Exercise 10.23. Since standard Brownian motion $B(t)$ is a Martingale and $T$ is a stopping time for $B(t)$, it follows from the martingale stopping theorem (Exercise 19) that

$$E(B(T)) = E(B(0)) = 0.$$ 

Since

$$B(t) = \frac{X(t) - \mu t}{\sigma},$$

it follows that

$$E(X(T) - \mu T) = 0$$

or

$$E(T) = \frac{1}{\mu}E(X(T)). \quad (1)$$

Let $p$ denote the probability that $\{X(t), t \geq 0\}$ hits $A$ before it hits $-B$. By the result of part $(b)$ of Exercise 22, we have

$$1 = E(\exp\{-2\mu X(T)/\sigma^2\})$$

$$= E(\exp\{-2\mu X(T)/\sigma^2\} | X(t) \text{ hits } A \text{ before } -B)p$$

$$+ E(\exp\{-2\mu X(T)/\sigma^2\} | X(t) \text{ hits } -B \text{ before } A)(1 - p)$$

$$= \exp\{-2\mu A/\sigma^2\}p + \exp\{2\mu B/\sigma^2\}(1 - p),$$

where the last equality follows from the definition of $T$. The above equation yields

$$p = \frac{1 - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}. \quad (2)$$

Now, from equation $(1)$ and $(2)$, we obtain

$$E(T) = \frac{1}{\mu}E(X(T))$$

$$= \frac{1}{\mu}[E(X(T)|X(T) = A)p + E(X(T)|X(T) = -B)(1 - p)]$$

$$= \frac{1}{\mu}[Ap - B(1 - p)]$$

$$= \frac{A + B - Ae^{2\mu B/\sigma^2} - Be^{-2\mu A/\sigma^2}}{\mu \left(e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}\right)}.$$

(a). Since $B(1/t)$ has a normal distribution with mean 0 and variance $1/t$, we have

$$P(Y(t) \leq y) = P(tB(1/t) \leq y) = P(B(1/t) \leq \frac{y}{t}) = P\left(\frac{B(1/t)}{\sqrt{1/t}} \leq \frac{y}{\sqrt{1/t}}\right)$$

$$= \Phi\left(\frac{y}{\sqrt{1/t}}\right) = \Phi\left(\frac{y}{t}\right),$$

where $\Phi$ is the standard normal distribution function. Thus, $Y(t)$ has a normal distribution with mean 0 and variance $t$.

(b). Since $E[Y(t)] = 0$ and

$$E[B(u)B(v)] = \min(u, v),$$

we have

$$Cov(Y(s), Y(t)) = E[Y(s)Y(t)] - E[Y(s)]E[Y(t)]$$

$$= E[Y(s)Y(t)]$$

$$= E[sB(1/s)tB(1/t)]$$

$$= stE[B(1/s)B(1/t)]$$

$$= st \min(\frac{1}{s}, \frac{1}{t})$$

$$= \min(t, s).$$

(c) Clearly $Y(t) = tB(1/t)$ has a continuous sample path, as $B$ has a continuous sample path. Second, as shown in part (a) the $Y(t)$ has normal distribution with mean 0 and variance $t$. Third, as shown in part (b) the process $Y(t)$ has the same covariance structure as the standard Brownian motion. Therefore, it also has the independent increments as

$$Cov(Y(s), Y(t)-Y(s)) = Cov(Y(s), Y(t))-Cov(Y(s), Y(s)) = \min(s, t)-s = 0, \ s < t,$$

and for normal random variables, the fact that the covariance equals to zero means independence. Putting things together, we conclude that $Y(t)$ satisfies the definition of the Brownian motion, and hence $Y(t)$ is the standard Brownian motion.