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ON THE PRICING OF CONTINGENT CLAIMS UNDER CONSTRAINTS

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We discuss the problem of pricing contingent claims, such as European call options, based on the fundamental principle of "absence of arbitrage" and in the presence of constraints on portfolio choice, for example, incomplete markets and markets with short-selling constraints. Under such constraints, we show that there exists an *arbitrage-free interval* which contains the celebrated Black–Scholes price (corresponding to the unconstrained case); no price in the interior of this interval permits arbitrage, but every price outside the interval does. In the case of convex constraints, the endpoints of this interval are characterized in terms of auxiliary stochastic control problems, in the manner of Cvitanić and Karatzas. These characterizations lead to explicit computations, or bounds, in several interesting cases. Furthermore, a unique *fair price* $\hat{p}$ is selected inside this interval, based on utility maximization and "marginal rate of substitution" principles. Again, characterizations are provided for $\hat{p}$, and these lead to very explicit computations. All these results are also extended to treat the problem of pricing contingent claims in the presence of a higher interest rate for borrowing. In the special case of a European call option in a market with constant coefficients, the endpoints of the arbitrage-free interval are the Black–Scholes prices corresponding to the two different interest rates, and the fair price coincides with that of Barron and Jensen.

1. Introduction and summary. The famous Black and Scholes (1973) formula provides the unique price of a European contingent claim in an ideal, complete and unconstrained market, as laid out in Sections 2 and 3 of the present paper, based on the fundamental principle of "absence of arbitrage opportunities." In other words, this price is the unique one for which there are no arbitrage opportunities by taking either a short or a long position in the claim and investing wisely in the market. This price coincides with the minimal initial capital, starting with which one can exactly duplicate the claim at the terminal time, and also with the expectation of the claim's discounted value under the unique, "risk-neutral" equivalent probability measure [cf. Merton (1973), Cox and Ross (1976), Cox and Rubinstein (1984), Harrison and Kreps (1979), Harrison and Pliska (1981) and Karatzas (1989); see also Section 4 of this paper for a brief survey].

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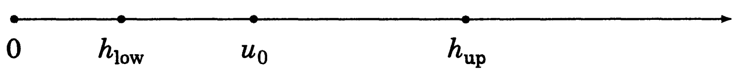
However, in the presence of *constraints* on portfolio choice (e.g., constraints on borrowing, on short-selling of stocks, even on accessing certain stocks at all, as in the case of "incomplete markets"), there ceases to exist a unique price for a contingent claim based solely on the principle of absence of arbitrage. Instead, there appears an "arbitrage-free" *interval* $[h_{\text{low}}, h_{\text{up}}]$ which contains the Black–Scholes price u_0; see the following figure:



Here, h_{up} represents the least price the seller can accept without risk, and h_{low} the greatest price the buyer can afford to pay without risk. This interval has the following properties:

1. Every price level outside the interval leads to an arbitrage opportunity.
2. There are no arbitrage opportunities for price levels in the interior of the interval.

These facts are demonstrated, to our knowledge for the first time, in Section 5 of this paper. Furthermore, if the constraints on portfolio choice are *convex*, it turns out that the endpoints of the arbitrage-free interval can be characterized as the values of certain suitable stochastic control problems, as in Cvitanić and Karatzas (1993), or El Karoui and Quenez (1995) for incomplete markets; see Section 6 and, in particular, Theorem 6.1. Roughly speaking, the upper (resp., lower) endpoint of the interval is equal to the supremum (resp., infimum) of the Black–Scholes prices of the claim over a family of auxiliary markets, which are slightly more complicated in structure but unconstrained.

There remains the question of how to choose a *unique price* for the claim in the presence of constraints on portfolio choice. There seems to be no definitive answer to this question, although several approaches have been suggested—most of them in the context of incomplete markets [e.g., Föllmer and Sondermann (1986), Foldes (1990), Föllmer and Schweizer (1991), Duffie and Skiadas (1991), Davis (1994), etc.] and some in different but related contexts [different interest rates for borrowing and saving, Barron and Jensen (1990); transaction costs, Hodges and Neuberger (1989)]. We adopt in Section 7 the approach of Davis (1994), which is based on utility maximization and on the principle of "zero marginal rate of substitution."

These considerations lead to the notion of a *fair price* $\hat{p}$ (Definition 7.3), which, under certain mild conditions (cf. Assumptions 7.1 and 7.2), is shown to lie within the arbitrage-free interval (Theorem 7.1). Counterexamples for which the fair price lies outside the arbitrage-free interval are also given in Section 8.3. In the special case of convex constraints, we show that the fair price admits a Black–Scholes representation under a certain "minimal" or "least-favorable" equivalent probability measure (Theorem 7.4). In the derivation of this latter result, we draw on the powerful results of Cvitanić and Karatzas (1992) for utility maximization under convex portfolio constraints [cf. Karatzas, Lehoczky, Shreve and Xu (1991) for the special case of incomplete markets]. The representation of Theorem 7.4 leads to *explicit computations* of the fair price $\hat{p}$ (Examples 7.1–7.4) for rather general portfolio constraints, including incomplete markets, short-selling or borrowing constraints, and so on. In particular, it is shown that $\hat{p}$ is independent of both initial wealth and utility function in a market with deterministic coefficients and in the presence of cone constraints on portfolios and, in this case, the corresponding equivalent martingale measure is also obtained by means of relative entropy minimization.

Section 8 offers a host of explicit computations for h_{low}, h_{up} and $\hat{p}$ in the special but important case of a European call option, for a market with constant coefficients and under various kinds of constraints; these computations are tabulated in Section 10, and constitute one of the main results of this paper. Explicit computations are also possible for a path-dependent (or "look-back") option; see Example 7.4.

A most interesting result, from a practical point of view, is that the same ideas and techniques can also treat the problem of pricing contingent claims in a market with *higher* interest rate for borrowing than for saving. More precisely, it is shown in Section 9 that in this case there also exists an arbitrage-free interval and a fair price $\hat{p}$ which always lies within that interval. In the special case of European call option in a market with constant coefficients, the endpoints of the arbitrage-free interval are the two Black–Scholes prices corresponding to the two different interest rates, and the fair price $\hat{p}$ coincides with the so-called minimax price in Barron and Jensen (1990) if a power-type utility function is employed.

2. The financial market model. In this paper we shall deal exclusively with a financial market $\mathscr{M}$ in which $d+1$ assets (or "securities") can be traded continuously. One of them is a nonrisky asset, called the *bond* (also frequently called "savings account"), with price $P_0(t)$ given by

$$dP_0(t) = P_0(t)r(t)\,dt, \qquad P_0(0) = 1. \tag{2.1}$$

The remaining d assets are risky; we shall refer to them as *stocks* and assume that the price $P_i(t)$ per share of the ith stock is governed by the linear stochastic differential equation

$$\begin{gathered} dP_i(t) = P_i(t)\left[b_i(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)\right], \\ P_i(0) = p_i, \ i = 1, 2, \ldots, d. \end{gathered} \tag{2.2}$$

In this model, $W(t) = (W_1(t), \ldots, W_d(t))^*$ is a standard Brownian motion in $\mathscr{R}^d$, whose components represent the external, independent sources of uncertainty in the market $\mathscr{M}$; with this interpretation, the *volatility coefficient* $\sigma_{ij}(\cdot)$ in (2.2) models the instantaneous intensity with which the jth source of uncertainty influences the price of the ith stock.

As is standard in the literature, $\mathscr{M}$ is assumed to be an *ideal market*; in other words, we have infinitely divisible assets, no constraints on consumption and no transaction costs or taxes. We shall allow, however, for *constraints on portfolio choice*, such as limitations on borrowing (from the savings account) or on short-selling (of stocks), and so on; see the examples in Section 6.

The probabilistic setting will be as follows: the Brownian motion W will be defined on a complete probability space $(\Omega, \mathscr{F}, \mathsf{P})$, and we shall denote by $\{\mathscr{F}_t\}$ the P-augmentation of the natural filtration $\mathscr{F}_t^W = \sigma(W(s);\ 0 \le s \le t)$. The coefficients of $\mathscr{M}$, that is, the *interest rate* process $r(t)$, the *appreciation rate* vector process $b(t) = (b_1(t), \dots, b_d(t))^*$ of the stocks and the *volatility* matrix-valued process $\sigma(t) = \{\sigma_{ij}(t)\}_{1 \le i,\, j \le d}$, will all be assumed to be progressively measurable with respect to $\{\mathscr{F}_t\}$ and *bounded uniformly* in $(t, \omega) \in [0, T] \times \Omega$. We shall also impose that the following strong nondegeneracy condition on the matrix $a(t) \triangleq \sigma(t)\sigma^*(t)$,

$$\xi^* a(t)\xi \ge \varepsilon \parallel \xi \parallel^2 \qquad \forall\ (t, \xi) \in [0, T] \times \mathscr{R}^d, \tag{2.3}$$

holds almost surely for a given real constant $\varepsilon > 0$. All processes encountered throughout the paper will be defined on the fixed, finite horizon $[0, T]$, and adapted to the filtration $\{\mathscr{F}_t\}$. We shall introduce also the "relative risk" process

$$\theta(t) \triangleq \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}], \tag{2.4}$$

where $\mathbf{1} = (1, 1, \dots, 1)^*$. The exponential martingale

$$Z_0(t) \triangleq \exp\left\{-\int_0^t \theta^*(s)\, dW(s) - \frac{1}{2}\int_0^t \parallel \theta(s) \parallel^2\, ds\right\}, \tag{2.5}$$

the discount process

$$\gamma_0(t) \triangleq \exp\left\{-\int_0^t r(s)\, ds\right\} \tag{2.6}$$

and the Brownian motion with drift

$$W_0(t) \triangleq W(t) + \int_0^t \theta(s)\, ds, \qquad 0 \le t \le T \tag{2.7}$$

will be employed quite frequently.

REMARK 2.1. It is a straightforward consequence of the strong nondegeneracy condition (2.3) that the matrices $\sigma(t)$, $\sigma^*(t)$ are invertible and that the norms of $(\sigma(t))^{-1}$ and $(\sigma^*(t))^{-1}$ are bounded above and below by δ and $1/\delta$, respectively, for some $\delta \in (1, \infty)$; compare with Karatzas and Shreve [(1991), page 372]. The boundedness of $b(\cdot)$, $r(\cdot)$ and $(\sigma(\cdot))^{-1}$ implies that of $\theta(\cdot)$; therefore, the process $Z_0(\cdot)$ of (2.5) is indeed a martingale and not just a local martingale.

3. Portfolio, consumption and wealth processes. Consider now a small economic agent, whose actions cannot affect market prices and who can decide, at any time $t \in [0, T]$, (1) how many shares of the bond $\phi_0(t)$ and how many shares of stocks $(\phi_1(t), \phi_2(t), \dots, \phi_d(t))^*$ to hold and (2) what amount of money $C(t+h) - C(t) \geq 0$ to withdraw for consumption during the interval $(t, t+h]$, $h > 0$. Of course, all these decisions can only be based on the current information $\mathscr{F}_t$, without anticipation of the future. More precisely, we have the following definitions.

DEFINITION 3.1. A *trading strategy* in the market $\mathscr{M}$ is a progressively measurable vector process $(\phi_0(t), \phi_1(t), \dots, \phi_d(t))$ such that $\int_0^T \phi_i^2(t)\,dt < \infty$, $0 \leq i \leq d$, almost surely.

The processes ϕ_0 and ϕ_i represent the number of shares of the bond and the ith stock, respectively, $1 \leq i \leq d$, which are held or shorted at any given time t. A short position in the bond (resp., the ith stock), that is, $\phi_0 < 0$ (resp., $\phi_i < 0$), should be thought of as a *loan*.

DEFINITION 3.2. A *cumulative consumption process* is a nonnegative progressively measurable process $\{C(t), 0 \leq t \leq T\}$ with paths on $(0, T]$ which are increasing, right continuous with left limits (RCLL) and with $C(0) = 0$, $C(T) < \infty$ a.s.

