E3106, Solutions to Homework 10

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Exercise 10.6. Since Brownian motion has independent and stationary increments, the probability that you recover your purchase price is the probability that a Brownian motion goes up c by time t. Hence the desired probability is

$$\begin{aligned}
&1 - P\left(\max_{0\le s\le t} B(s) \ge c\right) \\
=&1 - P(T_c \le t) \\
=&1 - \frac{2}{\sqrt{2\pi}}\int_{c/\sqrt{t}}^{\infty} e^{-y^2/2}dy = 1 - 2\Phi\left(-\frac{c}{\sqrt{t}}\right)
\end{aligned}$$

where the last equality follows from Equation (10.7) in the textbook.
Exercise 10.7. By conditioning on $B(t_1)$, we have

$$P\left(\max_{t_1\le s\le t_2} B(s) > x\right) = \int_{-\infty}^{\infty} P\left(\max_{t_1\le s\le t_2} B(s) > x \mid B(t_1) = y\right) f_{t_1}(y)dy$$

If $y > x$, it is obvious that

$$P\left(\max_{t_1\le s\le t_2} B(s) > x \mid B(t_1) = y\right) = 1 \tag{1}$$

If $y \le x$, it follows from the independent and stationary increments of Brownian motion that

$$\begin{aligned}
&P\left(\max_{t_1\le s\le t_2} B(s) > x \mid B(t_1) = y\right) \\
=&P\left(\max_{t_1\le s\le t_2} (B(s) - B(t_1)) > x - y \mid B(t_1) = y\right) \\
=&P\left(\max_{t_1\le s\le t_2} (B(s) - B(t_1)) > x - y\right) \quad \text{(by independent increments)} \\
=&P\left(\max_{0\le s\le t_2 - t_1} B(s) > x - y\right) \quad \text{(by stationary increments)} \\
=&\frac{2}{\sqrt{2\pi}}\int_{(x-y)/\sqrt{t_2-t_1}}^{\infty} e^{-u^2/2}du = 2\Phi\left(-\frac{x-y}{\sqrt{t_2-t_1}}\right)
\end{aligned}$$

where the equality before the last follows from the continuity of the distribution of $B(s)$, and the last equality follows from the result in Section 10.2 of the textbook. Now, by Equation (1), we have

$$\begin{aligned}
&P\left(\max_{t_1\le s\le t_2} B(s) > x\right)\\
&=\int_{-\infty}^{x} 2\Phi\left(\frac{x-y}{\sqrt{t_2-t_1}}\right) f_{t_1}(y)dy + P\left(B(t_1)\ge x\right)\\
&=\int_{-\infty}^{x} 2\Phi\left(-\frac{x-y}{\sqrt{t_2-t_1}}\right)\frac{1}{\sqrt{2\pi t_1}}e^{-y^2/2t_1}dy + \Phi\left(-\frac{x}{\sqrt{t_1}}\right)
\end{aligned}$$