E3106, Solutions to Homework 8

Columbia University

Exercise 2. The conditional density is given by

$$f_{s|t_1,t_2}(x|A,B) = P(B(s) = x|B(t_1) = A, B(t_2) = B)$$
$$= \frac{P(B(t_1) = A, B(s) = x, B(t_2) = B)}{P(B(t_1) = A, B(t_2) = B)}.$$

Using the independent increaments, we have that the denominator is

$$\begin{aligned} P(B(t_1) &= A, B(t_2) = B) \\ &= P(B(t_1) = A, B(t_2) - B(t_1) = B - A) \\ &= P(B(t_1) = A)P(B(t_2) - B(t_1) = B - A) \\ &= f_{t_1}(A) f_{t_2 - t_1}(B - A), \end{aligned}$$

where the two normal density functions are

$$\begin{aligned} f_{t_2-t_1}(B - A) &= \frac{1}{\sqrt{2\pi}\sqrt{t_2 - t_1}} \exp\left\{-\frac{(B - A)^2}{2(t_2 - t_1)}\right\}, \\ f_{t_1}(A) &= \frac{1}{\sqrt{2\pi}\sqrt{t_1}} \exp\left\{-\frac{A^2}{2t_1}\right\}. \end{aligned}$$

Similarly, we have the numerator is

$$P(B(t_1) = A, B(s) - B(t_1) = x - A, B(t_2) - B(s) = B - x) = f_{t_1}(A) f_{s-t_1}(x - A) f_{t_2-s}(B - x),$$

where

$$\begin{aligned} f_{t_2-s}(B - x) &= \frac{1}{\sqrt{2\pi}\sqrt{t_2 - s}} \exp\left\{-\frac{(B - x)^2}{2(t_2 - s)}\right\}, \\ f_{s-t_1}(x - A) &= \frac{1}{\sqrt{2\pi}\sqrt{s - t_1}} \exp\left\{-\frac{(x - A)^2}{2(s - t_1)}\right\}. \end{aligned}$$

Thus, putting things together we have

$$\begin{aligned}
& f_{s|t_1,t_2}(x|A,B) \\
= \;& \frac{f_{s-t_1}(x-A) f_{t_2-s}(B-x)}{f_{t_2-t_1}(B-A)} \\
= \;& K_1 \exp\left\{ -\frac{(x-A)^2}{2(s-t_1)} - \frac{(B-x)^2}{2(t_2-s)} + \frac{(B-A)^2}{2(t_2-t_1)} \right\} \\
= \;& K_2 \exp\left\{ -x^2 \left(\frac{1}{2(s-t_1)} + \frac{1}{2(t_2-s)} \right) + x \left(\frac{A}{s-t_1} + \frac{B}{t_2-s} \right) \right\} \\
= \;& K_2 \exp\left\{ -\frac{t_2-t_1}{2(s-t_1)(t_2-s)} \left(x^2 - \frac{A(t_2-s)+B(s-t_1)}{t_2-t_1} 2x \right) \right\} \\
= \;& K_3 \exp\left\{ -\frac{t_2-t_1}{2(s-t_1)(t_2-s)} \left(x - \frac{A(t_2-s)+B(s-t_1)}{t_2-t_1} \right)^2 \right\}
\end{aligned}$$

where K_1, K_2, K_3 do not depend on x.

Hence, the conditional distribution is normal with mean and variance given by

$$E[B(s)|B(t_1)=A, B(t_2)=B] = \frac{A(t_2-s)+B(s-t_1)}{t_2-t_1}$$

$$\mathrm{Var}[B(s)|B(t_1)=A, B(t_2)=B] = \frac{(s-t_1)(t_2-s)}{t_2-t_1}$$

Exercise 9. Since $X(t)$ has independent and stationary increments, it follows that the joint density of $X(s)$ and $X(t)$ is given by

$$\begin{aligned}
& f(x,y) \\
=& P(X(s)=x, X(t)=y) \\
=& P(X(s)=x, X(t)-X(s)=y-x) \\
=& P(X(s)=x) P(X(t)-X(s)=y-x) \\
=& P(X(s)=x) P(X(t-s)=y-x)
\end{aligned}$$

Let $g_s(x)$ denote the probability density function of $X(s)$. Since $X(s)$ is normally distributed with mean μs and variance $\sigma^2 s$, it follows that

$$P(X(s)=x) = \frac{1}{\sigma\sqrt{2\pi s}} \exp\left\{ -\frac{(x-\mu s)^2}{2s\sigma^2} \right\}.$$

Furthermore

$$P(X(t-s)=y-x) = \frac{1}{\sigma\sqrt{2\pi(t-s)}} \exp\left\{ -\frac{(y-x-\mu(t-s))^2}{2(t-s)\sigma^2} \right\}$$

Thus,

$$\begin{aligned}
& f(x,y) \\
=& \frac{1}{2\pi\sigma^2\sqrt{s(t-s)}} \exp\left\{ -\frac{(x-\mu s)^2}{2s\sigma^2} - \frac{(y-x-\mu(t-s))^2}{2(t-s)\sigma^2} \right\}.
\end{aligned}$$

Exercise 16. For any $0 \leq s \leq t$, we have

$$E[Y(t)|\mathcal{F}_s] = Y(s).$$

Taking expectation on both sides we have

$$E[Y(t)] = E[E[Y(t)|\mathcal{F}_s]] = E[Y(s)].$$

Letting $s = 0$ yields

$$E[Y(t)] = E[Y(0)].$$

Exercise 21. Since $B(t)$ is a martingale, by the definition of $X(t)$ we have

$$B(t) = \frac{X(t) - \mu t}{\sigma}$$

is also a martingale. Applying the martingale stopping theorem we have

$$E[B(T)] = B(0) = 0,$$

or

$$E[\frac{X(T) - \mu T}{\sigma}] = 0, \;\; E[X(T) - \mu T] = 0.$$

Since $X(T) = x$, we have

$$E[x - \mu T] = 0, \; E[T] = \frac{x}{\mu}.$$