E3106, Solutions to Homework 5

Columbia University

Exercise 12.

(a)

$$P(X_1 < X_2 < X_3) = P(X_1 = \min(X_1, X_2, X_3)) \cdot P(X_2 < X_3 | X_1 = \min(X_1, X_2, X_3))$$

It follows the result in p. 282 in Section 5.2.3 of the textbook (see also the last example in the Lecture note #9) that

$$P(X_1 = \min(X_1, X_2, X_3)) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$$

By the lack of memory property of exponential distribution, we have

$$P(X_2 < X_3 | X_1 = \min(X_1, X_2, X_3)) = P(X_2 < X_3) = \frac{\lambda_2}{\lambda_2 + \lambda_3}.$$

Hence,

$$P(X_1 < X_2 < X_3) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_2 + \lambda_3}.$$

(b)

$$\begin{aligned}
&P(X_1 < X_2 | X_3 = \max(X_1, X_2, X_3)) \\
=&\frac{P(X_1 < X_2 < X_3)}{P(X_3 = \max(X_1, X_2, X_3))} \\
=&\frac{P(X_1 < X_2 < X_3)}{P(X_1 < X_2 < X_3) + P(X_2 < X_1 < X_3)} \\
=&\frac{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \frac{\lambda_2}{\lambda_2+\lambda_3}}{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \frac{\lambda_2}{\lambda_2+\lambda_3} + \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \frac{\lambda_1}{\lambda_1+\lambda_3}} \\
=&\frac{\lambda_1 + \lambda_3}{\lambda_1 + \lambda_2 + 2\lambda_3},
\end{aligned}$$

where the third equality follows from Part (a).

(c)

$$\begin{aligned}
&E[\max X_i | X_1 < X_2 < X_3] \\
=&E[X_1 + (X_2 - X_1) + (X_3 - X_2) | X_1 < X_2 < X_3] \\
=&E[X_1 | X_1 < X_2 < X_3] + E[X_2 - X_1 | X_1 < X_2 < X_3] + E[X_3 - X_2 | X_1 < X_2 < X_3] \\
=&E[X_1 | X_1 < X_2 < X_3] + E[X_2 | X_2 < X_3] + E[X_3] \\
=&E[\min X_i | X_1 < X_2 < X_3] + E[\min\{X_2, X_3\} | X_2 < X_3] + E[X_3]
\end{aligned}$$

where the third equality follows from the memoryless property of exponential distribution. Then by the result in p. 282 of the textbook, $\min_i X_i$ is independent with the rank ordering of X_i, thus

$$E[\min_i X_i | X_1 < X_2 < X_3] = E[\min_i X_i] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}$$
$$E[\min\{X_2, X_3\} | X_2 < X_3] = E[\min\{X_2, X_3\}] = \frac{1}{\lambda_2 + \lambda_3}$$

Therefore,

$$E[\max X_i | X_1 < X_2 < X_3] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_3}.$$

(d) By conditioning on the rank ordering of X_i, we have

$$E[\max X_i] = \sum_{i \neq j \neq k} E[\max X_i | X_i < X_j < X_k] P(X_i < X_j < X_k),$$

where the sum is taken over all 6 permutations of 1, 2, 3. It follows from the results of Parts (a) and (c) that

$$E[\max X_i | X_i < X_j < X_k] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k}$$
$$P(X_i < X_j < X_k) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_j}{\lambda_j + \lambda_k}$$

Hence

$$E[\max X_i] = \sum_{i \neq j \neq k} \left[\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k} \right] \left[\frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_j}{\lambda_j + \lambda_k} \right]$$

Exercise 21

The expected total time in the system can be broken down as follows:

$$\begin{aligned} E[\text{time in system}] =& E[\text{waiting time at 1}] + E[\text{service time at 1}] \\ &+ E[\text{waiting time at 2}] + E[\text{service time at 2}], \end{aligned}$$

where we claim that

$$E[\text{waiting time at 1}] = \frac{1}{\mu_1},\ E[\text{service time at 1}] = \frac{1}{\mu_1}$$
$$E[\text{waiting time at 2}] = \frac{1}{\mu_2} \frac{\mu_1}{\mu_1 + \mu_2},\ E[\text{service time at 2}] = \frac{1}{\mu_2}.$$

The only thing we need to justify is how to compute E[waiting time at 2], as the other three terms follow from the definition of exponential random variables. To do this, conditioning on which server completes service first when both servers are busy, we get

$$E[\text{waiting time at 2}] = E[\text{waiting time at 2} \mid \text{1 completes first}] \cdot P[\text{1 completes first}] + E[\text{waiting time at 2} \mid \text{2 completes first}] \cdot P[\text{2 completes first}]$$

