E3106, Solutions to Homework 11

Columbia University

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Exercise 2. (a) By the fact that if X_1 and X_2 are two independent Poission random variables with mean μ_1 and μ_2 respectively, then X_1+X_2 is also Poission distributed with mean $\mu_1 + \mu_2$ (using the fact in the basic stat class W3600), we obtain that S_n is a Poission random variable with mean $n\mu$.

(b)

$$\begin{aligned} P(N(t) = n) =& P(N(t) \geq n) - P(N(t) \geq n+1) \\ =& P(S_n \leq t) - P(S_{n+1} \leq t) \\ =& P(S_n \leq [t]) - P(S_{n+1} \leq [t]) \\ =& \sum_{k=0}^{[t]} \frac{(n\mu)^k}{k!} e^{-n\mu} - \sum_{k=0}^{[t]} \frac{((n+1)\mu)^k}{k!} e^{-(n+1)\mu} \end{aligned}$$

where $[t]$ denote the largest integer not exceeding t.

Exercise 4. (a) No. Suppose, for instance, that the interarrival times of $N_1(t)$ are identically equal to 1 and $N_2(t)$ is a Poission process. If the first interarrival time of the process $\{N(t), t \geq 0\}$ is equal to 3/4, then we can be certain that the next one is less than or equal to 1/4, because there is at least another event from $N_1(t)$ will happen at time 1. Therefore, the interarrival times of $\{N(t), t \geq 0\}$ are in general not independent.

(b) No. We can use the same process in (a) as a counterexample. Let λ be the rate of the Poission process $N_2(t)$, $X_1, X_2, \ldots$ be its interarrival times, and $Z_1, Z_2, \ldots$ be the interarrival times of $\{N(t), t \geq 0\}$. Then we have

$$P(Z_1 = 1) = P(X_1 \geq 1) = e^{-\lambda}.$$

Now let us compute

$$P(Z_2 = 1) = P(Z_2 = 1 | X_1 > 1) P(X_1 > 1) + P(Z_2 = 1 | X_1 < 1) P(X_1 < 1).$$

Note that

$$P(Z_2 = 1 | X_1 < 1) = 0,$$

because the $Z_2 < 1 - X_1 < 1$ on the event $X_1 < 1$ as there must be a event at time 1. Futhermore,

$$P(Z_2 = 1 | X_1 > 1) = P(X_1 \geq 2 | X_1 > 1),$$

as on the set $X_1 > 1$, the event $Z_2 = 1$ means the first two events are all from $N_1(t)$ are times 1 and 2, and thus the first event from $N_2(t)$ must happens after time 2.

Putting together, we have

$$\begin{aligned} & P(Z_2 = 1) \\ =& P(X_1 \geq 2 | X_1 > 1) P(X_1 > 1) + 0 \\ =& P(X_1 \geq 2) = e^{-2\lambda}, \end{aligned}$$

which is in general not equal to $P(Z_1 = 1)$. So Z_1 and Z_2 are not identically distributed.

(c) No. From the results in (a) or (b), the interarrival times of $\{N(t), t \geq 0\}$ can be either not independent or not identically distributed, so $\{N(t), t \geq 0\}$ is in general not a renewal process.

Exercise 15. (a) Let X_i be the amount of time that the miner has to travel until he returns to the room or becomes free after his ith choice. Then $X_1, X_2, \ldots$ are independent and identically distributed with

$$P(X_i = 2) = P(X_i = 4) = P(X_i = 6) = \frac{1}{3}. \tag{1}$$

Define random variable

$$N = \min\{n \geq 1 : X_n = 2\} \tag{2}$$

Since $N = n$ if and only if $X_1 \neq 2, X_2 \neq 2, \ldots, X_{n-1} \neq 2$ and $X_n = 2$, it follows that the event $\{N = n\}$ is independent with $X_{n+1}, X_{n+2}, \ldots$. Thus, N is a stopping time for the sequence $X_1, X_2, \ldots$. And by the definition of T, we have

$$T = \sum_{i=1}^{N} X_i \tag{3}$$

(b) By the definition of N in Equation (2) and the distribution of X_i in Equation (1), N is a geometric random variable with parameter $p = P(X_1 = 2) = 1/3$. Thus

$$E[N] = \frac{1}{p} = 3$$

Since N is a stopping time for the sequence $X_1, X_2, \ldots$ and has finite mean, it follows from Wald's equation and Equation (3) that

$$E[T] = E[X_1]E[N] = (2 \times \frac{1}{3} + 4 \times \frac{1}{3} + 6 \times \frac{1}{3}) \times 3 = 12$$

(c)

$$
\begin{aligned}
&E\left[\sum_{i=1}^{N} X_i | N = n\right] \\
=&E\left[\sum_{i=1}^{n} X_i | N = n\right] \\
=&\sum_{i=1}^{n} E[X_i | N = n] \\
=&\sum_{i=1}^{n} E[X_i | X_1 \neq 2, X_2 \neq 2, \ldots, X_{n-1} \neq 2, X_n = 2] \\
=&\sum_{i=1}^{n-1} E[X_i | X_i \neq 2] + E[X_n | X_n = 2].
\end{aligned}
$$

Since

$$P(X_i = 4 | X_i \neq 2) = P(X_i = 6 | X_i \neq 2) = \frac{1}{2}$$

it follows that

$$E[X_i | X_i \neq 2] = 4 \times \frac{1}{2} + 6 \times \frac{1}{2} = 5$$

Therefore

$$E\left[\sum_{i=1}^{N} X_i | N = n\right] = 5(n-1) + 2 = 5n - 3$$

(d) From the result of (c), we have

$$E[T|N] = 5N - 3.$$

Hence

$$E[T] = E[E[T|N]] = 5E[N] - 3 = 5 \times 3 - 3 = 12.$$

Exercise 16. By the definition of N, exactly four of the random variables $X_1, X_2, \ldots, X_N$ are equal to 1 and all the others are equal to 0. Thus,

$$E\left[\sum_{i=1}^{N} X_i\right] = 4. \tag{4}$$

Suppose any sequence of the deck of cards are equally likely, then

$$P(X_i = 1) = \frac{4}{52} = \frac{1}{13}$$

Hence

$$E[X_i] = P(X_i = 1) = \frac{1}{13}. \tag{5}$$

We will show that Wald's equation cannot be true in this case. Suppose it is true. Then we have $E[N] = 4/(1/13) = 52$, which is clearly not ture, as we have only 52 cards and there is always a positive probability that we do not need to turn over all 52 cards until all 4 aces appear.

In fact, since

$$\sum_{i=1}^{52} X_i = 4$$

always holds, it follows that $X_1, X_2, \ldots, X_{52}$ are not independent with each other. Therefore, Wald's equation is not applicable.