Exercise 2. The conditional density is given by

\[ f_{s|t_1, t_2}(x|A, B) = \frac{P(B(s) = x|B(t_1) = A, B(t_2) = B) \cdot P(B(t_1) = A, B(t_2) = B)}{P(B(t_1) = A, B(t_2) = B)}. \]

Using the independent increments, we have that the denominator is

\[ P(B(t_1) = A, B(t_2) = B) = P(B(t_1) = A)P(B(t_2) - B(t_1) = B - A) = f_{t_1}(A)f_{t_2-t_1}(B - A), \]

where the two normal density functions are

\[ f_{t_2-t_1}(B - A) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left\{ \frac{-(B - A)^2}{2(t_2 - t_1)} \right\}, \]

\[ f_{t_1}(A) = \frac{1}{\sqrt{2\pi t_1}} \exp\left\{ \frac{-A^2}{2t_1} \right\}. \]

Similarly, we have the numerator is

\[ P(B(t_1) = A, B(s) - B(t_1) = x - A, B(t_2) - B(s) = B - x) = f_{t_1}(A)f_{s-t_1}(x-A)f_{t_2-s}(B-x), \]

where

\[ f_{t_2-s}(B - x) = \frac{1}{\sqrt{2\pi\sqrt{t_2 - s}}} \exp\left\{ \frac{-(B - x)^2}{2(t_2 - s)} \right\}, \]

\[ f_{s-t_1}(x - A) = \frac{1}{\sqrt{2\pi\sqrt{s - t_1}}} \exp\left\{ \frac{-(x - A)^2}{2(s - t_1)} \right\}. \]
Thus, putting things together we have
\[
    f_{s|t_1,t_2}(x|A,B) = \frac{f_{s-t_1}(x-A) f_{t_2-s}(B-x)}{f_{t_2-t_1}(B-A)}
\]
\[
= K_1 \exp \left\{ -\frac{(x-A)^2}{2(s-t_1)} - \frac{(B-x)^2}{2(t_2-s)} + \frac{(B-A)^2}{2(t_2-t_1)} \right\}
\]
\[
= K_2 \exp \left\{ -x^2 \left( \frac{1}{2(s-t_1)} + \frac{1}{2(t_2-s)} \right) + x \left( \frac{A}{s-t_1} + \frac{B}{t_2-s} \right) \right\}
\]
\[
= K_3 \exp \left\{ -\frac{t_2-t_1}{2(s-t_1)(t_2-s)} \left( x - \frac{A(t_2-s) + B(s-t_1)}{t_2-t_1} \right)^2 \right\}
\]
where $K_1, K_2, K_3$ do not depend on $x$.

Hence, the conditional distribution is normal with mean and variance given by
\[
    E[B(s)|B(t_1) = A, B(t_2) = B] = \frac{A(t_2-s) + B(s-t_1)}{t_2-t_1}
\]
\[
    \text{Var}[B(s)|B(t_1) = A, B(t_2) = B] = \frac{(s-t_1)(t_2-s)}{t_2-t_1}
\]

**Exercise 9.** Since $X(t)$ has independent and stationary increments, it follows that the joint density of $X(s)$ and $X(t)$ is given by
\[
    f(x, y)
\]
\[
= P(X(s) = x, X(t) = y)
\]
\[
= P(X(s) = x, X(t) - X(s) = y - x)
\]
\[
= P(X(s) = x) P(X(t) - X(s) = y - x)
\]
\[
= P(X(s) = x) P(X(t-s) = y - x)
\]
Let $g_s(x)$ denote the probability density function of $X(s)$. Since $X(s)$ is normally distributed with mean $\mu_s$ and variance $\sigma^2_s$, it follows that
\[
    P(X(s) = x) = \frac{1}{\sigma_s \sqrt{2\pi}} \exp \left\{ -\frac{(x - \mu_s)^2}{2\sigma_s^2} \right\}.
\]
Furthermore
\[
    P(X(t-s) = y - x) = \frac{1}{\sigma_{t-s} \sqrt{2\pi}} \exp \left\{ -\frac{(y - x - \mu(t-s))^2}{2(t-s)\sigma^2} \right\}
\]
Thus,
\[
    f(x, y)
\]
\[
= \frac{1}{2\pi\sigma^2 \sqrt{s(t-s)}} \exp \left\{ -\frac{(x - \mu_s)^2}{2s\sigma^2} - \frac{(y - x - \mu(t-s))^2}{2(t-s)\sigma^2} \right\}.
\]
Exercise 16. For any $0 \leq s \leq t$, we have

$$E[Y(t)|\mathcal{F}_s] = Y(s).$$

Taking expectation on both sides we have

$$E[Y(t)] = E[E[Y(t)|\mathcal{F}_s]] = E[Y(s)].$$

Letting $s = 0$ yields

$$E[Y(t)] = E[Y(0)].$$

Exercise 21. Since $B(t)$ is a martingale, by the definition of $X(t)$ we have

$$B(t) = \frac{X(t) - \mu t}{\sigma}$$

is also a martingale. Applying the martingale stopping theorem we have

$$E[B(T)] = B(0) = 0,$$

or

$$E\left[\frac{X(T) - \mu T}{\sigma}\right] = 0, \quad E[X(T) - \mu T] = 0.$$  

Since $X(T) = x$, we have

$$E[x - \mu T] = 0, \quad E[T] = \frac{x}{\mu}.$$