Midterm Solution. Fall 2005.
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1. The state space of the Markov chain is $\{(i,j)|i \geq 1, j \geq 1\}$. Suppose that the Markov chain is currently in the state (i,j). Let T denote the time the Markov chain stays in state (i,j), and let T_{kl} denote the time until the k-th male mates with the l-th female. Since they mate in any time interval of length h with probability $\lambda h + o(h)$, it follows that T_{kl} is exponentially distributed with rate λ. Since

$$T = \min_{1\leq k\leq i, 1\leq l\leq j} T_{kl}$$

and the ij random variables T_{kl} are independent with each other, it follows that T is exponentially distributed with rate $ij\lambda$, namely

$$v_{(i,j)} = ij\lambda.$$

Since each offspring is equally likely to be male or female, we conclude

$$P_{(i,j)(i,j+1)} = P_{(i,j)(i+1,j)} = \frac{1}{2}.$$

2. (a). Three. They are {0,1}, { 2}, and {3}. {2} is a transient state. The others are recurrent.

(b).

$$\pi_0 = .7\pi_0 + .25\pi_1$$
$$\pi_1 = .3\pi_0 + .75\pi_1$$
$$\pi_0 + \pi_1 = 1.$$

This yields $\pi_0 = \frac{5}{11}$, $\pi_1 = \frac{6}{11}$.

(c). $p = 0.10 * 1 + 0.20 * 1 + 0.40 * p + 0.30 * 0$. Thus, $p = 1/2$.

(d).

$$\lim_{n\to\infty} p_{20}^{(n)} = p\pi_0 = \frac{1}{2} * \frac{5}{11} = \frac{5}{22}$$
$$\lim_{n\to\infty} p_{21}^{(n)} = p\pi_1 = \frac{1}{2} * \frac{6}{11} = \frac{6}{22}.$$

(e).

$$\lim_{n\to\infty} p_{00}^{(n)} = \pi_0 = \frac{5}{11}, \lim_{n\to\infty} p_{10}^{(n)} = \pi_0 = \frac{5}{11},$$

$$\lim_{n\to\infty} p_{01}^{(n)} = \pi_1 = \frac{6}{11}, \ \lim_{n\to\infty} p_{11}^{(n)} = \pi_1 = \frac{6}{11}.$$

Since 0 and 1 can only make transitions between 0 and 1, we have

$$\lim_{n\to\infty} p_{02}^{(n)} = \lim_{n\to\infty} p_{03}^{(n)} = \lim_{n\to\infty} p_{12}^{(n)} = \lim_{n\to\infty} p_{13}^{(n)} = 0$$

Since 2 is a transient state, we have

$$\lim_{n\to\infty} p_{22}^{(n)} = 0,$$

and

$$\lim_{n\to\infty} p_{23}^{(n)} = 1 - \lim_{n\to\infty} p_{20}^{(n)} - \lim_{n\to\infty} p_{21}^{(n)} - \lim_{n\to\infty} p_{22}^{(n)} = 1 - \frac{5}{22} - \frac{6}{22} - 0 = \frac{1}{2}.$$

$\lim_{n\to\infty} p_{20}^{(n)}$ and $\lim_{n\to\infty} p_{21}^{(n)}$ are given in Part (d). Since 3 is an absorbing state, we have

$$\lim_{n\to\infty} p_{03}^{(n)} = \lim_{n\to\infty} p_{13}^{(n)} = \lim_{n\to\infty} p_{23}^{(n)} = 0, \ \lim_{n\to\infty} p_{33}^{(n)} = 1.$$

In summary, we have

$$\lim_{n\to\infty} \mathbf{P}^{(n)} = \begin{pmatrix} 5/11 & 6/11 & 0 & 0 \\ 5/11 & 6/11 & 0 & 0 \\ 5/22 & 6/22 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Let A be the event that machine 1 is till working at time t. Then

$$P(A) = e^{-\lambda_1 t}, \ P(A^c) = 1 - e^{-\lambda_1 t}.$$

Furthermore,

$$P(\text{machine 1 is the first machine to fail}|A^c) = 1.$$

In addition by the memoryless property,

$$\begin{aligned} & P(\text{machine 1 is the first machine to fail}|A) \\ = \ & P(\text{machine 1 fails before machine 2}) \\ = \ & \frac{\lambda_1}{\lambda_1 + \lambda_2}. \end{aligned}$$

In summary, we have

$$
\begin{aligned}
& P(\text{machine 1 is the first machine to fail}) \\
= \; & P(\text{machine 1 is the first machine to fail}|A)P(A) \\
& \quad + P(\text{machine 1 is the first machine to fail}|A^c)P(A^c) \\
= \; & \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-\lambda_1 t} + 1 - e^{-\lambda_1 t}.
\end{aligned}
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