E3106, Solutions to Homework 2

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Exercise 13. **Proof** For $\forall n > r$ and $\forall i, j$, according to Chapman-Kolmogorov equations, we have

$$P_{ij}^n = \sum_k P_{ik}^{n-r} P_{kj}^r.$$

Since

$$\sum_k P_{ik}^{n-r} = 1,$$

it follows that there exists some state k_0 such that $P_{ik_0}^{n-r} > 0$. And because $P_{k_0 j}^r > 0$, it follows that

$$P_{ij}^n \geq P_{ik_0}^{n-r} P_{k_0 j}^r > 0,$$

which completes the proof.

Exercise 14. (1) The classes of the states of the Markov chain with transition probability $\mathbf{P}_1$ is $\{0, 1, 2\}$. Because it is a finite-state Markov chain, all the states are recurrent.

(2)The classes of the states of the Markov chain with transition probability $\mathbf{P}_2$ is $\{0, 1, 2, 3\}$. And because it is a finite-state Markov chain, all the states are recurrent.

(3)The classes of the states of the Markov chain with transition probability $\mathbf{P}_3$ is $\{0, 2\}, \{1\}, \{3, 4\}$. It can be easily seen from a graph that $\{0, 2\}$ and $\{3, 4\}$ are recurrent, while $\{1\}$ is transient.

Here we also give a rigorous proof of this claim.

(3.1) States 0,2 are recurrent. Firstly, we prove by induction that for all n, $P_{00}^n = \frac{1}{2}$ and $P_{02}^n = \frac{1}{2}$. Obviously the result is true for $n = 1$. Then suppose $P_{00}^{n-1} = \frac{1}{2}$ and $P_{02}^{n-1} = \frac{1}{2}$ are true, it follows that

$$\begin{aligned} P_{00}^n &= P_{00}^{n-1} P_{00} + P_{02}^{n-1} P_{20} = \frac{1}{2} \\ P_{02}^n &= P_{00}^{n-1} P_{02} + P_{02}^{n-1} P_{22} = \frac{1}{2}. \end{aligned}$$

Hence the result is true for all n.

Secondly, as

$$\sum_{n=1}^{\infty} P_{00}^n = \sum_{n=1}^{\infty} \frac{1}{2} = \infty,$$

state 0 is recurrent.

(3.2) State 1 is transient, because

$$\sum_{n=1}^{\infty} P_{11}^{n} = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

(3.3) State 3,4 is recurrent, because we can prove, in the same way as in (3.1), that for all n,$P_{33}^{n} = \frac{1}{2}$. And hence

$$\sum_{n=1}^{\infty} P_{33}^{n} = \sum_{n=1}^{\infty} \frac{1}{2} = \infty.$$

(4) The classes of the states of the Markov chain $\{X_n, n \geq 0\}$ with transition probability $\mathbf{P}_4$ is $\{0,1\}, \{2\}, \{3\}, \{4\}$. Again, we can easily determine transient and recurent states via a graph.

Here we also give a rigorous proof of this.

(4.1) States 0,1 are recurrent. Let $\{Y_n, n \geq 0\}$ denote a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Because $\{Y_n, n \geq 0\}$ is irreducible and finite-state, state 0 is recurrent for $\{Y_n, n \geq 0\}$. Starting from state 0 or 1, the two Markov chains $\{X_n, n \geq 0\}$ and $\{Y_n, n \geq 0\}$ follow identical transition probabilities, so state 0 is also recurrent for Markov chain $\{X_n, n \geq 0\}$.

