E3106, Solutions to Homework 7

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Exercise 14. Letting the number of cars in the station be the state variable, we have a birth-death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 20,\ \mu_1 = \mu_2 = 12$$

$$\lambda_i = 0,\ i > 2.$$

Hence the limiting probabilities must satisfy, as the equations on p. 380-371

$$\begin{aligned} 12P_1 &= 20P_0 \\ 12P_2 &= 20P_1 \\ 12P_3 &= 20P_2 \\ P_0 + P_1 + P_2 + P_3 &= 1 \end{aligned}$$

which leads to

$$P_0 = \left(1 + \frac{5}{3} + \left[\frac{5}{3}\right]^2 + \left[\frac{5}{3}\right]^3\right)^{-1} = \frac{27}{272}.$$

(1) The fraction of the attendant's time spent servicing cars is equal to the fraction of time there are cars in the system and is therefore $1 - P_0 = 245/272$.

(2) The fraction of potential customers that are lost is equal to the fraction of customers that arrive when there are 3 cars in the station, and is therefore

$$P_3 = \left(\frac{5}{3}\right)^2 P_0 = \frac{125}{272}.$$

Exercise 15. Let $X(t)$ denote the number of customers in the service center at time t, then $\{X(t), t \geq 0\}$ is a birth and death process with state space $\{0, 1, 2, 3\}$ and rates

$$\lambda_0 = \lambda_1 = \lambda_2 = 3$$
$$\mu_1 = 2, \mu_2 = \mu_3 = 4.$$

From the results obtained in Section 6.5 of the textbook, the limiting probabilities of the Markov chain satisfy

$$\begin{aligned}
P_1 &= \frac{3}{2}P_0 \\
P_2 &= \frac{3}{4}P_1 = \frac{9}{8}P_0 \\
P_3 &= \frac{3}{4}P_2 = \frac{27}{32}P_0 \\
P_0 &+ P_1 + P_2 + P_3 = 1,
\end{aligned}$$

which solve to yield

$$P_0 = \frac{32}{143}, P_1 = \frac{48}{143}, P_2 = \frac{36}{143}, P_3 = \frac{27}{143}.$$

(a) The fraction of potential customers that enter the system is

$$1 - P_3 = \frac{116}{143}.$$

(b) If there was only a single server and the rate was 4 services per hour, then the new model would have

$$\begin{aligned}
\lambda_0 &= \lambda_1 = \lambda_2 = 3 \\
\mu_1 &= \mu_2 = \mu_3 = 4
\end{aligned}$$

and the limiting probabilities would satisfy

$$\begin{aligned}
P_1 &= \frac{3}{4}P_0 \\
P_2 &= \frac{3}{4}P_1 = \frac{9}{16}P_0 \\
P_3 &= \frac{3}{4}P_2 = \frac{27}{64}P_0 \\
P_0 &+ P_1 + P_2 + P_3 = 1,
\end{aligned}$$

yielding

$$P_0 = \frac{64}{175}, P_1 = \frac{48}{175}, P_2 = \frac{36}{175}, P_3 = \frac{27}{175}.$$

Thus the fraction of potential customers that enter the system would be

$$1 - P_3 = \frac{148}{175}.$$

Exercise 17. Let $X(t)$ denote the state of the machine at time t, which is one of the following:

0: the machine is up
1: the machine is down due to type 1 failure

2: the machine is down due to type 2 failure

Then $\{X(t), t \geq 0\}$ is a continuous time Markov chain with

$$\begin{aligned} &P_{01} = p = 1 - P_{02}, P_{10} = P_{20} = 1 \\ &v_0 = \lambda, v_1 = \mu_1, v_2 = \mu_2. \end{aligned}$$

From equations (6.18) in the textbook, the balance equations for the limiting probabilities are

$$\begin{aligned} \lambda P_0 &= \mu_1 P_1 + \mu_2 P_2 \\ \mu_1 P_1 &= p\lambda P_0 \\ \mu_2 P_2 &= (1-p)\lambda P_0 \end{aligned}$$

