IEOR 4106, Final Exam, Columbia University
Prof. S. Kou, Dec. 21, 2004

1. (10 pts). (a). Y has a Poisson distribution with parameter λt. $E[Y] = \lambda t$.
(b). X has an exponential distribution with rate λ. $E[X] = 1/\lambda$.

2. (10 pts). (M/M/1 queue). This is a birth-death process with $\lambda_n = \lambda$ and $\mu_n = \mu$. If $\lambda < \mu$, then from the formula sheet.

$$\begin{aligned} P_0 &= \frac{1}{1+\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}} = \frac{1}{1+\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}} \\ &= \frac{1}{1+\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n}} = \frac{1}{\frac{1}{1-\frac{\lambda}{\mu}}} = 1 - \frac{\lambda}{\mu}, \end{aligned}$$

$$P_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} P_0 = \frac{\lambda^n}{\mu^n}\left(1-\frac{\lambda}{\mu}\right), \ n \geq 1.$$

3. (10 pts). Let $X(t)$ denote the number of machines that are down at time t, then $\{X(t), t \geq 0\}$ is a birth and death process with state space $\{0, 1, 2, 3\}$ and rates

$$\begin{aligned} \lambda_0 &= \frac{3}{12}, \lambda_1 = \frac{2}{12}, \lambda_2 = \frac{1}{12} \\ \mu_1 &= \frac{1}{10}, \ \mu_2 = \mu_3 = \frac{2}{10}. \end{aligned}$$

Then the balance equations reduce to

$$\begin{aligned} P_1 &= \frac{\lambda_0}{\mu_1} P_0 = \frac{3/12}{1/10} P_0 = \frac{5}{2} P_0 \\ P_2 &= \frac{\lambda_1}{\mu_2} P_1 = \frac{2/12}{2/10} P_1 = \frac{10}{12} P_1 = \frac{25}{12} P_0 \\ P_3 &= \frac{\lambda_2}{\mu_3} P_2 = \frac{1/12}{2/10} P_2 = \frac{5}{12} P_2 = \frac{125}{144} P_0. \end{aligned}$$

Using $P_0 + P_1 + P_2 + P_3 = 1$, we obtain

$$\left(1 + \frac{5}{2} + \frac{25}{12} + \frac{125}{144}\right) P_0 = 1,$$

$$P_0 = \frac{144}{929}, P_1 = \frac{5}{2}\frac{144}{929} = \frac{360}{929}, P_2 = \frac{25}{12}\frac{144}{929} = \frac{300}{929}, P_3 = \frac{125}{144}\frac{144}{929} = \frac{125}{929}.$$

(a) The average number of machines not in use is

$$P_1 + 2P_2 + 3P_3 = \frac{360}{929} + 2 \times \frac{300}{929} + 3 \times \frac{125}{929} = 1.437.$$

(b) The proportion of time both repairmen are busy is

$$P_2 + P_3 = \frac{300}{929} + \frac{125}{929} = 0.457\,48.$$

4. (10 pts). By conditioning on X_1 of the first renewal:

$$\mathsf{E}(\gamma_t) = \int_0^\infty \mathsf{E}(\gamma_t | X_1 = x)\, dF(x)$$

Since

$$\mathsf{E}(\gamma_t | X_1 = x) = \begin{cases} x - t & \text{if } x > t, \\ \mathsf{E}(\gamma_{t-x}), & \text{if } 0 < x \le t. \end{cases}$$

we have the renewal equation.

$$\mathsf{E}(\gamma_t) = \int_0^t \mathsf{E}(\gamma_{t-x})\, dF(x) + \int_t^\infty (x - t)\, dF(x).$$

5. (10 pts). For every $0 < s < t < \infty$,

$$\mathsf{E}[B_t B_s] = \mathsf{E}[\mathsf{E}[B_t B_s | \mathcal{F}_s]] = \mathsf{E}[B_s \mathsf{E}[B_t | \mathcal{F}_s]] = \mathsf{E}[B_s^2] = s,$$

as $\mathsf{E}[B_t|\mathcal{F}_s] = B_s + \mathsf{E}[B_t - B_s|\mathcal{F}_s] = B_s + \mathsf{E}[B_t - B_s] = B_s$.

6. (10 pts). Let $\tau(m) = \min\{t : B(t) = m\}$. From the formula sheet

$$P(B(T) \le x, \tau(m) \le T) = \Phi(-\frac{2m - x}{\sqrt{T}}), \;\; P(\tau(m) \le T) = 2\Phi(-\frac{m}{\sqrt{T}}).$$

$$\begin{aligned} P(M(T) &\ge m, B(T) \ge x) = P(B(T) \ge x, \tau(m) \le T) \\ &= P(\tau(m) \le T) - P(B(T) \le x, \tau(m) \le T) \\ &= 2\Phi(-\frac{m}{\sqrt{T}}) - \Phi(-\frac{2m - x}{\sqrt{T}}). \end{aligned}$$

7. (10 pts) Since B_t is a martingale, we have by the martingale stopping theorem

$$0 = E[B(0)] = E[B(\tau)] = E[2 - 4\tau].$$

Thus,

$$E[\tau] = \frac{1}{2}.$$

8. (10 pts) Consider the two end notes first (S_u and S_d). If we sell one share of the option, and buy α shares of the stock. Then at time T the setting the payoff at the these two nodes being equal yields

$$\alpha u S_0 - C_u = \alpha d S_0 - C_d,$$

or

$$\alpha = \frac{C_u - C_d}{(u - d)\, S_0},$$

resulting in a riskless profit

$$\begin{aligned}\frac{C_u - C_d}{(u-d)\,S_0} uS_0 - C_u &= \frac{C_u uS_0 - C_d uS_0 - C_u\,(u-d)\,S_0}{(u-d)\,S_0} \\ &= \frac{dC_u - uC_d}{u-d}\end{aligned}$$

To have the same payoff at the middle node S_0, we must have

$$\alpha S_0 - C_0 = \frac{dC_u - uC_d}{u-d},$$

or

$$\frac{C_u - C_d}{(u-d)\,S_0} S_0 - C_0 = \frac{dC_u - uC_d}{u-d},$$

implying

$$-C_d + (1-d)C_u + uC_d = C_0\,(u-d)\,,$$

$$(1-d)C_u - (1-d)C_d + (u-d)C_d = C_0\,(u-d)\,,$$

or in summary

$$\frac{C_u - C_d}{C_0 - C_d} = \frac{u-d}{1-d}.$$