A basic assumption in the market $\mathscr{M}$ is that trading and consumption strategies should satisfy the so-called self-financing condition

$$\sum_{i=0}^{d} \phi_i(t)P_i(t) = \sum_{i=0}^{d} \phi_i(0)P_i(0) + \sum_{i=0}^{d} \int_0^t \phi_i(u)\,dP_i(u) - C(t), \qquad 0 \leq t \leq T, \tag{3.1}$$

almost surely. The meaning of the equation is that, starting with an initial amount $x = \phi_0(0) + \sum_{i=1}^{d} \phi_i(0)p_i$ of wealth, all changes in wealth are due to capital gains (appreciation of stocks and interest from the bond) minus the amount consumed.

For both economic and mathematical considerations, it is useful to introduce wealth and portfolio processes.

DEFINITION 3.3. A *portfolio process* is a progressively measurable process $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_d(\cdot))$: $[0, T] \times \Omega \to \mathscr{R}^d$.

DEFINITION 3.4. For a given initial capital x, a portfolio process $\pi(\cdot)$ as in Definition 3.3 and a cumulative consumption process $C(\cdot)$ as in Definition 3.1, consider the *wealth equation*

$$
\begin{aligned}
dX(t) &= X(t)\Big[1-\sum_{i=1}^{d}\pi_i(t)\Big]\frac{dP_0(t)}{P_0(t)}+\sum_{i=1}^{d}X(t)\pi_i(t)\frac{dP_i(t)}{P_i(t)}-dC(t)\\
&= X(t)\Big[1-\sum_{i=1}^{d}\pi_i(t)\Big]r(t)\,dt\\
&\quad+\sum_{i=1}^{d}X(t)\pi_i(t)\Big[b_i(t)\,dt+\sum_{j=1}^{d}\sigma_{ij}(t)\,dW_j(t)\Big]-dC(t),\\
&= X(t)r(t)\,dt+X(t)\pi^*(t)\sigma(t)\,dW_0(t)-dC(t),\qquad X(0)=x,
\end{aligned}\tag{3.2}
$$

or equivalently

$$
\begin{aligned}
\gamma_0(t)X(t) &= x-\int_0^t\gamma_0(s)\,dC(s)\\
&\quad+\int_0^t\gamma_0(s)X(s)\pi^*(s)\sigma(s)\,dW_0(s),\qquad 0\le t\le T,
\end{aligned}\tag{3.3}
$$

in the notation of (2.1), (2.2) and (2.5)–(2.7). If this equation has a unique solution $X(\cdot)\equiv X^{x,\pi,C}(\cdot)$, this is then called the *wealth process* corresponding to the triple (x,π,C).

The interpretation here is that the components of $\pi(\cdot)$ represent the *proportions* of the wealth $X(\cdot)$ which are invested in the respective stocks $i=1,\dots,d$.

REMARK 3.1. In the setup of Definition 3.4, notice that for the stochastic integral to be well defined we must have $\int_0^T X^2(t)\|\pi(t)\|^2\,dt<\infty$, a.s. Furthermore, if we define

$$
\phi_i(t)=\begin{cases} X(t)\pi_i(t)/P_i(t), & i=1,\dots,d,\\ X(t)\big(1-\sum_{j=1}^{d}\pi_j(t)\big)/P_0(t), & i=0,\end{cases}\qquad \text{for } 0\le t\le T,
$$

then $\phi(\cdot)=(\phi_0(\cdot),\phi_1(\cdot),\dots,\phi_d(\cdot))^*$ constitutes a trading strategy in the sense of Definition 3.1 and we have

$$
X(t)=\sum_{i=0}^{d}\phi_i(t)P_i(t),\qquad 0\le t\le T,\tag{3.4}
$$

as well as the self-financing condition (3.1), which follows then from the wealth equation (3.2). Notice that the wealth process $X(\cdot)$ can clearly take both positive and negative values.

Equation (3.3) leads us to consider the process

$$
\begin{aligned}
N_0(t) &\triangleq \gamma_0(t)X(t) + \int_0^t \gamma_0(s)\,dC(s) \\
&= x + \int_0^t \gamma_0(s)X(s)\pi^*(s)\sigma(s)\,dW_0(s), \qquad 0 \le t \le T,
\end{aligned}
\tag{3.5}
$$

which is seen to be a continuous local martingale under the so-called risk-neutral probability measure (or "equivalent martingale measure")

$$
\mathsf{P}^0(A) \triangleq \mathsf{E}[Z_0(T)1_A], \qquad A \in \mathscr{F}_T, \tag{3.6}
$$

in the notation of (2.5).

DEFINITION 3.5. A portfolio–consumption process pair (π, C) is called *admissible* for the initial capital $x \in \mathscr{R}$, and we write $(\pi, C) \in \mathscr{A}(x)$, if the following statements hold:

(i) The pair $\pi(\cdot), C(\cdot)$ obeys the conditions of Definitions 3.2–3.4.
(ii) The solution $X^{x,\pi,C}(\cdot) \equiv X(\cdot)$ of (3.2) satisfies, almost surely,

$$
X^{x,\pi,C}(t) \ge -\Lambda \qquad \forall\, 0 \le t \le T. \tag{3.7}
$$

Here, Λ is a nonnegative random variable with $\mathsf{E}^0(\Lambda^p) < \infty$ for some $p > 1$.

The admissibility requirements in Definition 3.5 are imposed in order to prevent pathologies like *doubling strategies* [cf. Harrison and Pliska (1981) and Karatzas and Shreve (1997)]; such strategies achieve arbitrarily large levels of wealth at $t = T$, but require $X(\cdot)$ to be unbounded from below on $[0, T]$.

If $(\pi, C) \in \mathscr{A}(x)$, the P^0-local martingale $N_0(\cdot)$ of (3.5) is also bounded uniformly from below and is thus a P^0-supermartingale. Consequently,

$$
\mathsf{E}^0\left[\gamma_0(T)X^{x,\pi,C}(T) + \int_0^T \gamma_0(t)\,dC(t)\right] \le x \qquad \forall\, (\pi, C) \in \mathscr{A}(x). \tag{3.8}
$$

Here E^0 denotes the expectation operator corresponding to the probability measure P^0 of (3.6); under this measure the process $W_0(\cdot)$ of (2.7) is standard Brownian motion, by the Girsanov theorem [e.g., Karatzas and Shreve (1991), Section 3.5], and the discounted stock processes $\gamma_0(\cdot)P_i(\cdot)$ are martingales, since

$$
dP_i(t) = P_i(t)\left[r(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_0^{(j)}(t)\right], \qquad P_i(0) = p_i,\ i = 1, \dots, d, \tag{3.9}
$$

from (2.2) and (2.7), where $W_0^{(j)}$ is the jth component of W_0.

REMARK 3.2. For any $x \in \mathscr{R}$ and $(\pi, C) \in \mathscr{A}(x)$, let $F = X^{x,\pi,C}(T)$. Then for any $a \neq 0$, we have $X^{ax,\pi,aC}(\cdot) = a \cdot X^{x,\pi,C}(\cdot)$ from (3.2). In particular:

(i) If $a > 0$, $(\pi, aC) \in \mathscr{A}(ax)$ and $X^{ax,\pi,aC}(T) = aF$ a.s.
(ii) If $a = -1$ and $C(\cdot) \equiv 0$, $X^{-x,\pi,0}(T) = -F$.

4. Contingent claims and arbitrage in the unconstrained market. The dynamics of the market $\mathscr{M}$ become more interesting once we introduce *contingent claims* such as options. Suppose, in particular, that at time $t = 0$ we sign a contract which gives us the right (but not the obligation, whence the term *option*) to buy, at the specified time T ("expiration date"), one share of the stock $i = 1$ at a specified price q ("exercise price"). At expiration $t = T$, if the price $P_1(T, \omega)$ of the share is below the exercise price, the contract is worthless to us; on the other hand, if $P_1(T, \omega) > q$, we can exercise our option at time $t = T$, which means to buy one share of the stock at the exercise price q and then sell the share immediately in the market for $P_1(T, \omega)$. In other words, this contract entitles its holder to a payment of $B(T) \equiv B(T, \omega) = (P_1(T, \omega) - q)^+$ at time $t = T$; it is called a *European call option*, in contradistinction to an "American call option" that can be exercised at any stopping time (with values) in $[0, T]$. See Myneni (1992) for a survey on the pricing of American options with unconstrained portfolios. In this paper we shall deal primarily with the pricing problem under constraints on portfolio choice and confine ourselves to European options; the similar problem for American options will be treated elsewhere.

The following definition generalizes the concept of European call option.

DEFINITION 4.1. A *European contingent claim* (ECC) is a financial instrument consisting of a payment $B(T)$ at maturity time T; here, $B(T)$ is a nonnegative, $\mathscr{F}_T$-measurable random variable with $\mathsf{E}[(B(T))^{1+\varepsilon}] < \infty$ for some $\varepsilon > 0$.

We shall denote the price at time $t = 0$ of the ECC by $B(0)$. The main purpose of this paper is to find out what $B(0)$ should be in the market $\mathscr{M}$; in other words, how much an agent should charge for selling such a contractual obligation and how much another agent could afford to pay for it.

It turns out that the answer depends on the structure of the market $\mathscr{M}$. In this section, we consider the simplest case: that of a complete, unconstrained market, that is, one in which every asset can be traded and unlimited short-selling of both the bond and stocks is also permitted (subject to the admissibility requirements of Definition 3.5). More precisely, $\pi_i(\cdot)$ takes values in $\mathscr{R}$ for each $1 \le i \le d$. In this case the answer to the pricing problem is well known. A standard approach to this problem is to utilize the concept of *arbitrage* in the market $\mathscr{M}$ with the ECC, denoted by $(\mathscr{M}, B)$ for short, with B standing for the pair $(B(0),\ B(T))$.

DEFINITION 4.2. There is an *arbitrage opportunity* in $(\mathscr{M}, B)$ if there exist an initial wealth $x \ge 0$ (respectively, $x \le 0$), an admissible pair $(\pi, C) \in \mathscr{A}(x)$ and a constant $a = -1$ (respectively, $a = 1$), such that

$$x + aB(0) = X^{x,\pi,C}(0) + aB(0) < 0$$

at time $t = 0$ and

$$X^{x,\pi,C}(T) + aB(T) \ge 0 \quad \text{a.s.}$$

at time $t = T$. The values $a = \pm 1$ indicate long or short positions in the ECC, respectively.

This definition of arbitrage is standard in the literature; see, for example, Duffie [(1992), Chapter 6] and Myneni (1992). Such an arbitrage opportunity represents a *riskless* source of generating profit, strictly greater than the profit from the bond, by the combination of a trading–consumption strategy and the ECC. Furthermore, from the scaling properties in Remark 3.2, we know then that the profits from such a scheme are limitless. Such opportunities should *not* exist in a well-behaved, rational market.

