Solutions to In-Class Homework Problems 4, 5, 6.

4. Suppose we have one server and unlimited space for the waiting room. The arrival rate is λ and the service rate is μ. Write down the Kolmogorov backward equations for $P_{ij}(t)$.

Solution: This is a birth-death process with $\lambda_n = \lambda$ and $\mu_n = \mu$. Thus, the equation is

$$\begin{aligned} P'_{0j}(t) &= \lambda[P_{1j}(t) - P_{0j}(t)] \\ P'_{ij}(t) &= \lambda P_{i+1,j}(t) + \mu P_{i-1,j}(t) - (\lambda+\mu)P_{ij}(t), \quad i > 0. \end{aligned}$$

5. Problem 13, Ch. 6.

Solution:

Let $X(t)$ denote the number of customers in the shop at time t, then $\{X(t), t \geq 0\}$ is a birth and death process with state space $\{0, 1, 2\}$ and rates

$$\lambda_0 = \lambda_1 = 3,\ \mu_1 = \mu_2 = 4.$$

The limiting probabilities of the Markov chain satisfy

$$\begin{aligned} &4P_1 = 3P_0 \\ &4P_2 = 3P_1 \\ &P_0 + P_1 + P_2 = 1, \end{aligned}$$

yielding

$$P_0 = \frac{16}{37}, P_1 = \frac{12}{37}, P_2 = \frac{9}{37}.$$

The average number of customers in the shop is

$$P_1 + 2P_2 = \frac{12}{37} + 2 \times \frac{9}{37} = \frac{30}{37}.$$

6. People come to the server according to Poisson process at rate λ per hour, and there is only one server with the service time being exponentially distributed with rate μ. The waiting room can have as many as people as we want. Compute P_n, the steady state probability that there are exactly n customers in the system, where $n = 0, 1, 2, ...$

Solution 1. This is a birth-death process with $\lambda_n = \lambda$ and $\mu_n = \mu$. Thus, equation in the notes gives

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}} = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n}} = \frac{1}{1 + \frac{\lambda/\mu}{1-\frac{\lambda}{\mu}}} = \frac{1 - \frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu} + \lambda/\mu} = 1 - \frac{\lambda}{\mu},$$

and

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} P_0 = \frac{\lambda^n}{\mu^n} P_0 = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right).$$

Solution 2: This is a birth-death process with $\lambda_n = \lambda$ and $\mu_n = \mu$. Thus, the balance equations are

$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ \lambda P_1 &= \mu P_2 \\ &\cdots \\ \lambda P_n &= \mu P_{n+1} \\ &\cdots \end{aligned}$$

Thus,

$$P_1 = \frac{\lambda}{\mu} P_0,\ P_2 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0,\ ..., P_n = \left(\frac{\lambda}{\mu}\right)^n P_0, ...$$

Therefore,

$$P_0 = (1 + \frac{\lambda}{\mu} + \cdots + \left(\frac{\lambda}{\mu}\right)^n + \cdots)^{-1} = \left(\frac{1}{1 - \frac{\lambda}{\mu}}\right)^{-1} = 1 - \frac{\lambda}{\mu},$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right).$$