day.}
\end{cases}
\]

Then \( \{X_n, n \geq 0\} \) is a Markov chain with probability transition matrix

\[
P = \begin{bmatrix}
0.7 & 0.3 \\
0.6 & 0.4
\end{bmatrix}.
\]
Since Friday is 4 days after Monday and state 0 corresponds to flipping heads, the quantity we are interested in is \( P_{00}^4 \). We can compute \( P \) by squaring twice:

\[
P^2 = \begin{bmatrix} 0.67 & 0.33 \\ 0.66 & 0.34 \end{bmatrix}
\]

\[
P^4 = (P^2)^2 = \begin{bmatrix} 0.6667 & 0.3333 \\ 0.6666 & 0.3334 \end{bmatrix},
\]

giving us \( P_{00}^4 = 0.6667 \).

**Exercise 10.** Let \( \{X_n, n \geq 0\} \) be the Markov chain defined in Example 4.3, whose transition probability matrix is

\[
P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}.
\]

By transforming the state 2 into an absorbing state, we obtain a new transition probability matrix

\[
Q = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Because the original Markov chain and the new Markov chain follows identical transition probabilities until the state 2 is entered, it follows that, starting in state 0 or 1 at time \( n \), the probability the original Markov chain never enters state 2 by time \( n + k \) is equal to the probability the new Markov chain is not in state 2 at time \( n + k \). Therefore, the desired probability is equal to

\[
P(X_{n+3} \neq 2|X_n = 0) = 1 - P(X_{n+3} = 2|X_n = 0) = 1 - Q^{(3)}_{02}.
\]

Since

\[
Q^{(3)} = Q^3 = \begin{bmatrix} 0.293 & 0.292 & 0.415 \\ 0.219 & 0.22 & 0.561 \\ 0 & 0 & 1 \end{bmatrix},
\]

the desired probability is \( 1 - 0.415 = 0.585 \).

**Exercise 11.** Let \( \{X_n, n \geq 0\} \) and \( P \) be defined the same as in the above solution. By transforming the state 0 into an absorbing state, we obtain a new transition probability matrix

\[
Q = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}.
\]

Because the original Markov chain and the new Markov chain follows identical transition probabilities until the state 0 is entered, it follows that, starting in state 1 or 2 at time \( n - k \), the probability the original Markov chain never enters
state 0 by time \( n - 1 \) is equal to the probability the transformed Markov chain is not in state 0 at time \( n - 1 \). Therefore, the desired probability is equal to

\[
P(X_n = 2 | X_{n-4} = 2, X_{n-k} \neq 0, k = 1, 2, \ldots, 7)
\]

\[
= P(X_n = 2 | X_{n-1} \neq 0, X_{n-4} = 2)
\]

\[
= \frac{P(X_n = 2, X_{n-1} \neq 0 | X_{n-4} = 2)}{P(X_{n-1} \neq 0 | X_{n-4} = 2)}
\]

\[
= \frac{P(X_n = 2 | X_{n-4} = 2)}{1 - Q_{20}^4}
\]

Since

\[
Q^{(3)} = Q^3 = \begin{bmatrix}
1 & 0 & 0 \\
0.609 & 0.181 & 0.21 \\
0.539 & 0.21 & 0.251
\end{bmatrix}
\]

and

\[
Q^{(4)} = Q^4 = \begin{bmatrix}
1 & 0 & 0 \\
0.7053 & 0.1354 & 0.1593 \\
0.6522 & 0.1593 & 0.1885
\end{bmatrix},
\]

the desired probability is

\[
\frac{0.1885}{1 - 0.539} = 0.409.
\]