One of the most interesting "classical" results on option pricing is that by only excluding such arbitrage opportunities, the price of the ECC can be uniquely determined, namely, as

$$u_0 \triangleq \mathsf{E}^0[\gamma_0(T)B(T)] = \mathsf{E}[\gamma_0(T)B(T)Z_0(T)]. \tag{4.1}$$

More precisely, if the ECC has a price $B(0) > u_0$ at time $t = 0$, then there is an arbitrage opportunity involving a trading–consumption strategy $(\phi_0, \dots, \phi_d, 0)^*$ and a short position in the ECC. Conversely, for any ECC having price $B(0) < u_0$, there is also an arbitrage opportunity using exactly $(-\phi_1, \dots, -\phi_d, 0)^*$ and taking a long position in the ECC. Hence the price for the ECC has to be u_0 if no arbitrage is allowed in $\mathscr{M}$. This price is called the *arbitrage-free price*, also known as the *Black–Scholes price*. Furthermore, corresponding to the Black–Scholes price u_0, there is a "hedging portfolio" process $\pi(\cdot)$ [hence also a corresponding trading process $\phi(\cdot)$] and a consumption process $C(\cdot) \equiv 0$, such that

$$X^{u_0, \pi, 0}(T) = B(T). \tag{4.2}$$

With the same portfolio $\pi(\cdot)$ [hence the opposite trading strategy $-\phi(\cdot)$], we have

$$X^{-u_0, \pi, 0}(T) = -B(T). \tag{4.3}$$

REMARK 4.1. We have $u_0 < \infty$ in (4.1); indeed, with $c < \infty$ denoting a common upper bound on $||\theta(\cdot)||$ and $|r(\cdot)|$, and with $p = 1 + \varepsilon$, $1/p + 1/q = 1$,

$$\begin{aligned} u_0 &\le \exp(-cT)(\mathsf{E}(B(T))^p)^{1/p}(\mathsf{E}(Z_0(T))^q)^{1/q} \\ &\le \exp(-cT + (q-1)c^2T/2)(\mathsf{E}(B(T))^p)^{1/p} < \infty. \end{aligned}$$

If $\mathscr{M}$ is a market with *constant coefficients* b, r, σ in (2.1) and (2.2), then explicit calculations are possible for u_0 of (4.1) in the following cases.

EXAMPLE 4.1. *European call option*, $B(T) = (P_1(T) - q)^+$. Then

$$u_0 = p\Phi(\mu_+(T, p)) - qe^{-rT}\Phi(\mu_-(T, p)), \qquad p = P_1(0), \tag{4.4}$$

where

$$\mu_{\pm}(t,p) \triangleq \frac{1}{\sigma\sqrt{t}}\left[\log\left(\frac{p}{q}\right) + \left(r \pm \left(\frac{\sigma^2}{2}\right)\right)t\right] \quad \text{and}$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left(\frac{-u^2}{2}\right)du \tag{4.5}$$

is the cumulative standard normal distribution function; we have set $\sigma = \sigma_{11} > 0$. Furthermore, the portfolio process in (4.2) and (4.3) satisfies

$$\pi_1(t) > 1 \quad \text{and} \quad \pi_i(t) = 0, \qquad 2 \le i \le d \quad \text{a.s.} \tag{4.6}$$

We refer the reader to Harrison and Pliska (1981), Cox and Rubinstein (1984), Karatzas (1989), Karatzas and Shreve (1997) or Duffie (1992) for details.

EXAMPLE 4.2. *Path-dependent* ("look-back") *option*, $B(T) = \max_{0\le t\le T} P_1(t)$ of Goldman, Sosin and Gatto (1979). Then the price of (4.1) is given by

$$u_0 = pe^{-rT}\int_0^\infty f(T,\xi;\rho)e^{\sigma\xi}\,d\xi, \qquad p = P_1(0),$$

where $\sigma = \sigma_{11} > 0$, $\rho \triangleq r/\sigma - \sigma/2$ and

$$f(t,\xi;\rho) \triangleq 1 - \Phi\left(\frac{\xi - \rho t}{\sqrt{t}}\right) + e^{2\xi\rho}\left[1 - \Phi\left(\frac{\xi + \rho t}{\sqrt{t}}\right)\right]. \tag{4.7}$$

Further, the portfolio $\pi(\cdot)$ of (4.2) and (4.3) is given by $\pi_i(t) \equiv 0$, $i = 2,\dots,d$, and

$$\pi_1(t) = \frac{e^{\sigma Y(t)}f(T-t, Y(t);\rho) + \sigma\int_{Y(t)}^\infty f(T-t,\xi;\rho)e^{\sigma\xi}\,d\xi}{e^{\sigma Y(t)} + \sigma\int_{Y(t)}^\infty f(T-t,\xi;\rho)e^{\sigma\xi}\,d\xi} \tag{4.8}$$

for $0 \le t \le T$, where

$$Y(t) \triangleq \max_{0\le s\le t}(W_0(s) + \rho s) - (W_0(t) + \rho t) = \frac{1}{\sigma}\log\left(\frac{\max_{0\le s\le t} P_1(s)}{P_1(t)}\right).$$

We refer the reader to Karatzas and Shreve [(1997), Section 2.4] for the details.

A main drawback in the above classical argument is its dependence on the assumptions of completeness and unconstrainedness for the market $\mathcal{M}$. More to the point, as we have seen in the above discussion, it is critical to be able to use $-\phi$ as a trading strategy, if ϕ is permitted in the market, and to trade in all $(d+1)$ assets if necessary. However, if we are in a constrained market, for instance, a market in which short-selling of stocks is prohibited [i.e., with $\phi_i(\cdot) \ge 0$ for each $i = 1,\dots,d$], then the admissibility of the strategy $(\phi_0,\dots,\phi_d)^*$ does not imply that of $(-\phi_0,\dots,-\phi_d)^*$. Furthermore, in an incomplete market, not all the assets are accessible.

A general arbitrage argument is needed to cover these cases as well as the classical unconstrained case.

5. Upper and lower arbitrage prices. Let us introduce further constraints on portfolio choice, in addition to those of Definition 3.5. Suppose that we are given two nonempty Borel subsets K_+ and K_- of $\mathscr{R}^d$; for any $x \in \mathscr{R}$, we shall consider portfolio–consumption pairs in the class

$$\mathscr{A}'(x) \triangleq \{(\pi, C) \in \mathscr{A}(x)\colon \pi(t) \in K_+ \text{ if } X^{x,\pi,C}(t) > 0, \text{ and } \\ \pi(t) \in K_- \text{ if } X^{x,\pi,C}(t) < 0, \forall\, t \in [0,T) \text{ a.s.}\}. \tag{5.1}$$

In other words, K_+ (resp., K_-) represents our constraint on portfolio choice when the wealth is positive (resp., negative). We shall see examples in Section 6 where such different constraints on portfolio, depending on the sign of the level of wealth, arise quite naturally.

DEFINITION 5.1. Given a European contingent claim $B(T)$ as in Definition 4.1, introduce the *lower hedging class*

$$\mathscr{L} \triangleq \{x \geq 0\colon \exists\, (\check{\pi}, \check{C}) \in \mathscr{A}_-(-x) \text{ such that } X^{-x,\check{\pi},\check{C}}(T) \geq -B(T) \text{ a.s.}\} \tag{5.2}$$

and the *upper hedging class*

$$\mathscr{U} \triangleq \{x \geq 0\colon \exists\, (\hat{\pi}, \hat{C}) \in \mathscr{A}_+(x), \text{ such that } X^{x,\hat{\pi},\hat{C}}(T) \geq B(T) \text{ a.s.}\}. \tag{5.3}$$

Here we have set

$$\mathscr{A}_-(y) \triangleq \{(\check{\pi}, \check{C}) \in \mathscr{A}(y)\colon \check{\pi}(t) \in K_- \text{ and } \\ X^{y,\check{\pi},\check{C}}(t) \leq 0,\ \forall\, 0 \leq t < T \text{ a.s.}\} \quad \text{for } y \leq 0,$$

$$\mathscr{A}_+(z) \triangleq \{(\hat{\pi}, \hat{C}) \in \mathscr{A}(z)\colon \hat{\pi}(t) \in K_+ \text{ and } \\ X^{z,\hat{\pi},\hat{C}}(t) \geq 0,\ \forall\, 0 \leq t < T, \text{ a.s.}\} \quad \text{for } z \geq 0.$$

The elements $(\check{\pi}, \check{C})$ [resp., $(\hat{\pi}, \hat{C})$] in the definitions of the classes $\mathscr{L}$ and $\mathscr{U}$ are called *lower* (resp., *upper*) *hedging strategies* for the ECC.

Clearly, the set $\mathscr{L}$ contains the origin. On the other hand, it is a straightforward consequence of Definition 5.1 that both sets $\mathscr{L}$ and $\mathscr{U}$ are (connected) intervals. More precisely, we have the following result.

PROPOSITION 5.1. *For any $x_1 \in \mathscr{L}$, $0 \leq y_1 \leq x_1$ implies $y_1 \in \mathscr{L}$. Similarly, for any $x_2 \in \mathscr{U}$, $y_2 \geq x_2$ implies $y_2 \in \mathscr{U}$.*

PROOF. Suppose $(\pi_2, C_2) \in \mathscr{A}(x_2)$ satisfies the conditions of (5.3). Then, with $y_2 \geq x_2$, one "just consumes immediately the amount $y_2 - x_2$"; in other words, with $\hat{C}_2(t) = C_2(t) + (y_2 - x_2)\mathbf{1}_{(0,T]}(t)$, we have $X^{y_2,\pi_2,\hat{C}_2}(t) \equiv X^{x_2,\pi_2,C_2}(t)$ for all $0 < t \leq T$, and thus $y_2 \in \mathscr{U}$. A similar argument works for $\mathscr{L}$. □

The purpose of this section is to show that, in the presence of constraints as in (5.1), the Black–Scholes price $u_0 = \mathsf{E}^0[\gamma_0(T)B(T)]$ is replaced by an interval $[h_{\mathrm{low}}, h_{\mathrm{up}}]$ which contains u_0 and is defined by (5.4) below, in the following sense: (i) if $B(0)$, the price of the ECC at time $t = 0$, does not belong to $[h_{\mathrm{low}}, h_{\mathrm{up}}]$, then there exists an arbitrage opportunity (Theorem 5.2); (ii) if it belongs to the interior $(h_{\mathrm{low}}, h_{\mathrm{up}})$ of this interval or if $B(0) = h_{\mathrm{low}} = h_{\mathrm{up}}$, then arbitrage opportunities do not exist (Theorem 5.3 and Corollary 5.1).

DEFINITION 5.2. The *lower arbitrage* and the *upper arbitrage* prices are defined by

$$h_{\mathrm{low}} \triangleq \sup\{x\colon\ x \in \mathscr{L}\}, \qquad h_{\mathrm{up}} \triangleq \inf\{x\colon\ x \in \mathscr{U}\}, \tag{5.4}$$

respectively.

Here we adopt the convention that $\inf\varnothing = +\infty$. In Section 6 we shall provide characterizations of the numbers h_{low}, h_{up} in terms of suitable stochastic control problems, which lead to explicit computation in several interesting special cases (cf. Section 8).

REMARK 5.1. Heuristically, the upper arbitrage price may be viewed as the minimal amount necessary for the seller of the ECC to set aside at time $t = 0$, in order to make sure that he will be able to cover his obligation at time $t = T$. Similarly, the lower arbitrage price can be viewed as the maximal amount that the buyer of the ECC is willing to pay at $t = 0$ and still be sure that he will be able to cover, at time $t = T$, the debt he incurred at $t = 0$ by purchasing the ECC.

This intuition suggests that the lower arbitrage price h_{low} cannot be larger than the upper arbitrage price h_{up}. The following theorem shows, in fact, that for general constraint sets K_+ and K_-, a stronger result holds.

THEOREM 5.1. *We have for any nonempty constraint sets K_+ and K_- in $\mathscr{B}(\mathscr{R}^d)$,*

$$0 \le h_{\mathrm{low}} \le u_0 \le h_{\mathrm{up}},$$

where $u_0 = \mathsf{E}^0[\gamma_0(T)B(T)]$ is the Black–Scholes price of (4.1).

PROOF. By (3.8) and the definition of $\mathscr{U}$, we get

$$x \ge \mathsf{E}^0\left[\gamma_0(T)X^{x,\hat{\pi},\hat{C}}(T) + \int_0^T \gamma(s)\,d\hat{C}(s)\right] \ge \mathsf{E}^0[\gamma_0(T)B(T)] = u_0 \qquad \forall\, x \in \mathscr{U}.$$

Hence, $h_{\mathrm{up}} \ge u_0$. Similarly,

$$\begin{aligned} -y &\ge \mathsf{E}^0\left[\gamma_0(T)X^{-y,\check{\pi},\check{C}}(T) + \int_0^T \gamma_0(s)\,d\check{C}(s)\right] \\ &\ge \mathsf{E}^0[\gamma_0(T)(-B(T))] = -u_0, \qquad \forall\, y \in \mathscr{L}, \end{aligned}$$

whence $y \le u_0$ and $h_{\mathrm{low}} \le u_0$. $\square$

One feature of the above theorem is that it holds for *any* constraint sets; therefore it is applicable to many situations. For instance, in the case of a European call option $B(T) = (P_1(T) - q)^+$ on the first stock, assuming that this stock can be traded, we have $P_1(0) \in \mathscr{U}$ and thus $0 \le h_{\text{low}} \le u_0 \le h_{\text{up}} \le P_1(0) < \infty$.

