E3106, Solutions to Homework 4

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Problem 56. This is equivalent to the gambler's ruin problem discussed in Section 4.5.1 with $N = n + m$. Thus, the desired probability is just the probability P_m defined in the gambler's ruin problem, which is

$$P_m = \begin{cases} \frac{1-(q/p)^m}{1-(q/p)^{m+n}}, & \text{if } p \neq \frac{1}{2} \\ \frac{m}{m+n}, & \text{if } p = \frac{1}{2} \end{cases}$$

given by Equation (4.14) in the textbook.

Problem 57. Define

$$P_0 = P(X_n \text{ starts in } 0\text{, and visits all states before returning to } 0).$$

By conditioning on the first transition we obtain

$$\begin{aligned} P_0 =& P(\text{visits all states before returning to } 0 \,|\, \text{starts in } 0, X_1 = 1) \\ & \times P(\text{starts in } 0, X_1 = 1) \\ & + P(\text{visits all states before returning to } 0 \,|\, \text{starts in } 0, X_1 = n) \\ & \times P(\text{starts in } 0, X_1 = n) \\ =& P(\text{reaches state } n \text{ before returning to } 0 \,|\, X_1 = 1)p \\ & + P(\text{reaches state } 1 \text{ before returning to } 0 \,|\, X_1 = n)q \end{aligned} \tag{1}$$

The conditional probability

$$P(\text{reaches state } n \text{ before returning to } 0 \,|\, X_1 = 1)$$

is equal to the probability that a gambler, initially with 1 unit of money, will eventually have n units of money. Hence it follows from Equation (4.14) that

$$\begin{aligned} & P(\text{reaches state } n \text{ before returning to } 0 \,|\, X_1 = 1) \\ =& \begin{cases} \frac{1-(q/p)}{1-(q/p)^n} & \text{if } p \neq \frac{1}{2} \\ \frac{1}{n} & \text{if } p = \frac{1}{2} \end{cases} \end{aligned} \tag{2}$$

And we can obtain from similar argument that

$$\begin{aligned} & P(\text{reaches state } 1 \text{ before returning to } 0 \,|\, X_1 = n) \\ =& \begin{cases} \frac{1-(p/q)}{1-(p/q)^n} & \text{if } p \neq \frac{1}{2} \\ \frac{1}{n} & \text{if } p = \frac{1}{2} \end{cases} \end{aligned} \tag{3}$$

Finally, it follows from Equation (1), (2), (3) that

$$P_0 = \begin{cases} p\left(\frac{1-(q/p)}{1-(q/p)^n}\right) + q\left(\frac{1-(p/q)}{1-(p/q)^n}\right) & \text{if } p \neq \frac{1}{2} \\ \frac{1}{n} & \text{if } p = \frac{1}{2} \end{cases}$$

Problem 58. Using the hint, the desired probability is

$$P(X_{n+1} = i+1 | X_n = i, \lim_{m\to\infty} X_m = N) = \frac{P(X_{n+1} = i+1, \lim_{m\to\infty} X_m = N | X_n = i)}{P(\lim_{m\to\infty} X_m = N | X_n = i)}.$$

First of all,

$$P(\lim_{m\to\infty} X_m = N | X_n = i) = P(\text{reach } N \text{ before } 0 | X_n = i) = P_i,$$

using the notation in Equation (4.14) on p. 215. Secondly,

$$\begin{aligned} P(X_{n+1} &= i+1, \lim_{m\to\infty} X_m = N | X_n = i) \\ &= P(\lim_{m\to\infty} X_m = N | X_{n+1} = i+1, X_n = i) P(X_{n+1} = i+1 | X_n = i) \\ &= P(\lim_{m\to\infty} X_m = N | X_{n+1} = i+1) p \\ &= P(\text{reach } N \text{ before } 0 | X_n = i+1) p \\ &= P_{i+1} p, \end{aligned}$$

where the second equality follows from the Markov property, and the last one uses the notation in Equation (4.14). In summary we have the desired probability is

$$\frac{P_{i+1}p}{P_i} = \left\{ \begin{array}{ll} p\frac{1-(q/p)^{i+1}}{1-(q/p)^i}, & \text{if } p \neq \frac{1}{2} \\ p\frac{(i+1)}{i} = \frac{i+1}{2i}, & \text{if } p = \frac{1}{2} \end{array} \right\}$$

Problem 61. (a) With $P(0) = 0$ and $P(N) = 1$, we have a recursion

$$P(i) = \alpha_i P(i+1) + (1-\alpha_i) P_{i-1}, \ i = 1, ..., N-1.$$

(b) The latter equation can be rewritten as

$$P(i+1) - P(i) = \frac{1-\alpha_i}{\alpha_i}(P(i) - P(i-1)).$$

Repeating the same technique as in the gambler's ruin problem, we have

$$\begin{aligned} P(2) - P(1) &= \frac{1-\alpha_1}{\alpha_1}(P(1) - P(0)) = \frac{1-\alpha_1}{\alpha_1} P(1) \\ P(3) - P(2) &= \frac{1-\alpha_2}{\alpha_2}(P(2) - P(0)) = \frac{1-\alpha_2}{\alpha_2}\frac{1-\alpha_1}{\alpha_1} P(1) \\ &\dots \\ P(i) - P(i-2) &= \frac{1-\alpha_{i-1}}{\alpha_{i-1}}(P(i-1) - P(i-2)) = \frac{1-\alpha_{i-1}}{\alpha_{i-1}} \cdots \frac{1-\alpha_2}{\alpha_2}\frac{1-\alpha_1}{\alpha_1} P(1). \end{aligned}$$