If 1 completes service first, by the memoryless property it is as if service restarts at server 2, and the expected wait time at server 2 is $\frac{1}{\mu_2}$. If 2 completes service first, our wait time is 0. Also, we know from properties of the exponential distribution that $P[\text{1 completes first}] = \frac{\mu_1}{\mu_1+\mu_2}$. So we have

$$E[\text{waiting time at 2}] = \frac{1}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2} + 0 * \frac{\mu_2}{\mu_1+\mu_2} = \frac{1}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2}.$$

Putting these quantities together, we get

$$E[\text{time in system}] = \frac{2}{\mu_1} + \frac{1}{\mu_2}\left(1 + \frac{\mu_1}{\mu_1+\mu_2}\right).$$

Exercise 25

(a) Our expected total time in the system can be written as

$$E[\text{time in system}] = E[\text{waiting time}] + E[\text{service time}].$$

The waiting time is simply the minimum of three exponentially distributed random variables with rates μ_1, μ_2, and μ_3. Therefore,

$$E[\text{waiting time}] = \frac{1}{\mu_1+\mu_2+\mu_3}.$$

To compute the expected service time, we condition on which server frees up first. Since

$$P(\text{Server } i \text{ finishes first}) = P(X_i = \min(X_1, X_2, X_3)) = \frac{\mu_i}{\mu_1+\mu_2+\mu_3},$$

(see the lecture notes #9, or p. 282 of the textbook), we have

$$E[\text{service time}] = \sum_{i=1}^{3} \frac{1}{\mu_i}\frac{\mu_i}{\mu_1+\mu_2+\mu_3} = \sum_{i=1}^{3} \frac{1}{\mu_1+\mu_2+\mu_3}.$$

Combining these two terms, we get

$$E[\text{time in system}] = \frac{1}{\mu_1+\mu_2+\mu_3} + \sum_{i=1}^{3} \frac{1}{\mu_1+\mu_2+\mu_3} = \frac{4}{\mu_1+\mu_2+\mu_3}.$$

(b) If one customer is waiting in front of us, we must first wait for this customer to enter service, and then we are in the same situation as in part (a):

$$\begin{aligned} E[\text{ time in system }] &= E[\text{ time until first customer enters service }] + (\text{ result from part (a) }) \\ &= \frac{1}{\mu_1+\mu_2+\mu_3} + \frac{4}{\mu_1+\mu_2+\mu_3} \\ &= \frac{5}{\mu_1+\mu_2+\mu_3} \end{aligned}$$

Exercise 42.

(a)

$$E[S_4] = E\left[\sum_{i=1}^{4} T_i\right] = \sum_{i=1}^{4} E\left[T_i\right] = \frac{4}{\lambda}.$$

(b)

$E\left[S_4 | N\left(1\right) = 2\right] = 1 + E$[time for 2 more events after time 1 — $N\left(1\right) = 2$]=1+ E[time for 2 more events after time 1]= $1 + \frac{2}{\lambda}$, using the memoryless property.

(c) Since $\{N(t), t \geq 0\}$ has independent increments, we have

$$E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)] = (4-2)\lambda = 2\lambda.$$

Exercise 47

(a) We write the expectation as follows:

E[time until next entry] $=$ E[time until a server frees]+E[time until an arrival].

The time until a server frees up is simply the minimum of two exponentially distributed random variables, both with rate μ so

$$E[\text{time until a server frees}] = \frac{1}{\mu + \mu} = \frac{1}{2\mu}.$$

By the memoryless property of the exponential distribution, when a server frees up, its as if the Poisson process of arrivals restarts. Therefore, the expected time until another arrival is

$$E[\text{time until an arrival}] = \frac{1}{\lambda}.$$

Putting these together,

$$E[\text{time until next entry}] = \frac{1}{2\mu} + \frac{1}{\lambda}.$$

(b) Let T_i denote the time until both servers are busy, starting with i busy servers, for $i = 0, 1$. Then, we have the relationship

$$E[T_0] = \frac{1}{\lambda} + E[T_1],$$

since it takes an expected time of $\frac{1}{\lambda}$ for a customer to enter the system.