We define the notion of arbitrage with portfolios constrained as in (5.1), by analogy with Definition 4.2.

DEFINITION 5.3. We say that there exists in $(\mathscr{M}, B)$ an *arbitrage opportunity with constrained portfolios* if there exists an initial wealth $x \ge 0$ (resp., $x \le 0$), an admissible portfolio–consumption process pair (π, C) in the class $\mathscr{A}_+(x)$ [resp., $\mathscr{A}_-(x)$] of Definition 5.1 and a constant $a = -1$ (resp., $a = 1$) such that

$$x + aB(0) = X^{x,\pi,C}(0) + aB(0) < 0 \tag{5.5}$$

and

$$X^{x,\pi,C}(T) + aB(T) \ge 0 \quad \text{a.s.} \tag{5.6}$$

Again, the values $a = \pm 1$ represent long or short positions in the ECC, respectively.

THEOREM 5.2. *For any ECC price $B(0) > h_{\text{up}}$, there exists an arbitrage opportunity in the sense of Definition* 5.3; *similarly for any ECC price $B(0) < h_{\text{low}}$.*

PROOF. Suppose that $B(0) > h_{\text{up}}$. Then for any $x_1 \in (h_{\text{up}}, B(0))$ we know that $x_1 \in \mathscr{U}$, by the definition of h_{up}. Thus there exists a $(\hat{\pi}, \hat{C}) \in \mathscr{A}_+(x_1)$ such that

$$X^{x_1,\hat{\pi},\hat{C}}(0) - B(0) = x_1 - B(0) < 0$$

and

$$X^{x_1,\hat{\pi},\hat{C}}(T) - B(T) \ge B(T) - B(T) = 0 \quad \text{a.s.}$$

Hence (5.5) and (5.6) in Definition 5.3 are satisfied with $a = -1$.

For the case $B(0) < h_{\text{low}}$, there is an arbitrage opportunity which satisfies (5.5) and (5.6) with $a = 1$. The argument is similar to the first one and we omit the details. □

THEOREM 5.3. *For any ECC price $B(0) \notin (\mathscr{U} \cup \mathscr{L})$, there is no arbitrage in $(\mathscr{M}, B)$ with constrained portfolios.*

PROOF. We shall prove this by contradiction. Suppose $B(0) \notin \mathscr{U}$, $B(0) \notin \mathscr{L}$ and that there is an arbitrage opportunity in $(\mathscr{M}, B)$ with constrained portfolios. Two cases may arise.

Case 1. The arbitrage opportunity satisfies (5.5) and (5.6) with $a = -1$. In this case, there exist an initial wealth $x \in [0, \infty)$ and a pair $(\pi_1, C_1) \in \mathscr{A}_+(x)$

such that

$$x = X^{x,\pi_1,C_1}(0) < B(0) \quad \text{and} \\ X^{x,\pi_1,C_1}(T) \ge B(T) \quad \text{a.s.} \tag{5.7}$$

From (5.7) and the definition of $\mathscr{U}$ we know that $x = X^{x,\pi_1,C_1}(0) \in \mathscr{U}$, whence $B(0) \in \mathscr{U}$, thanks to $x < B(0)$ and Proposition 5.1, a contradiction.

Case 2. The arbitrage opportunity satisfies (5.5) and (5.6) with $a = 1$. The proof is similar to that of Case 1, so we omit the details. □

COROLLARY 5.1. *If $h_{\text{low}} < h_{\text{up}}$, then for any price $B(0) \in (h_{\text{low}}, h_{\text{up}})$ of the ECC there is no arbitrage opportunity in $(\mathscr{M}, B)$ with constrained portfolios.*

In view of Theorems 5.2 and Corollary 5.1, the interval $[h_{\text{low}}, h_{\text{up}}]$ is the best possible interval for the ECC price that one can obtain by using only arbitrage arguments. We shall call $[h_{\text{low}}, h_{\text{up}}]$ the *arbitrage-free interval.*

REMARK 5.2. In an unconstrained market, that is, with $K_+ = K_- = \mathscr{R}^d$, we know from the classical results that the Black–Scholes price u_0 belongs to both the lower hedging class $\mathscr{L}$ and the upper hedging class $\mathscr{U}$ [see Chapter 6 in Duffie (1992)]; thus we have $h_{\text{low}} = h_{\text{up}} = u_0$ according to Theorem 5.1.

REMARK 5.3. If the option price is equal to one of the two endpoints h_{low} or h_{up}, it may well be that in some situations there is no arbitrage, while in others there may be an arbitrage opportunity, depending on the consumption process. For example, in the unconstrained case, if $B(0) = h_{\text{up}} = u_0$, there is no arbitrage, as it can be shown that the consumption process for the hedging strategy is almost surely zero (see KS, page 378). On the other hand, if $B(0) = h_{\text{up}}$, $h_{\text{up}} \in \mathscr{U}$ and $\hat{C}(T) > 0$ a.s. (for instance, as in Remark 8.1), then this consumption can be viewed as a kind of arbitrage opportunity.

Within the arbitrage-free interval, a unique fair price might be determined by considerations based on utility maximization or on a stochastic game between the buyer and the seller. An approach using utility maximization, originally due to Davis (1994), is discussed in detail in Section 7.

6. Representations for convex constraints. We shall concentrate in this section on the important special case where the constraint sets K_+ and K_- of (5.1) are nonempty closed, convex sets in $\mathscr{R}^d$. For such sets, we shall obtain in this section representations of h_{low} and h_{up} in terms of auxiliary stochastic control problems [cf. (6.7) and (6.8)], which will lead in turn to explicit computations in Section 8.

We start by introducing the functions

$$\delta(x) \triangleq \sup_{\pi \in K_+} (-\pi^* x): \mathscr{R}^d \mapsto \mathscr{R} \cup \{+\infty\}$$

and

$$\tilde{\delta}(x) \triangleq \inf_{\pi \in K_-} (-\pi^* x): \mathscr{R}^d \mapsto \mathscr{R} \cup \{-\infty\}.$$

In the terminology of convex analysis, $\delta(\cdot)$ and $-\tilde{\delta}(\cdot)$ are the *support functions* of the convex sets $-K_+$ and K_-, respectively; they are closed, positively homogeneous, proper convex functions on $\mathscr{R}^d$ [Rockafellar (1970), page 114]. The support functions $\delta(\cdot)$ and $-\tilde{\delta}(\cdot)$ are finite on their *effective domains* $\tilde{K}_+$ and $\tilde{K}_-$, respectively, where,

$$\begin{aligned}
\tilde{K}_+ &\triangleq \{x \in \mathscr{R}^d;\ \exists\ \beta \in \mathscr{R} \text{ s.t. } -\pi^* x \leq \beta,\ \forall\ \pi \in K_+\} \\
&= \{x \in \mathscr{R}^d;\ \delta(x) < \infty\}, \\
\tilde{K}_- &\triangleq \{x \in \mathscr{R}^d;\ \exists\ \beta \in \mathscr{R} \text{ s.t. } -\pi^* x \geq \beta,\ \forall\ \pi \in K_-\} \\
&= \{x \in \mathscr{R}^d;\ \tilde{\delta}(x) > -\infty\}.
\end{aligned}$$

Notice that both $\tilde{K}_+$ and $\tilde{K}_-$ are convex cones. The following two assumptions will be imposed throughout this section.

ASSUMPTION 6.1. The functions $\delta(\cdot)$ and $\tilde{\delta}(\cdot)$ are continuous on $\tilde{K}_+$ and $\tilde{K}_-$, respectively.

ASSUMPTION 6.2. The function $\delta(\cdot)$ is bounded uniformly from below by some real constant.

These two assumptions are satisfied by all of the examples below. In particular, Theorem 10.2 in Rockafellar [(1970), page 84] guarantees that Assumption 1 is satisfied, if $\tilde{K}_+$ and $\tilde{K}_-$ are locally simplicial, and Assumption 6.2 is satisfied if and only if K_+ contains the origin.

The convex constraints are perhaps among the most important constraints that arise in practice. A few of them are listed below.

EXAMPLE 6.1. All of the following examples satisfy Assumptions 6.1 and 6.2.

(i) *Unconstrained case*: $\phi \in \mathscr{R}^{d+1}$. In other words, $K_+ = K_- = \mathscr{R}^d$. Then $\tilde{K}_+ = \tilde{K}_- = \{0\}$ and $\delta = \tilde{\delta} \equiv 0$ on $\tilde{K}_+$ and $\tilde{K}_-$, respectively.

(ii) *Prohibition of short-selling of stocks*: $\phi_i \geq 0$, $1 \leq i \leq d$. In other words, $K_+ = [0, \infty)^d$ and $K_- = (-\infty, 0]^d$. Then $\tilde{K}_+ = \tilde{K}_- = [0, \infty)^d$ and $\delta \equiv 0$ on $\tilde{K}_+$, $\tilde{\delta} \equiv 0$ on $\tilde{K}_-$.

(iii) *Constraints on the short-selling of stocks.* A generalization of (ii) is $K_+ = [-k, \infty)^d$ for some $k \geq 0$ and $K_- = (-\infty, l]^d$ for some $l \in \mathscr{R}$. Then $\tilde{K}_+ = \tilde{K}_- = [0, \infty)^d$ and $\delta(x) = k \sum_{i=1}^d x_i$, $\tilde{\delta}(x) = -l \sum_{i=1}^d x_i$ on $\tilde{K}_+$ and $\tilde{K}_-$, respectively.

(iv) *Incomplete market, in which only the first m stocks can be traded*: $\phi_i = 0$, $\forall\ i = m+1, \dots, d$, *for some fixed* $m \in \{1, \dots, d-1\}$, $d \geq 2$. In other words, $K_+ = K_- = \{\pi \in \mathscr{R}^d;\ \pi_i = 0,\ \forall\ i = m+1, \dots, d\}$. Then $\tilde{K}_+ = \tilde{K}_- = \{x \in \mathscr{R}^d;\ x_i = 0,\ \forall\ i = 1, \dots, m\}$ and $\delta = \tilde{\delta} \equiv 0$ on $\tilde{K}_+$ and $\tilde{K}_-$.

(v) *Incomplete market, with prohibition of investment in the first m stocks*: $\phi_i = 0,\ 1 \le i \le m$ *for some* $1 \le m \le d$, $d \ge 2$. In other words, $K_+ = K_- = \{\pi \in \mathscr{R}^d;\ \pi_i = 0,\ 1 \le i \le m\}$. Then $\tilde{K}_+ = \tilde{K}_- = \{x \in \mathscr{R}^d;\ x_{m+1} = \cdots = x_d = 0\}$ and $\delta = \tilde{\delta} \equiv 0$ on $\tilde{K}_+$ and $\tilde{K}_-$.

(vi) *Both* K_+ *and* K_- *are closed, convex cones in* $\mathscr{R}^d$. Then $\tilde{K}_+(\tilde{K}_-) = \{x \in \mathscr{R}_d;\ \pi^* x \ge 0,\ \forall\ \pi \in K_+(K_-)\}$ and $\delta\ (\tilde{\delta}) \equiv 0$ on $\tilde{K}_+\ (\tilde{K}_-)$. This clearly generalizes all the previous examples except (iii).