(4.2) State 2 is recurrent, because

$$\sum_{n=1}^{\infty} P_{22}^{n} = \sum_{n=1}^{\infty} 1 = \infty.$$

(4.3) State 3 is transient, because

$$\sum_{n=1}^{\infty} P_{33}^{n} = \sum_{n=1}^{\infty} (\frac{2}{3})^n < \infty.$$

(4.4) State 4 is transient, because

$$\sum_{n=1}^{\infty} P_{44}^{n} = \sum_{n=1}^{\infty} 0 < \infty.$$

Exercise 15. **Proof** Since state j can be reached from state i, there exists m such that $P_{ij}^{m} > 0$. Let $i_0 = i$ and $i_m = j$. As

$$P_{ij}^{m} = \sum_{i_1 i_2 \cdots i_{m-1}} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{m-2} i_{m-1}} P_{i_{m-1} i_m} > 0,$$

there exists a path $i_0,\, i_1, \ldots, i_{m-1},\, i_m$ from i to j such that

$$P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{m-2} i_{m-1}} P_{i_{m-1} i_m} > 0.$$

Suppose that there exist two indices $1 \le k < l \le m$ such that $i_k = i_l$, then $i_0 = i, i_1, \ldots, i_k, i_{l+1}, \ldots, i_m = j$ is also a path from i to j, and

$$P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{k-1} i_k} P_{i_k i_{l+1}} \cdots P_{i_{m-1} i_m} > 0.$$

Hence the shortest possible transition path must contain all distinctive states. Therefore, the shortest number of transitions must contain M steps or less. (If i and j are the same, then it could contain M steps; if i and j are different, then it can have at most $M-1$ steps.)

Exercise 17. Let $\{X_n, n \ge 0\}$ be the Markov chain defined in Example 4.15. Define

$$Y_n = X_n - X_{n-1},\ n \ge 1.$$

First we want to show that $\{Y_n, n \ge 1\}$ are independent identically distributed random variables.

For $\forall n \ge 1$

$$\begin{aligned}
&P(Y_n = 1) = 1 - P(Y_n = -1) \\
=&P(X_n - X_{n-1} = 1) = \sum_i P(X_n = i+1 | X_{n-1} = i) P(X_{n-1} = i) \\
=&p \sum_i P(X_{n-1} = i) \\
=&p.
\end{aligned}$$

This shows that $\{Y_n, n \ge 1\}$ all have the same distribution.

For $\forall n > k$,

$$\begin{aligned}
&P(Y_n = 1, Y_k = 1) \\
=&P(X_n - X_{n-1} = 1, X_k - X_{k-1} = 1) \\
=&\sum_{i,j} P(X_n = i+1, X_{n-1} = i, X_k = j+1, X_{k-1} = j) \\
=&\sum_{i,j} P(X_n = i+1 | X_{n-1} = i) P(X_{n-1} = i | X_k = j+1) P(X_k = j+1 | X_{k-1} = j) P(X_{k-1} = j) \\
=&\sum_{i,j} p^2 P_{j+1,i}^{n-1-k} P(X_{k-1} = j) \\
=&\sum_{j} p^2 P(X_{k-1} = j) \\
=&p^2 \\
=&P(Y_n = 1) P(Y_k = 1).
\end{aligned}$$

Similar arguments hold for $P(Y_n = i, Y_k = j) = P(Y_n = i) P(Y_k = j), \forall i, j \in \{1, -1\}$. So Y_n and Y_k are independent.

In summary $Y_1, Y_2, \ldots$ are i.i.d random variables.

Secondly, suppose that $\{X_n, n \geq 0\}$ starts in state 0, *i.e.*, $X_0 = 0$. Then

$$X_n = \sum_{k=1}^{n} Y_k.$$

Since $Y_1, Y_2, \ldots$ are i.i.d random variables and $E[Y_n] = 2p - 1$, it follows from strong law of large numbers that

$$P(\lim_{n\to\infty} \frac{X_n}{n} = E[Y_1] = 2p - 1) = 1.$$

If $p > \frac{1}{2}$, then $2p - 1 > 0$, it follows from the above equality that

$$P(\lim_{n\to\infty} X_n = \infty) = 1,$$

Hence

$$P(X_n \text{ visits 0 infinitely often}) \leq P(\lim_{n\to\infty} X_n \neq \infty) = 0.$$

Thus,

$$P(X_n \text{ visits 0 infinitely often}) = 0.$$

So state 0 is a transient state.

A similar argument holds when $p < \frac{1}{2}$. Consequently, when $p \neq \frac{1}{2}$, state 0 is a transient state of the Markov chain $\{X_n, n \geq 0\}$.