E3106, Solutions to Homework 9

Columbia University

Exercise 10.23. Since standard Brownian motion $B(t)$ is a Martingale and T is a stopping time for $B(t)$, it follows from the martingale stopping theorem (Exercise 19) that

$$E(B(T)) = E(B(0)) = 0.$$

Since

$$B(t) = \frac{X(t) - \mu t}{\sigma},$$

it follows that

$$E(X(T) - \mu T) = 0$$

or

$$E(T) = \frac{1}{\mu} E(X(T)). \tag{1}$$

Let p denote the probability that $\{X(t), t \geq 0\}$ hits A before it hits $-B$. By the result of part (b) of Exercise 22, we have

$$\begin{aligned}
1 =& E(\exp\{-2\mu X(T)/\sigma^2\}) \\
=& E(\exp\{-2\mu X(T)/\sigma^2\} | X(t) \text{ hits } A \text{ before } -B)p \\
&+ E(\exp\{-2\mu X(T)/\sigma^2\} | X(t) \text{ hits } -B \text{ before } A)(1-p) \\
=& \exp\{-2\mu A/\sigma^2\}p + \exp\{2\mu B/\sigma^2\}(1-p),
\end{aligned}$$

where the last equality follows from the definition of T. The above equation yields

$$p = \frac{1 - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}. \tag{2}$$

Now, from equation (1) and (2), we obtain

$$\begin{aligned}
E(T) &= \frac{1}{\mu} E(X(T)) \\
=& \frac{1}{\mu}[E(X(T)|X(T) = A)p + E(X(T)|X(T) = -B)(1-p)] \\
=& \frac{1}{\mu}[Ap - B(1-p)] \\
=& \frac{A + B - Ae^{2\mu B/\sigma^2} - Be^{-2\mu A/\sigma^2}}{\mu\left(e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}\right)}.
\end{aligned}$$

Exercise 10.26.

(a). Since $B(1/t)$ has a normal distribution with mean 0 and variance $1/t$, we have

$$\begin{aligned} P(Y(t) &\leq y) = P(tB(1/t) \leq y) = P(B(1/t) \leq \frac{y}{t}) = P\left(\frac{B(1/t)}{\sqrt{1/t}} \leq \frac{\frac{y}{t}}{\sqrt{1/t}}\right) \\ &= \Phi\left(\frac{\frac{y}{t}}{\sqrt{1/t}}\right) = \Phi\left(\frac{y}{\sqrt{t}}\right), \end{aligned}$$

where Φ is the standard normal distribution function. Thus, $Y(t)$ has a normal distribution with mean 0 and variance t.

(b). Since $E[Y(t)] = 0$ and

$$E[B(u)B(v)] = \min(u, v),$$

we have

$$\begin{aligned} Cov(Y(s), Y(t)) &= E[Y(s)Y(t)] - E[Y(s)]E[Y(t)] \\ &= E[Y(s)Y(t)] \\ &= E[sB(1/s)tB(1/t)] \\ &= stE[B(1/s)B(1/t)] \\ &= st\min(\frac{1}{s}, \frac{1}{t}) \\ &= \min(t, s). \end{aligned}$$

(c) Clearly $Y(t) = tB(1/t)$ has a continuous sample path, as B has a continuous sample path. Second, as shown in part (a) the $Y(t)$ has normal distribution with mean 0 and variance t. Third, as shown in part (b) the process $Y(t)$ has the same covariance structure as the standard Brownian motion. Therefore, it also has the independent increments as

$$Cov(Y(s), Y(t)-Y(s)) = Cov(Y(s), Y(t))-Cov(Y(s), Y(s)) = \min(s, t)-s = 0, \quad s < t,$$

and for normal random variables, the fact that the covariance equals to zero means independence. Putting things together, we conclude that $Y(t)$ satisfies the definition of the Brownian motion, and hence $Y(t)$ is the standard Brownian motion.