(vii) *Prohibition of borrowing*: $\phi_0 \ge 0$. In other words, $K_+ = \{\pi \in \mathscr{R}^d\colon \sum_{i=1}^d \pi_i \le 1\}$ and $K_- = \{\pi \in \mathscr{R}^d;\ \sum_{i=1}^d \pi_i \ge 1\}$. Then $\tilde{K}_+ = \tilde{K}_- = \{x \in \mathscr{R}^d\colon x_1 = x_2 = \cdots = x_d \le 0\}$ and $\delta(x) = -x_1$ on $\tilde{K}_+$, $\tilde{\delta}(x) = -x_1$ on $\tilde{K}_-$.

(viii) *Constraints on borrowing.* A generalization of (vii) is $K_+ = \{\pi \in \mathscr{R}^d\colon \sum_{i=1}^d \pi_i \le k\}$ for some $k \ge 1$ and $K_- = \{\sum_{i=1}^d \pi_i \ge l\}$ for some $l \in \mathscr{R}$. Then $\tilde{K}_+ = \tilde{K}_- = \{x \in \mathscr{R}^d\colon x_1 = \cdots = x_d \le 0\}$, $\delta(x) = -kx_1$ on $\tilde{K}_+$ and $\tilde{\delta}(x) = -lx_1$ on $\tilde{K}_-$.

Explicit formulae or bounds for h_{low} and h_{up} for all these examples, in the case of a European call option in a market with constant coefficients, will be presented in detail in Section 8. It is interesting to notice that, for all these examples, $\tilde{K}_+$ is equal to $\tilde{K}_-$ (in this connection, see also Proposition 7.2). In general, this will not be the case; see Example 8.8.

The technique to handle such convex constraints is developed in Cvitanić and Karatzas (1993). The basic idea is to introduce a family of auxiliary markets, in which the unconstrained (hedging) problem is relatively easy to solve, and then try to come back to the original market. This basic idea will help us here to give representations for the lower arbitrage price h_{low} and the upper arbitrage price h_{up}, in terms of appropriate stochastic control problems which involve optimization with respect to "parameters" of the auxiliary markets.

In order to introduce these families of auxiliary markets, the notation of Sections 5 and 6 in Cvitanić and Karatzas (1993) will be carried over here for K_+; in addition, we shall consider the analogous notation for K_-. Define the class $\mathscr{H}$ (resp., $\tilde{\mathscr{H}}$) to be the set of progressively measurable processes $\nu = \{\nu(t),\ 0 \le t \le T\}$ with values in $\tilde{K}_+$ (resp., $\tilde{K}_-$), which satisfy $\mathsf{E}\int_0^T (\|\nu(t)\|^2 + \delta(\nu(t)))\,dt < \infty$ [resp., $\mathsf{E}\int_0^T (\|\nu(t)\|^2 + \tilde{\delta}(\nu(t)))\,dt < \infty$]; also introduce, for every $\nu \in \mathscr{H} \cup \tilde{\mathscr{H}}$, the analogues

$$\theta_\nu(t) \triangleq \theta(t) + \sigma^{-1}(t)\nu(t),$$

$$\gamma_\nu(t) \triangleq \exp\left[-\int_0^t \{r(s) + \delta(\nu(s))\}\,ds\right], \tag{6.1}$$

$$\tilde{\gamma}_\nu(t) \triangleq \exp\left[-\int_0^t \{r(s) + \tilde{\delta}(\nu(s))\}\,ds\right],$$

$$Z_\nu(t) \triangleq \exp\left[-\int_0^t \theta_\nu^*(s)\,dW(s) - \tfrac{1}{2}\int_0^t \|\theta_\nu(s)\|^2\,ds\right], \tag{6.2}$$

$$W_\nu(t) \triangleq W(t) + \int_0^t \theta_\nu(s)\,ds, \tag{6.3}$$

of the processes in (2.4)–(2.7), as well as the measure

$$\mathsf{P}^{\nu}(A) \triangleq \mathsf{E}[Z_{\nu}(T)\mathbf{1}_A] = \mathsf{E}^{\nu}[\mathbf{1}_A], \qquad A \in \mathscr{F}_T, \tag{6.4}$$

by analogy with (3.6). Finally, denote by $\mathscr{D}$ (resp., $\tilde{\mathscr{D}}$) the subset consisting of the processes $\nu \in \mathscr{H}$ (resp., $\tilde{\mathscr{H}}$) such that ν is bounded uniformly in $(t, \omega) \in [0, T] \times \Omega$:

$$\sup_{(t,\omega)\in[0,T]\times\Omega} \|\nu(t, \omega)\| < \infty. \tag{6.5}$$

Therefore, for every $\nu \in \mathscr{D} \cup \tilde{\mathscr{D}}$, the exponential local martingale $Z_{\nu}(\cdot)$ of (6.2) is actually a martingale, from which we conclude that the measure P^{ν} of (6.4) is a probability measure and the process $W_{\nu}(\cdot)$ of (6.3) is a P^{ν}-Brownian motion, by the Girsanov theorem. In terms of this new Brownian motion $W_{\nu}(\cdot)$, the stock price (2.2) can be rewritten as

$$dP_i(t) = P_i(t)\left[(r(t) - \nu_i(t))\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_{\nu}^{(j)}(t)\right], \qquad i = 1, \dots, d. \tag{6.6}$$

In the special case of an *incomplete market* [Example 6.1(iv)], this equation shows that the discounted prices $\gamma_0(\cdot)P_i(\cdot)$, $i = 1, \dots, n$, are martingales under every probability measure in the class $\{\mathsf{P}^{\nu}\}_{\nu\in\mathscr{D}}$ of (6.4).

THEOREM 6.1. *With the above notation, we have the following statements*:

(i) *The lower arbitrage price is given by*

$$h_{\text{low}} = \inf_{\nu\in\tilde{\mathscr{D}}} \mathsf{E}^{\nu}[\tilde{\gamma}_{\nu}(T)B(T)] =: g \tag{6.7}$$

provided that the function $\tilde{\delta}(\cdot)$ *is bounded uniformly from below by some real constant.*

(ii) *The upper arbitrage price is given by*

$$h_{\text{up}} = \sup_{\nu\in\mathscr{D}} \mathsf{E}^{\nu}[\gamma_{\nu}(T)B(T)] \tag{6.8}$$

and if the right-hand side of (6.8) *is finite, then* $h_{\text{up}} \in \mathscr{U}$.

In particular, taking $\nu \equiv 0$ in (6.7) and (6.8), we recover the result $0 \le h_{\text{low}} \le u_0 \le h_{\text{up}}$ of Theorem 5.1. For $\nu \in \mathscr{D}$ (resp. $\nu \in \tilde{\mathscr{D}}$), observe that the number $\mathsf{E}^{\nu}[\gamma_{\nu}(T)B(T)]$ (resp. $\mathsf{E}^{\nu}[\tilde{\gamma}_{\nu}(T)B(T)]$) is exactly the Black–Scholes price of the contingent claim in a new auxiliary market with unconstrained portfolios.

Notice that $\tilde{\delta} \ge 0$ in all the cases of Example 6.1, except in (iii) when $l > 0$ and in (viii) when $l < 0$. We shall treat these two cases separately (see Examples 8.1 and 8.2) by employing the definition of h_{low} directly.

The representation (6.8) for h_{up} is proved as in Cvitanić and Karatzas (1993), although a set *bigger* than our $\mathscr{D}$ is used there, so we only need to establish (6.7). As in Cvitanić and Karatzas (1993), the proof uses the martingale representation and Doob–Meyer decomposition theorems and relies on the construction of a submartingale with regular sample paths.

Let us denote by $\mathscr{S}$ the set of all $\{\mathscr{F}_t\}$-stopping times τ with values in $[0, T]$, and by $\mathscr{S}_{\rho,\xi}$ the subset of $\mathscr{S}$ consisting of stopping times τ such that $\rho(\omega) \le \tau(\omega) \le \xi(\omega)$, $\forall\ \omega \in \Omega$, for any two stopping times $\rho \in \mathscr{S}$ and $\xi \in \mathscr{S}$ such that $\rho \le \xi$ a.s. For every $\tau \in \mathscr{S}$, consider also the $\mathscr{F}_\tau$-measurable random variables

$$(6.9)\qquad \tilde{V}(\tau) \triangleq \operatorname{ess\,inf}_{\nu\in\tilde{\mathscr{D}}} \mathsf{E}^\nu\left[B(T)\exp\left\{-\int_\tau^T (r(s)+\tilde{\delta}(\nu(s)))\,ds\right\}\middle|\mathscr{F}_\tau\right]$$

and

$$(6.10)\quad \tilde{Q}_\nu(\tau) \triangleq \tilde{V}(\tau)\exp\left(-\int_0^\tau (r(u)+\tilde{\delta}(\nu(u)))\,du\right) = \tilde{V}(\tau)\tilde{\gamma}_\nu(\tau), \qquad \nu \in \tilde{\mathscr{D}}.$$

LEMMA 6.1. *For every $\nu \in \tilde{\mathscr{D}}$, $\tau \in \mathscr{S}$, $\alpha \in \mathscr{S}_{\tau,T}$ we have the submartingale property*

$$\tilde{Q}_\nu(\tau) \le \mathsf{E}^\nu[\tilde{Q}_\nu(\alpha)|\mathscr{F}_\tau] \quad a.s.$$

LEMMA 6.2. *There exists a RCLL modification $\tilde{V}^+(\cdot)$ of the process $\tilde{V}(\cdot)$. Furthermore, if we define $\tilde{Q}_\nu^+(\cdot)$ by analogy with* (6.10), *then $\{\tilde{Q}_\nu^+(t), \mathscr{F}_t, 0 \le t \le T\}$ is a P^ν-submartingale with RCLL paths.*

The proofs of Lemmas 6.1 and 6.2 are carried out in a manner similar to that of the Appendix in Cvitanić and Karatzas (1993).

LEMMA 6.3. *For the processes $\tilde{V}(\cdot)$ and $\tilde{Q}_\nu(\cdot)$ of* (6.9) *and* (6.10) *we have*

$$(6.11)\qquad \mathsf{E}^0\left[\sup_{0\le t\le T}(\tilde{V}(t))^p\right] < \infty \qquad \forall\ p \in (1, 1+\varepsilon),$$

$$(6.12)\qquad \mathsf{E}^\nu\left[\sup_{0\le t\le T}\tilde{Q}_\nu(t)\right] < \infty \qquad \forall\ \nu \in \tilde{D}.$$

In particular, $\mathsf{E}^\nu[\tilde{\gamma}_\nu(T)B(T)] = \mathsf{E}^\nu(\tilde{Q}_\nu(T)) < \infty\ \forall\ \nu \in \tilde{\mathscr{D}}$.

PROOF. From (6.9) it follows that

$$(6.13)\quad 0 \le \tilde{V}(t) \le \mathsf{E}^0\left[B(T)\exp\left(-\int_t^T r(s)\,ds\right)\middle|\mathscr{F}_t\right] \le \exp(cT)B(t), \quad 0 \le t \le T,$$

holds almost surely in the notation of Remark 4.1 and with $B(t) \triangleq \mathsf{E}^0[B(T)|\mathscr{F}(t)]$. Now with $1 < p < 1+\varepsilon$, $r = (1+\varepsilon)/p$ and $1/r + 1/s = 1$, we have from the Hölder inequality and the Doob maximal inequality,

$$\begin{aligned}\mathsf{E}^0\left[\sup_{0\le t\le T}(B(t))^p\right] &\le \text{const.}\ \mathsf{E}^0(B(T))^p = \text{const.}\ \mathsf{E}[Z_0(T)(B(T))^p]\\ &\le \text{const.}\ (\mathsf{E}[(B(T))^{pr}])^{1/r}(\mathsf{E}[(Z_0(T))^s])^{1/s}.\end{aligned}$$

Therefore,

$$\mathsf{E}^0\Big[\sup_{0\le t\le T}(B(t))^p\Big] \le \text{const.}\,(\mathsf{E}[(B(T))^{1+\varepsilon}])^{1/r}\exp\Big(\frac{s-1}{2}c^2T\Big) < \infty, \tag{6.14}$$

which proves (6.11) in conjuction with (6.13).