Solving in terms of P_0 yields

$$P_1 = \frac{p\lambda}{\mu_1} P_0, P_2 = \frac{(1-p)\lambda}{\mu_2} P_0$$

which implies, since $P_0 + P_1 + P_2 = 1$, that

$$P_0 \left[1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2} \right] = 1$$

or

$$P_0 = \frac{\mu_1\mu_2}{\mu_1\mu_2 + p\lambda\mu_2 + (1-p)\lambda\mu_1},$$

which is the the proportion of time the machine is up. The proportion of time the machine is down due to type 1 failure is

$$P_1 = \frac{p\lambda\mu_2}{\mu_1\mu_2 + p\lambda\mu_2 + (1-p)\lambda\mu_1}.$$

The proportion of time the machine is down due to type 2 failure is

$$P_2 = \frac{(1-p)\lambda\mu_1}{\mu_1\mu_2 + p\lambda\mu_2 + (1-p)\lambda\mu_1}$$

Exercise 18. Let state 0 be the case that the machine is working, and state i be that it is in repair phase i, $i = 1, 2, ..., k$. The balance equations for limiting probabilities are

$$\begin{aligned} \lambda P_0 &= \mu_k P_k \\ \mu_1 P_1 &= \lambda P_0 \\ \mu_i P_i &= \mu_{i-1} P_{i-1},\ i = 2, ..., k \\ P_0 + P_1 + \cdots + P_k &= 1. \end{aligned}$$

To solve it, note that

$$\mu_i P_i = \mu_{i-1} P_{i-1} = \cdots = \mu_1 P_1 = \lambda P_0.$$

Hence

$$P_i = \frac{\lambda}{\mu_i} P_0,$$

and

$$1 = P_0 + P_1 + \cdots + P_k = P_0 \left[1 + \frac{\lambda}{\mu_1} + \cdots + \frac{\lambda}{\mu_k} \right].$$

Thus,

$$P_0 = \frac{1}{1 + \frac{\lambda}{\mu_1} + \cdots + \frac{\lambda}{\mu_k}}, \ P_i = \frac{\frac{\lambda}{\mu_i}}{1 + \frac{\lambda}{\mu_1} + \cdots + \frac{\lambda}{\mu_k}}.$$

The answer to part (a) is P_i and the answer to part (b) is P_0.

Exercise 21. We have a birth-death process with

$$\lambda_1 = \lambda_2 = \lambda, \ \ \mu_1 = \mu, \ \mu_2 = 2\mu.$$

Thus,

$$\begin{aligned} P_1 &= \frac{\lambda}{\mu} P_0 \\ P_2 &= \frac{\lambda}{2\mu} P_1 \\ P_0 + P_1 + P_2 &= 1. \end{aligned}$$

Solving it yields

$$P_0 = \frac{1}{1 + (\lambda/\mu) + \frac{\lambda^2}{2\mu^2}}, \ P_1 = \frac{\lambda}{\mu} P_0 = \frac{\lambda/\mu}{1 + (\lambda/\mu) + \frac{\lambda^2}{2\mu^2}}.$$

So the desired probability is

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + (\lambda/\mu) + \frac{\lambda^2}{2\mu^2}}.$$

Exercise 23. Let $X(t)$ denote the number of machines that are down at time t, then $\{X(t), t \geq 0\}$ is a birth and death process with state space $\{0, 1, 2, 3\}$ and rates

$$\lambda_0 = \frac{3}{10}, \lambda_1 = \frac{2}{10}, \lambda_2 = \frac{1}{10}$$
$$\mu_1 = \frac{1}{8}, \mu_2 = \mu_3 = \frac{1}{4}.$$

Then the balance equations reduce to

$$P_1 = \frac{3/10}{1/8} P_0 = \frac{12}{5} P_0$$
$$P_2 = \frac{2/10}{1/4} P_1 = \frac{48}{25} P_0$$
$$P_3 = \frac{1/10}{1/4} P_2 = \frac{96}{125} P_0.$$

Using $P_0 + P_1 + P_2 + P_3 = 1$, we obtain

$$P_0 = \frac{125}{761}, P_1 = \frac{300}{761}, P_2 = \frac{240}{761}, P_3 = \frac{96}{761}.$$

(a) The average number of machines not in use is

$$P_1 + 2P_2 + 3P_3 = \frac{300}{761} + 2 \times \frac{240}{761} + 3 \times \frac{96}{761} = \frac{1068}{761}.$$

(b) The proportion of time both repairmen are busy is

$$P_2 + P_3 = \frac{240}{761} + \frac{96}{761} = \frac{336}{761}.$$