Now we would like an expression for $E[T_1]$. Let X be the time until the first event (arrival of a new customer or departure of the existing customer) and let Y be the additional time after X until both servers are busy. Now, we can compute $E[T_1]$ by conditioning Y on whether the service of the one customer finishes before another arrival or vice versa:

$$\begin{aligned}
E[T_1] =& E[X] + E[Y] \\
=& \frac{1}{\lambda+\mu} + E[Y|\text{ departure first}] \cdot P[\text{departure first}] \\
&+ E[Y|\text{ arrival occurs first}] \cdot P[\text{arrival occurs first}] \\
=& \frac{1}{\lambda+\mu} + E[T_0]\frac{\mu}{\lambda+\mu} + 0 * P[\text{arrival occurs first}].
\end{aligned}$$

The last equality follows by noticing: (1) Conditional on the departure completing first, the expected time until both servers are busy is simply $E[T_0]$, as by the Markov property, after the first event occurs it is as if all random times above restart. (2)If an arrival occurs first Y is 0, as both servers will be busy immediately.

In summary we have two equations

$$E[T_0] = \frac{1}{\lambda} + E[T_1], \;\; E[T_1] = \frac{1}{\lambda+\mu} + E[T_0]\frac{\mu}{\lambda+\mu}$$

Now, solving this system of two equations gives us

$$E[T_0] = \frac{2\lambda+\mu}{\lambda^2}, \; E[T_1] = \frac{\lambda+\mu}{\lambda^2}.$$

(c) Let L_i denote the time until a customer is lost, starting with i busy servers, for $i = 1, 2$. The desired answer is simply L_2, because the customers can only get lost if both servers are busy, and if one customer is lost, the whole system is back to the case of two busy servers due to the memoryless property.

Similar to Part (b), let D be the time to the first event until either a departure or an arrival, and let Z be the time after the first event (arrival or departure) until a customer is lost. Then we can write:

$$\begin{aligned}
E[L_2] = E[D] &+ E[Z|\text{ server frees first }]P[\text{ server frees first }] \\
&+ E[Z|\text{ arrival occurs first }]P[\text{ arrival occurs first }] \\
&= \frac{1}{\lambda+\mu+\mu} + (E[T_1] + E[L_2])\left(\frac{2\mu}{\lambda+2\mu}\right) \\
&= \frac{1}{\lambda+2\mu} + \left(\frac{\lambda+\mu}{\lambda^2} + E[L_2]\right)\left(\frac{2\mu}{\lambda+2\mu}\right).
\end{aligned}$$

The second equality above follows from noticing that given that a server frees first, the expected additional time until a loss occurs is the expected time until both servers become busy again, or $E[T_1]$, plus the total expected time until a loss occurs starting with both servers busy, or $E[L_2]$. Now, we have an equation for $E[L_2]$, which we can easily solve to get

$$E[L_2] = \frac{1}{\lambda} + \frac{2\mu(\lambda+\mu)}{\lambda^3}.$$

Exercise 52. Let $\{N_i(t), t \geq 0\}$ denote the Poisson process of team i, and let

$$N(t) = N_1(t) + N_2(t), t \geq 0.$$

Since $N_1(t)$ are independent with $N_2(t)$, it follows that $\{N(t), t \geq 0\}$ is a Poisson with rate $\lambda_1 + \lambda_2$. Let

$$I_n = \begin{cases} 1 & \text{if the } n\text{-th point of } N(t) \text{ belongs to team 1} \\ -1 & \text{if the } n\text{-th point of } N(t) \text{ belongs to team 2} \end{cases}$$

then $\{I_n, n \geq 1\}$ are i.i.d random variables with

$$\begin{aligned} & P(I_n = 1) \\ =& P(\text{team 1 scores before team 2 after the } (n-1)\text{-th point}) \\ =& \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

Thus, the match ends when the first time $\sum_{i=1}^{n} I_i = k$ or $-k$, and team 1 wins when the match ends with $\sum_{i=1}^{n} I_i = k$.

Hence the problem is equivalent to the gambler's ruin problem in Section 4.5.1 of textbook; and the desired probability is equal to the probability that, initially having k units of wealth, the gambler's fortune will reach $2k$ before reaching 0. Since the gambler has probability $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ of winning one unit at each play, it follows from Equation (4.14) in textbook that the desired probability is equal to

$$P_k = \begin{cases} \frac{1-(\lambda_2/\lambda_1)^k}{1-(\lambda_2/\lambda_1)^{2k}} & \text{if } \lambda_1 \neq \lambda_2 \\ \frac{1}{2} & \text{if } \lambda_1 = \lambda_2 \end{cases}$$