On the other hand, from (6.13), (6.10) and the assumption that $\tilde{\delta}(\cdot)$ is uniformly bounded from below by some real constant, we obtain that

$$\begin{aligned} 0 \le \tilde{Q}_\nu(t) &= \tilde{V}(t)\exp\Big[-\int_0^t (r(s)+\tilde{\delta}(\nu(s)))\,ds\Big] \\ &\le \text{const.}\,B(t), \qquad 0\le t\le T, \end{aligned} \tag{6.15}$$

also holds almost surely. With $1 < p < 1+\varepsilon$ and $1/p + 1/q = 1$ we get then, for any fixed $\nu \in \tilde{D}$,

$$\begin{aligned} \mathsf{E}^\nu\Big[\sup_{0\le t\le T} B(t)\Big] &= \mathsf{E}^0\Big[\frac{Z_\nu(T)}{Z_0(T)}\sup_{0\le t\le T} B(t)\Big] \\ &\le \Big(\mathsf{E}^0\Big[\sup_{0\le t\le T}(B(t))^p\Big]\Big)^{1/p}\Big(\mathsf{E}^0\Big(\frac{Z_\nu(T)}{Z_0(T)}\Big)^q\Big)^{1/q} < \infty. \end{aligned}$$

We have used again the Hölder and Doob inequalities, (6.14), as well as the uniform boundedness of the process $\sigma^{-1}(\cdot)\nu(\cdot)$ in

$$\frac{Z_\nu(t)}{Z_0(t)} = \exp\Big\{-\int_0^t (\sigma^{-1}(s)\nu(s))^*\,dW_0(s) - \frac{1}{2}\int_0^t \|\sigma^{-1}(s)\nu(s)\|^2\,ds\Big\}, \qquad 0\le t\le T.$$

In conjuction with (6.15), this leads then to (6.12). □

Proof of Theorem 6.1. The proof is similar to Cvitanić and Karatzas (1993). From now on we consider only the RCLL modifications of $\tilde{V}$ and $\tilde{Q}_\nu$; hence we can assume that these processes do have RCLL paths.

Part 1. We shall first prove the inequality $h_{\text{low}} \ge g$. This is obvious if $g = 0$ so let us assume, for the remainder of this part of the proof, that $g > 0$ and try to show that $g \in \mathscr{L}$. From Lemma 6.2 and (6.12), $\tilde{Q}_\nu(\cdot)$ is a submartingale of class $\mathscr{D}[0,T]$ under P^ν, for every $\nu \in \tilde{D}$. Thus from the martingale representation theorem [Section 3.4 in Karatzas and Shreve (1991)] and the Doob–Meyer decomposition for submartingales [Section 1.4 in Karatzas and Shreve (1991)], we have for every $\nu \in \tilde{\mathscr{D}}$,

$$\begin{aligned} \tilde{Q}_\nu(t) &= \tilde{V}(0) + M_\nu(t) + A_\nu(t) \\ &= g + \int_0^t \psi_\nu^*(s)\,dW_\nu(s) + A_\nu(t), \qquad 0\le t\le T, \end{aligned} \tag{6.16}$$

where $M_\nu(t) = \int_0^t \psi_\nu^*(s)\,dW_\nu(s)$, $0\le t\le T$, is an $(\{\mathscr{F}_t\}, \mathsf{P}^\nu)$-martingale, $\psi_\nu(\cdot)$ is an $\mathscr{R}^d$-valued, $\{\mathscr{F}_t\}$-progressively measurable and almost surely square integrable process and $A_\nu(\cdot)$ is $\{\mathscr{F}_t\}$-predictable with increasing, RCLL paths and $A_\nu(0) = 0$, $\mathsf{E}^\nu A_\nu(T) < \infty$. Introduce the negative process

$$\check{X}(t) \triangleq -\tilde{V}(t) = -\frac{\tilde{Q}_\nu(t)}{\tilde{\gamma}_\nu(t)}, \qquad 0 \le t \le T, \text{ for every } \nu \in \tilde{\mathscr{D}}. \tag{6.17}$$

Then

$$\check{X}(0) = -\tilde{V}(0) = -g \quad \text{and} \quad \check{X}(T) = -B(T).$$

Hence, in order to show $g \in \mathscr{L}$, it is enough to find an admissible pair $(\check{\pi}, \check{C}) \in \mathscr{A}_-(-g)$ such that $\check{X}(\cdot) = X^{-g, \check{\pi}, \check{C}}(\cdot)$. Recall from (6.11) that $\tilde{V}(\cdot) = -\check{X}(\cdot)$ is dominated by the random variable $\Lambda = \sup_{0 \le t \le T} \tilde{V}(t) \ge 0$ with $\mathsf{E}^0(\Lambda^p) < \infty$ for some $p > 1$.

Let us start by observing that for any $\mu \in \tilde{\mathscr{D}}$, $\nu \in \tilde{\mathscr{D}}$, we have from (6.10)

$$\tilde{Q}_\mu(t) = \tilde{Q}_\nu(t) \exp\left[\int_0^t \tilde{\delta}(\nu(u))\, du - \int_0^t \tilde{\delta}(\mu(u))\, du\right], \qquad 0 \le t \le T.$$

Thus, from the differential form of (6.16) we get

$$\begin{aligned} d\tilde{Q}_\mu(t) &= \exp\left[\int_0^t \tilde{\delta}(\nu(s))\, ds - \int_0^t \tilde{\delta}(\mu(s))\, ds\right] \\ &\quad \times [\tilde{Q}_\nu(t)\{\tilde{\delta}(\nu(t)) - \tilde{\delta}(\mu(t))\}\, dt + \psi_\nu^*(t)\, dW_\nu(t) + dA_\nu(t)] \\ &= \exp\left[\int_0^t \tilde{\delta}(\nu(s))\, ds - \int_0^t \tilde{\delta}(\mu(s))\, ds\right] \\ &\quad \times [-\check{X}(t)\tilde{\gamma}_\nu(t)\{\tilde{\delta}(\nu(t)) - \tilde{\delta}(\mu(t))\}\, dt + dA_\nu(t) \\ &\qquad + \psi_\nu^*(t)\sigma^{-1}(t)(\nu(t) - \mu(t))\, dt + \psi_\nu^*(t)\, dW_\mu(t)], \end{aligned} \tag{6.18}$$

where the last equation comes from the definition of $\check{X}(\cdot)$ and the connection between $W_\mu(\cdot)$ and $W_\nu(\cdot)$ [cf. (6.3)]. Comparing (6.18) with the Doob–Meyer decomposition

$$d\tilde{Q}_\mu(t) = \psi_\mu^*(t)\, dW_\mu(t) + dA_\mu(t), \tag{6.19}$$

we conclude from the uniqueness of this decomposition that

$$\psi_\nu(t) \exp\left[\int_0^t \tilde{\delta}(\nu(s))\, ds\right] = \psi_\mu(t) \exp\left[\int_0^t \tilde{\delta}(\mu(s))\, ds\right], \qquad 0 \le t \le T,$$

so that the process

$$h(t) \triangleq \psi_\nu(t) \exp\left[\int_0^t \tilde{\delta}(\nu(s))\, ds\right], \qquad 0 \le t \le T \tag{6.20}$$

does not depend on ν. We claim that we also have, almost surely,

$$\int_0^T \mathbf{1}_{\{\check{X}(t)=0\}} \|h(t)\|^2\, dt = 0. \tag{6.21}$$

Indeed, consider the nonnegative P^0-submartingale $Q(\cdot) \equiv \tilde{Q}_0(\cdot)$ of (6.16). From the Tanaka–Meyer formula [Meyer (1976), page 365, equations (12.1) and (12.3)] we have

$$Q(t) = g + \int_0^t \mathbf{1}_{\{Q(s-)>0\}}\, dQ(s) + \Lambda(t) + \sum_{0<s\le t} \mathbf{1}_{\{Q(s-)=0\}}\, \Delta Q(s),$$

where $\Lambda(\cdot)$ is the local time of $Q(\cdot)$ at the origin: a continuous increasing process, flat off the set $\{0 \le t \le T;\ Q(t) = 0\}$ a.s. Comparing this expression with (6.16), we obtain that

$$\begin{aligned} M(t) &\triangleq \int_0^t \mathbf{1}_{\{Q(s)=0\}}\, dM_0(s) \\ &= \Lambda(t) + \sum_{0<s\le t} \mathbf{1}_{\{Q(s-)=0\}}\, \Delta Q(s) - \int_0^t \mathbf{1}_{\{Q(s-)=0\}}\, dA_0(s), \qquad 0 \le t \le T, \end{aligned}$$

is a continuous martingale of bounded variation. Thus, its quadratic variation

$$\langle M\rangle(T) = \int_0^T \mathbf{1}_{\{Q(t)=0\}}\, d\langle M_0\rangle(t) = \int_0^T \mathbf{1}_{\{Q(t)=0\}} \|h(t)\|^2\, dt$$

is almost surely equal to zero, and (6.21) follows [recall here that $M_0(t) = \int_0^t \psi_0^*(s)\, dW_0(s) = \int_0^t h^*(s)\, dW_0(s)$ from (6.16) and (6.20)].

Therefore, if we fix an arbitrary $\breve{\pi} \in K_-$ and define

$$\breve{\pi}(t) \triangleq \frac{-1}{\gamma_0(t)\breve{X}(t)} (\sigma^*(t))^{-1} h(t) \mathbf{1}_{\{\breve{X}(t)<0\}} + \breve{\pi}\mathbf{1}_{\{\breve{X}(t)=0\}}, \tag{6.22}$$

we obtain a portfolio process that satisfies almost surely

$$-\gamma_0(t)\breve{X}(t)\breve{\pi}^*(t)\sigma(t) = h^*(t) \quad \text{a.e. on } [0, T].$$

From this and from (6.18)–(6.20), we have

$$\begin{aligned} \exp\!\left[\int_0^t \tilde{\delta}(\nu(s))\, ds - \int_0^t \tilde{\delta}(\mu(s))\, ds\right] & \\ \times [-\breve{X}(t)\tilde{\gamma}_\nu(t)\{\tilde{\delta}(\nu(t)) - \tilde{\delta}(\mu(t))\}\, dt & \\ + dA_\nu(t) + \psi_\nu^*(t)\sigma^{-1}(t)(\nu(t) - \mu(t))\, dt] &= dA_\mu(t), \end{aligned}$$

whence

$$\begin{aligned} \exp\!\left[\int_0^t \tilde{\delta}(\nu(s))\, ds - \int_0^t \tilde{\delta}(\mu(s))\, ds\right] & \\ \times [-\breve{X}(t)\tilde{\gamma}_\nu(t)\{\tilde{\delta}(\nu(t)) + \breve{\pi}^*(t)\nu(t) & \\ - \tilde{\delta}(\mu(t)) - \breve{\pi}^*(t)\mu(t)\}\, dt + dA_\nu(t)] &= dA_\mu(t), \end{aligned}$$

thanks to (6.22). Therefore, the process $\breve{C}(\cdot)$ defined as

$$\breve{C}(t) \triangleq \int_0^t \tilde{\gamma}_\nu^{-1}(s)\, dA_\nu(s) - \int_0^t \breve{X}(s)[\tilde{\delta}(\nu(s)) + \nu^*(s)\breve{\pi}(s)]\, ds, \qquad 0 \le t \le T, \tag{6.23}$$

is independent of $\nu \in \tilde{\mathscr{D}}$. In particular, taking $\nu \equiv 0$, we see that

$$\check{C}(t) = \int_0^t \gamma_0^{-1}(s)\, dA_0(s), \qquad 0 \le t \le T,$$

is an increasing, adapted, RCLL process with $\check{C}(0) = 0$ and $\check{C}(T) < \infty$ almost surely. In other words, $\check{C}(\cdot)$ is a consumption process.

