Midterm Solution. Fall 2005.
E3106. Prof. S. Kou, Columbia University

1. The state space of the Markov chain is \( \{(i, j) | i \geq 1, j \geq 1\} \). Suppose that the Markov chain is currently in the state \((i, j)\). Let \( T \) denote the time the Markov chain stays in state \((i, j)\), and let \( T_{kl} \) denote the time until the \( k \)-th male mates with the \( l \)-th female. Since they mate in any time interval of length \( h \) with probability \( \lambda h + o(h) \), it follows that \( T_{kl} \) is exponentially distributed with rate \( \lambda \).

Since
\[
T = \min_{1 \leq k \leq i, 1 \leq l \leq j} T_{kl}
\]
and the \( ij \) random variables \( T_{kl} \) are independent with each other, it follows that \( T \) is exponentially distributed with rate \( ij \lambda \), namely
\[
v(i, j) = ij \lambda.
\]

Since each offspring is equally likely to be male or female, we conclude
\[
P_{(i, j)(i, j + 1)} = P_{(i, j)(i + 1, j)} = \frac{1}{2}.
\]

2. (a). Three. They are \( \{0, 1\} \), \( \{2\} \), and \( \{3\} \). \( \{2\} \) is a transient state. The others are recurrent.
(b).
\[
\begin{align*}
\pi_0 &= 0.7\pi_0 + 0.25\pi_1 \\
\pi_1 &= 0.3\pi_0 + 0.75\pi_1 \\
\pi_0 + \pi_1 &= 1
\end{align*}
\]
This yields \( \pi_0 = \frac{5}{11} \), \( \pi_1 = \frac{6}{11} \).
(c). \( p = 0.10 \times 1 + 0.20 \times 1 + 0.40 \times p + 0.30 \times 0 \). Thus, \( p = 1/2 \).
(d).
\[
\begin{align*}
\lim_{n \to \infty} p^{(n)}_{20} &= p\pi_0 = \frac{1}{2} \times \frac{5}{11} = \frac{5}{22} \\
\lim_{n \to \infty} p^{(n)}_{21} &= p\pi_1 = \frac{1}{2} \times \frac{6}{11} = \frac{6}{22}
\end{align*}
\]
(e).
\[
\begin{align*}
\lim_{n \to \infty} p^{(n)}_{00} &= \pi_0 = \frac{5}{11}, \quad \lim_{n \to \infty} p^{(n)}_{10} = \pi_0 = \frac{5}{11},
\end{align*}
\]
\[ \lim_{n \to \infty} p_{01}^{(n)} = \pi_1 = \frac{6}{11}, \quad \lim_{n \to \infty} p_{11}^{(n)} = \pi_1 = \frac{6}{11}. \]

Since 0 and 1 can only make transitions between 0 and 1, we have
\[ \lim_{n \to \infty} p_{02}^{(n)} = \lim_{n \to \infty} p_{03}^{(n)} = \lim_{n \to \infty} p_{12}^{(n)} = \lim_{n \to \infty} p_{13}^{(n)} = 0. \]

Since 2 is a transient state, we have
\[ \lim_{n \to \infty} p_{22}^{(n)} = 0, \]
and
\[ \lim_{n \to \infty} p_{23}^{(n)} = 1 - \lim_{n \to \infty} p_{20}^{(n)} - \lim_{n \to \infty} p_{21}^{(n)} - \lim_{n \to \infty} p_{22}^{(n)} = 1 - \frac{5}{22} - \frac{6}{22} - 0 = \frac{1}{2}. \]

\[ \lim_{n \to \infty} p_{20}^{(n)} \text{ and } \lim_{n \to \infty} p_{21}^{(n)} \text{ are given in Part (d). Since 3 is an absorbing state, we have} \]
\[ \lim_{n \to \infty} p_{03}^{(n)} = \lim_{n \to \infty} p_{13}^{(n)} = \lim_{n \to \infty} p_{23}^{(n)} = 0, \quad \lim_{n \to \infty} p_{33}^{(n)} = 1. \]

In summary, we have
\[ \lim_{n \to \infty} P^{(n)} = \begin{pmatrix} \frac{5}{11} & \frac{6}{11} & 0 & 0 \\ \frac{5}{11} & \frac{6}{11} & 0 & 0 \\ \frac{5}{22} & \frac{6}{22} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

3. Let \( A \) be the event that machine 1 is till working at time \( t \). Then
\[ P(A) = e^{-\lambda_1 t}, \quad P(A^c) = 1 - e^{-\lambda_1 t}. \]

Furthermore,
\[ P(\text{machine 1 is the first machine to fail}|A^c) = 1. \]

In addition by the memoryless property,
\[ P(\text{machine 1 is the first machine to fail}|A) \]
\[ = P(\text{machine 1 fails before machine 2}) \]
\[ = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \]
In summary, we have

\[ P(\text{machine 1 is the first machine to fail}) = P(\text{machine 1 is the first machine to fail}|A)P(A) \\
+ P(\text{machine 1 is the first machine to fail}|A^c)P(A^c) \\
= \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_1 t} + 1 - e^{-\lambda_1 t}. \]