Thus, adding these equations together we have

$$P(i) - P(1) = P(1)\left[\frac{1-\alpha_1}{\alpha_1} + \frac{1-\alpha_2}{\alpha_2}\frac{1-\alpha_1}{\alpha_1} + \cdots + \prod_{k=1}^{i-1}\frac{1-\alpha_k}{\alpha_k}\right],$$

or

$$P(i) = P(1)\left[1 + C_1 + C_2 + \cdots + C_{i-1}\right],$$

where

$$C_j = \prod_{k=1}^{j}\frac{1-\alpha_k}{\alpha_k}.$$

Since $P(N) = 1$, we have

$$P(1) = \frac{P_N}{1 + C_1 + C_2 + \cdots + C_{N-1}} = \frac{1}{1 + C_1 + C_2 + \cdots + C_{N-1}},$$

and

$$P(i) = \frac{1 + C_1 + C_2 + \cdots + C_{i-1}}{1 + C_1 + C_2 + \cdots + C_{N-1}},\ i = 1, ..., N.$$

(c) This is equal to $P(N-i)$ in the above equation with $\alpha_{N-k} = k/N$, as when there are $N-k$ balls in urn 2, the probability of winning is equal to that of moving a ball from urn 1 to urn 2. Thus,

$$\alpha_k = \frac{N-k}{N}.$$

In this case

$$\begin{aligned} C_j &= \prod_{k=1}^{j}\frac{1-\alpha_k}{\alpha_k} = \prod_{k=1}^{j}\frac{k}{N-k} = \frac{1\cdot 2\cdots j}{(N-1)\cdots(N-j)} \\ &= \frac{j!(N-j-1)!}{(N-1)!} = \frac{1}{\binom{N-1}{j}}, \end{aligned}$$

and

$$P(i) = \frac{1 + \frac{1}{\binom{N-1}{1}} + \frac{1}{\binom{N-1}{2}} + \cdots + \frac{1}{\binom{N-1}{i-1}}}{1 + \frac{1}{\binom{N-1}{1}} + \frac{1}{\binom{N-1}{2}} + \cdots + \frac{1}{\binom{N-1}{N-1}}},\ i = 1, ..., N.$$

Problem 62. (a) By the fourth interpretation of π_i, we have the desired is

$$\frac{1}{\pi_i} = \frac{1}{1/5} = 5.$$

(b). There are two cases. Suppose the first move is to the right. Then the probability is just the probability that a gambler who wins with probability p will reach 4 before going broke, starting with 1; this is equal to

$$\frac{1-q/p}{1-(q/p)^4},\ \text{if } p \neq \frac{1}{2};\ \frac{1}{4},\ \text{if } p = \frac{1}{2}.$$

Similarly, if the first move is to left, then the probability is, with p and q reversed, given by

$$\frac{1-p/q}{1-(p/q)^4}, \text{ if } p \neq \frac{1}{2}; \ \frac{1}{4}, \text{ if } p = \frac{1}{2}.$$

Therefore, the overall probability is when $p \neq \frac{1}{2}$

$$p\frac{1-q/p}{1-(q/p)^4} + q\frac{1-p/q}{1-(p/q)^4} = \frac{p-q}{1-(q/p)^4} + \frac{q-p}{1-(p/q)^4},$$

and is $1/4$ if $p = 1/2$.

Problem 63. The transient states of the Markov chain are $\{1, 2, 3\}$, and the transition probabilities between the these states are

$$\mathbf{P_T} = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

So

$$\mathbf{S} = (\mathbf{I} - \mathbf{P_T})^{-1} = \begin{bmatrix} 0.6 & -0.2 & -0.1 \\ -0.1 & 0.5 & -0.2 \\ -0.3 & -0.4 & 0.8 \end{bmatrix}^{-1} = \begin{bmatrix} 2.2069 & 1.3793 & 0.6207 \\ 0.9655 & 3.1034 & 0.8966 \\ 1.3103 & 2.0690 & 1.9310 \end{bmatrix}.$$

Hence

$$s_{13} = 0.6207, s_{23} = 0.8966, s_{33} = 1.9310$$

and

$$f_{13} = \frac{s_{13} - 0}{s_{33}} = 0.3214, f_{23} = \frac{s_{23} - 0}{s_{33}} = 0.4643, f_{33} = \frac{s_{33} - 1}{s_{33}} = 0.4821$$