The same argument as on page 664 of Cvitanić and Karatzas (1992) shows then that

$$\tilde{\delta}(\nu(s)) + \nu^*(s)\check{\pi}(s) \le 0, \qquad 0 \le s \le T,$$

holds almost surely, for every $\nu \in \tilde{\mathscr{D}}$. Therefore, the proof in Cvitanić and Karatzas [(1992), pages 782–783], and Theorem 13.1 in Rockafellar [(1970), page 112], give us $\check{\pi}(\cdot) \in K_-$ a.s. Notice that for these arguments to work we need the continuity of $\tilde{\delta}(\cdot)$ as well as the condition that $\tilde{\delta}(\cdot)$ be bounded uniformly from below by some real constant.

Now putting the various pieces together, we obtain

$$\begin{aligned} d(-\check{X}(t)\tilde{\gamma}_\nu(t)) &= d\tilde{Q}_\nu(t) = \psi_\nu^*(t)\, dW_\nu(t) + dA_\nu(t) \\ &= \tilde{\gamma}_\nu(t)\big[d\check{C}(t) + \check{X}(t)[\tilde{\delta}(\nu(t)) + \nu^*(t)\check{\pi}(t)]\, dt \\ &\qquad - \check{X}(t)\check{\pi}^*(t)\sigma(t)\, dW_\nu(t)\big], \end{aligned}$$

so that

$$\begin{aligned} d(\check{X}(t)\tilde{\gamma}_\nu(t)) &= -\tilde{\gamma}_\nu(t)\, d\check{C}(t) - \tilde{\gamma}_\nu(x)\check{X}(t)[\tilde{\delta}(\nu(t)) + \nu^*(t)\check{\pi}(t)]\, dt \\ &\quad + \tilde{\gamma}_\nu(t)\check{X}(t)\check{\pi}^*(t)\sigma(t)\, dW_\nu(t). \end{aligned} \tag{6.24}$$

Taking $\nu \equiv 0$ in (6.24), we obtain the wealth equation (3.2) in the form

$$d(\gamma_0(t)\check{X}(t)) = -\gamma_0(t)\, d\check{C}(t) + \gamma_0(t)\check{X}(t)\check{\pi}^*(t)\sigma(t)\, dW_0(t), \qquad \check{X}(0) = -g,$$

whence $\check{X}(\cdot) = X^{-g, \check{\pi}, \check{C}}(\cdot)$. The proof of $h_{\text{low}} \ge g$ is now complete.

Part 2. Let us consider the proof of the reverse inequality $h_{\text{low}} \le g$. This is obvious if $h_{\text{low}} = 0$, so we assume that $h_{\text{low}} > 0$. Thus we have $\mathscr{L} \neq \varnothing$ in (5.2), and for any $x \in \mathscr{L}$ there exists $(\pi, C) \in \mathscr{A}_-(-x)$ such that $X^{-x, \pi, C}(T) \ge -B(T)$ almost surely. It is easy to see from (3.3) and (6.1) that the analogue of (6.24) holds and thus

$$\begin{aligned} &\tilde{\gamma}_\nu(t)X^{-x, \pi, C}(t) + \int_0^t \tilde{\gamma}_\nu(s)\, dC(s) \\ &\quad + \int_0^t \tilde{\gamma}_\nu(s)X^{-x, \pi, C}(s)[\tilde{\delta}(\nu(s)) + \pi^*(s)\nu(s)]\, ds, \qquad 0 \le t \le T, \end{aligned}$$

is actually a P^ν-local martingale, whence a *supermartingale*. This is because $\tilde{\gamma}_\nu(\cdot)X^{-x, \pi, C}(\cdot)$ is bounded from below by a P^ν-integrable random variable, thanks to (3.7), (6.5) and the Hölder inequality. Therefore,

$$\begin{aligned}-x \geq \mathsf{E}^{\nu}\bigg[&\tilde{\gamma}_{\nu}(T)X^{-x,\pi,C}(T) + \int_0^T \tilde{\gamma}_{\nu}(s)\,dC(s) \\ &+ \int_0^T \tilde{\gamma}_{\nu}(s)X^{-x,\pi,C}(s)(\tilde{\delta}(\nu(s)) + \pi^*(s)\nu(s))\,ds\bigg] \\ \geq \mathsf{E}^{\nu}&[\tilde{\gamma}_{\nu}(T)(-B(T))]\end{aligned}$$

for every $x \in \mathscr{L}$, $\nu \in \tilde{\mathscr{D}}$, or equivalently, $x \leq \mathsf{E}^{\nu}[\tilde{\gamma}_{\nu}(T)B(T)]$, from which $h_{\mathrm{low}} \leq g$ follows. □

7. A fair price. We have seen that if the upper arbitrage price h_{up} is strictly bigger than the lower arbitrage price h_{low}, then the arbitrage argument alone is not enough to determine a unique price for the contingent claim. Several approaches have been proposed to get around this problem in the special case of incomplete markets [as in Example 6.1(iv)]; see, for example, Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), Duffie and Skiadas (1991), Foldes (1990) and Davis (1994). There are also some approaches that have been suggested in different, but related, contexts, such as pricing in the presence of transaction costs [Hodges and Neuberger (1989)] or under different interest rates for borrowing and saving [Barron and Jensen (1990)]. Although perhaps none of these approaches is totally satisfactory, we shall try in this section to generalize one of them, the Davis (1994) approach, to the constrained setup of Section 5.

The purpose of this section is not to solve the problem completely (because it might turn out that, from a practical point of view, the most convenient price to use is still the Black–Scholes price u_0; cf. Remark 11.4 in Section 11), but rather to see when the generalization works and when it does not, and hopefully to focus attention on the study of possible connections between arbitrage and utility maximization.

7.1. *Definition.* Davis's "fair price" is only defined for an agent with positive wealth and involves the concept of utility function. Before presenting the definition of the fair price, we shall briefly recall that of utility function.

DEFINITION 7.1. A function U: $(0, \infty) \to \mathscr{R}$ will be called a utility function if it is strictly increasing, strictly concave, of class C^1 and satisfies

$$U'(0+) \triangleq \lim_{x\downarrow 0} U'(x) = \infty, \qquad U'(\infty) \triangleq \lim_{x\to\infty} U'(x) = 0.$$

We shall denote by $I(\cdot)$ the inverse of the function $U'(\cdot)$. Notice that the function $I(\cdot)$ maps $(0, \infty)$ onto itself and satisfies

$$I(0+) = \infty, \qquad I(\infty) = 0, \qquad U'(I(x)) = x.$$

Consider the following "constrained portfolio" optimization problem,

$$V(x) \triangleq \sup_{(\pi, C)\in \mathscr{A}_+(x)} \mathsf{E}[U(X^{x,\pi,C}(T))], \qquad 0 < x < \infty, \tag{7.1}$$

where one tries to maximize expected terminal utility over portfolio–consumption pairs in the class $\mathscr{A}_+(x)$ of Definition 5.1. Clearly, we have

$$V(x) \ge \mathsf{E}U\left(x \exp\left[\int_0^T r(t)\,dt\right]\right) \ge U(x\exp(r_0 T)) > -\infty,$$

where r_0 is a lower bound on $r(\cdot)$.

ASSUMPTION 7.1. For all $x > 0$, the value $V(x)$ of (7.1) is attainable; in other words,

$$V(x) = \mathsf{E}[U(X_*^x(T))] \quad \text{where } X_*^x(T) \triangleq X^{x,\pi^*,C^*}(T) \tag{7.2}$$

for some $(\pi^*, C^*) \in \mathscr{A}_+(x)$, and we assume that the derivative of $V(\cdot)$ exists and is strictly positive: $V'(\cdot) > 0$ on $(0, \infty)$.

This assumption is satisfied in many interesting cases, in particular with $C^*(\cdot) = 0$. Indeed, it holds for all convex constraint sets K_+, subject to the rather mild Assumptions 6.1 and 6.2; see Section 7.3.

Suppose that at time $t = 0$, the price of the contingent claim is $p = B(0)$ and one diverts an amount δ, $|\delta| < x$, of money into the contingent claim B (i.e, buys δ/p shares of the contingent claim). Then one chooses an optimal portfolio–consumption strategy to achieve maximal expected utility from terminal wealth. Formally, one solves the stochastic control problem

$$W(\delta, p, x) \triangleq \sup_{(\pi, C)\in \mathscr{A}'(x-\delta)} \mathsf{E}U\left(X^{x-\delta,\pi,C}(T) + \frac{\delta}{p}B(T)\right), \qquad |\delta| < x, \tag{7.3}$$

where we formally set $U(x) = -\infty$ for $x < 0$. Notice that $W(0, p, \cdot)$ coincides with the function $V(\cdot)$ of (7.1) for every $p > 0$, and that we can actually take $X^{x-\delta,\pi,C}(T) > 0$ in (7.3) above. If the contingent claim price p is set so that this small diversion of funds has a neutral effect on W, in the sense

$$\frac{\partial W}{\partial \delta}(0, p, x) = 0, \tag{7.4}$$

then we tend to call this p the "fair price" of the contingent claim. Indeed, Davis (1994) uses exactly (7.4) to define the fair price. However, the differentiability of $W(\cdot, p, x)$ is often difficult to check directly. Here we shall use a requirement weaker than differentiability and reminiscent of the notion of "viscosity solutions" as in Crandall and Lions (1983).

DEFINITION 7.2. For a given $x > 0$, we call p a *weak solution* of (7.4) if, for every differentiable function $\phi(\cdot, p, x)$ satisfying $\phi(\delta, p, x) \ge W(\delta, p, x)$ for all $\delta \in (-x, x)$ and $\phi(0, p, x) = W(0, p, x) \equiv V(x)$, we have

$$\frac{\partial \phi}{\partial \delta}(0, p, x) = 0.$$

Notice the similarity of this notion with that of "viscosity subsolution" [see, for example, Definition 7.2 in Shreve and Soner (1994) or Fleming and Soner (1993), page 66].

DEFINITION 7.3. Suppose that for any given $x > 0$, the weak solution $p = \hat{p}(x) > 0$ of (7.4) is *unique*. Then we call this $\hat{p}(x)$ the *fair price for the contingent claim* at time $t = 0$, corresponding to initial wealth $x > 0$.

In economic terms, the requirement (7.4) postulates a "zero marginal rate of substitution" for $W(\cdot, \hat{p}(x), x)$ at $\delta = 0$. Generally speaking, Davis's fair price depends on the utility function $U(\cdot)$ and on the particular initial wealth $x > 0$. However, for convex constraint sets K_+ and K_-, we shall present in Section 7.3 conditions under which $\hat{p}(x)$ can be rendered independent of the utility function $U(\cdot)$ and/or the initial wealth $x > 0$.

7.2. *Connections with arbitrage.* An immediate question that we have to settle is whether there exist any arbitrage opportunities in $(\mathscr{M}, B)$ if the contingent claim price $B(0)$ is set to be $\hat{p}(x)$. In other words, whether $\hat{p}(x)$ belongs or not to the interval $[h_{\text{low}}, h_{\text{up}}]$, for every initial wealth $x > 0$. In general, the answer can be affirmative or negative, depending on the constraint sets K_+ and K_- (indeed, several counterexamples are given in Section 8.3); however, if we adopt the fairly general Assumption 7.2 below, then the answer is always affirmative.

