E3106. Prof. Kou, Fall 2005.

Solutions to In-Class Homework Problems 7, 8, 9.

7. Let $Y(t) = \sigma B(t) + \mu t$. Show that the process $A(t)$ is a martingale, where $A(t) = \exp\{Y(t) - \mu t - \frac{1}{2}\sigma^2 t\}$.

Solution:

$$\begin{aligned} E[A(t)|F_s] &= E[\exp\{Y(t) - \mu t - \frac{1}{2}\sigma^2 t\}|F_s] \\ &= E[\exp\{Y(t) - Y(s) - \mu t - \frac{1}{2}\sigma^2 t\}e^{Y(s)}|F_s] \\ &= e^{Y(s)} e^{-\mu t - \frac{1}{2}\sigma^2 t} E[\exp\{Y(t) - Y(s)\}|F_s] \\ &= e^{Y(s)} e^{-\mu t - \frac{1}{2}\sigma^2 t} E[\exp\{Y(t) - Y(s)\}] \\ &= e^{Y(s)} e^{-\mu t - \frac{1}{2}\sigma^2 t} e^{\mu(t-s)+\sigma^2(t-s)/2} \\ &= e^{Y(s)} e^{-\mu s - s\sigma^2/2} = A(s). \end{aligned}$$

Thus, $A(t)$ is a martingale.

8. Consider a European call option, which is written on a stock whose current value is \$8 at time 0, with the strike price \$9 and expiration date in one month. Assume for simplicity that one month later, when the option can be exercised, the stock price can either appreciate to \$10 or depreciate to \$6. Assume that the risk-free interest rate is zero. What is the price of the call option at time 0?

Solution: We know that the option price is give by

$$C(0) = \frac{1}{R}\left(C_u \frac{R-d}{u-d} + C_d \frac{u-R}{u-d}\right).$$

In our case, $C_u = (10-9)^+ = 1$, $C_d = (6-9)^+ = 0$, $R = 1$, $u = \frac{10}{8}$, $d = \frac{6}{8}$. Thus

$$C(0) = \frac{1}{1}\left(1\frac{1-\frac{6}{8}}{\frac{10}{8}-\frac{6}{8}} + 0\frac{\frac{10}{8}-1}{\frac{10}{8}-\frac{6}{8}}\right) = 0.5.$$

9. Is it true that $N(t) \geq n+1$ if and only if $S_n < t$?

Solution: No. As a counterexample, suppose $S_n = t$ and $S_{n+1} = t$. Then $N(t) \geq n+1$, but $S_n = t$.