ASSUMPTION 7.2. Suppose that $(\pi^{(1)}, C^{(1)}) \in \mathscr{A}'(x)$ and $(\pi^{(2)}, C^{(2)}) \in \mathscr{A}'(y)$, for arbitrary but fixed $x \in \mathscr{R}$, $y \in \mathscr{R}$. Then there exists a $(\pi, C) \in \mathscr{A}'(x + y)$ such that the corresponding terminal wealth $X^{x+y, \pi, C}(T)$ is obtained by superposition:

$$X^{x+y, \pi, C}(T) = X^{x, \pi^{(1)}, C^{(1)}}(T) + X^{y, \pi^{(2)}, C^{(2)}}(T) \quad \text{a.s.}$$

THEOREM 7.1. *Suppose that Assumptions* 7.1 *and* 7.2 *are satisfied and that the fair price* $\hat{p}(x)$ *exists for every* $x > 0$. *Then*

$$\forall\, x > 0, \qquad h_{\text{low}} \le \hat{p}(x) \le h_{\text{up}}. \tag{7.5}$$

The meaning of Assumption 7.2 is that whenever an agent chooses to invest in two different accounts $X_1(\cdot) \equiv X^{x, \pi^{(1)}, C^{(1)}}(\cdot)$ and $X_2(\cdot) \equiv X^{y, \pi^{(1)}, C^{(2)}}(\cdot)$ separately, then this is equivalent, in terms of terminal wealth, to investing and consuming according to some strategy (π, C) which is admissible for the initial wealth level $x + y$, for arbitrary real numbers x and y. This assumption holds, in particular, if the pair

$$\pi = (\pi^{(1)} X_1 + \pi^{(2)} X_2)/(X_1 + X_2), \qquad C = C^{(1)} + C^{(2)}$$

is indeed in $\mathscr{A}'(x+y)$. A sufficient condition for Assumption 7.2 to hold in the case of convex sets $K_{\pm}$ is given along these lines in Proposition 7.1.

PROOF OF THEOREM 7.1. We establish the upper bound first. Suppose that $\hat{p}(x) > 0$ is the fair price of Definition 7.3 for the initial wealth $x > 0$. For arbitrary $y \in \mathscr{U}$, we want to show that $\hat{p}(x) \le y$. Now for any $\delta \in (-x\hat{p}(x)/y, 0) \cap (-x, 0)$, by Remark 3.2 and the definition of the class $\mathscr{U}$ in (5.3), there exists an admissible pair $(\pi^{(1)}, C^{(1)}) \in \mathscr{A}_+(\zeta)$ with $\zeta \triangleq (-\delta y/\hat{p}(x)) \in (0, x)$, such that

$$X^{\zeta, \pi^{(1)}, C^{(1)}}(T) \ge (-\delta/\hat{p}(x))B(T) \tag{7.6}$$

holds almost surely. On the other hand, by Assumption 7.1, there is an admissible pair $(\pi^{(2)}, C^{(2)}) \in \mathscr{A}_+(w)$ which is optimal for the problem of (7.1) with the initial wealth

$$w = x - \delta + \delta y/\hat{p}(x) = x - \delta - \zeta > x - \zeta > 0 \quad \text{(recall that } \delta < 0),$$

that is, the resulting wealth process $X^{w, \pi^{(2)}, C^{(2)}}(\cdot) \ge 0$ satisfies

$$V(w) = V(x - \delta - \zeta) = \mathsf{E}\big[U\big(X^{w, \pi^{(2)}, C^{(2)}}(T)\big)\big]. \tag{7.7}$$

Thus, from Assumption 7.2, we know that there is an admissible pair $(\pi^{(3)}, C^{(3)}) \in \mathscr{A}'(x - \delta)$ such that

$$\begin{aligned} X^{x-\delta, \pi^{(3)}, C^{(3)}}(T) &= X^{\zeta, \pi^{(1)}, C^{(1)}}(T) + X^{w, \pi^{(2)}, C^{(2)}}(T) \\ &\ge X^{w, \pi^{(2)}, C^{(2)}}(T) - \frac{\delta}{\hat{p}(x)} B(T) \end{aligned} \tag{7.8}$$

by (7.6). Hence, by the definition of W in (7.3),

$$\begin{aligned} W(\delta, \hat{p}(x), x) &\ge \mathsf{E}U\big(X^{x-\delta, \pi^{(3)}, C^{(3)}}(T) + (\delta/\hat{p}(x))B(T)\big) \\ &\ge \mathsf{E}U\big(X^{w, \pi^{(2)}, C^{(2)}}(T)\big) = V\big(x - \delta + \delta y/\hat{p}(x)\big) \end{aligned}$$

thanks to (7.8) and (7.7). Therefore, for any function ϕ as in Definition 7.2, we have

$$\begin{aligned} \frac{\phi(\delta, \hat{p}(x), x) - \phi(0, \hat{p}(x), x)}{\delta} &\le \frac{W(\delta, \hat{p}(x), x) - V(x)}{\delta} \\ &\le \frac{V(x - \delta + \delta y/\hat{p}(x)) - V(x)}{\delta}, \end{aligned}$$

since $\delta < 0$, and in the limit, as $\delta \uparrow 0$,

$$0 = \frac{\partial \phi}{\partial \delta}(0, \hat{p}(x), x) \le \left(\frac{y}{\hat{p}(x)} - 1\right) V'(x).$$

Since $V'(x) > 0$ by Assumption 7.1, we obtain $y \ge \hat{p}(x)$, from which the upper bound in (7.5) follows.

Now consider the lower bound. For arbitrary $z \in \mathscr{L}$, we want to show $z \le \hat{p}(x)$. Given any $\delta \in (0, x)$, again by Remark 3.2 and the definition of $\mathscr{L}$, we know that there exists a pair $(\pi^{(4)}, C^{(4)}) \in \mathscr{A}_-(-\xi)$ with $\xi \triangleq \delta z/\hat{p}(x) > 0$ such that

$$X^{-\xi,\pi^{(4)},C^{(4)}}(T) \geq (\delta/\hat{p}(x))(-B(T)) \quad \text{a.s.} \tag{7.9}$$

Also by Assumption 7.1, there exists a pair $(\pi^{(5)}, C^{(5)}) \in \mathscr{A}_+(\eta)$, where $\eta \triangleq x - \delta + \xi = x - \delta + \delta z/\hat{p}(x) > 0$, with corresponding wealth process $X^{\eta,\pi^{(5)},C^{(5)}}(\cdot) \geq 0$ which satisfies

$$V(\eta) = V(x-\delta+\delta z/\hat{p}(x)) = \mathsf{E}\big[U\big(X^{\eta,\pi^{(5)},C^{(5)}}(T)\big)\big]. \tag{7.10}$$

From Assumption 7.2, we know that there exists a pair $(\pi^{(6)}, C^{(6)}) \in \mathscr{A}'(x-\delta)$ such that

$$\begin{aligned} X^{x-\delta,\pi^{(6)},C^{(6)}}(T) &= X^{-\xi,\pi^{(4)},C^{(4)}}(T) + X^{\eta,\pi^{(5)},C^{(5)}}(T) \\ &\geq X^{\eta,\pi^{(5)},C^{(5)}}(T) - \frac{\delta}{\hat{p}(x)}B(T) \quad \text{a.s.} \end{aligned} \tag{7.11}$$

Therefore,

$$\begin{aligned} W(\delta, \hat{p}(x), x) &\geq \mathsf{E}U\big(X^{x-\delta,\pi^{(6)},C^{(6)}}(T) + (\delta/\hat{p}(x))B(T)\big) \\ &\geq \mathsf{E}U\big(X^{\eta,\pi^{(5)},C^{(5)}}(T)\big) = V(x-\delta+\delta z/\hat{p}(x)) \end{aligned}$$

via (7.9), (7.10) and (7.11). Thus, for any function ϕ as in Definition 7.2, we have

$$\frac{\phi(\delta, \hat{p}(x), x) - \phi(0, \hat{p}(x), x)}{\delta} \geq \frac{V(x-\delta+\delta z/\hat{p}(x)) - V(x)}{\delta} \qquad \forall\, \delta \in (0, x),$$

and in the limit, as $\delta \downarrow 0$,

$$0 = \frac{\partial \phi}{\partial \delta}(0, \hat{p}(x), x) \geq \left(\frac{z}{\hat{p}(x)} - 1\right)V'(x).$$

Again, $V'(x) > 0$ leads to the lower bound $z \leq \hat{p}(x)$ of (7.5). □

REMARK 7.1. It is readily seen that the first part of the proof of Theorem 7.1 goes through and thus the upper bound $\hat{p}(x) \leq h_{\text{up}}$ of (7.5) is valid, even in the absence of Assumption 7.2, provided that the set K_+ is convex.

PROPOSITION 7.1. *If the constraint sets K_+ and K_- are convex, then a sufficient condition for the validity of Assumption* 7.2 *is*

$$\forall\, \pi_+ \in K_+,\ \pi_- \in K_-, \qquad \lambda\pi_+ + (1-\lambda)\pi_- \in \begin{cases} K_+, & \text{if } \lambda \geq 1, \\ K_-, & \text{if } \lambda \leq 0. \end{cases} \tag{7.12}$$

PROOF. For $x_i \in \mathscr{R}$ and $(\pi^{(i)}, C^{(i)}) \in \mathscr{A}'(x_i)$, let $X_i(\cdot) \equiv X^{x_i,\pi^{(i)},C^{(i)}}(\cdot)$, $i = 1, 2$, be the corresponding wealth processes and define $C(\cdot) \triangleq C^{(1)}(\cdot) + C^{(2)}(\cdot)$,

$x = x_1 + x_2$, $X(\cdot) \triangleq X_1(\cdot) + X_2(\cdot)$. Then it is not hard to see from the wealth equation (3.2) that $X(\cdot) = X^{x,\pi,C}(\cdot)$, where the portfolio $\pi(\cdot)$ is given by

$$\text{(7.13)} \quad \pi(t) \triangleq [\lambda(t)\pi^{(1)}(t) + (1-\lambda(t))\pi^{(2)}(t)]\mathbf{1}_{(X(t)\neq 0)}, \qquad \lambda(t) = X_1(t)/X(t).$$

To show that $(\pi, C) \in \mathscr{A}'(x)$, we have to check that

$$\text{(7.14)} \qquad \pi(t) \in K_+ \text{ on } \{X(t) > 0\} \quad \text{and} \quad \pi(t) \in K_- \text{ on } \{X(t) < 0\}.$$

Now on $\{X_1(t) > 0, X_2(t) = 0\}$ we have $\pi(t) = \pi^{(1)}(t) \in K_+$ in (7.13); similarly, $\pi(t) = \pi^{(2)}(t) \in K_+$ on $\{X_1(t) = 0, X_2(t) > 0\}$. By analogy, we have $\pi(t) \in K_-$ on $\{X(t) < 0,\ X_1(t)X_2(t) = 0\}$. It remains to see what happens on $\{X_1(t)X_2(t) \neq 0\}$. We distinguish several cases.

(i) $\{X_1(t) > 0,\ X_2(t) > 0\}$: on this event, $\pi^{(i)}(t) \in K_+$ $(i = 1, 2)$ and $0 < \lambda(t) < 1$, so $\pi(t) \in K_+$ by the convexity of K_+;

(ii) $\{X_1(t) < 0,\ X_2(t) < 0\}$: by similar arguments, $\pi(t) \in K_-$;

(iii) $\{X_1(t) > 0 > X_2(t),\ X(t) > 0\}$: then $\pi_1(t) \in K_+$, $\pi_2(t) \in K_-$, $\lambda(t) > 1$ and $\pi(t) \in K_+$ by (7.12);

(iv) $\{X_1(t) > 0 > X_2(t),\ X(t) < 0\}$: here $\lambda(t) < 0$, and (7.12) gives $\pi(t) \in K_-$;

(v) $\{X_2(t) > 0 > X_1(t),\ X(t) > 0\}$ and

(vi) $\{X_2(t) > 0 > X_1(t),\ X(t) < 0\}$ can be treated by analogy with (iii) and (iv).

In all these cases, (7.14) holds. □

Condition (7.12) is satisfied in the context of Examples 6.1, for the cases (i), (ii) and (iii) with $l \leq -k$, (iv), (v) and (vi) with $K_- = -K_+$ and (vii) and (viii) with $l \geq k$. For a discussion of how things can go wrong in (7.5) if the condition (7.12) fails, see the examples of subsection 8.3.

PROPOSITION 7.2. *For any two closed convex sets K_+ and K_- that satisfy (7.12), we have $\tilde{K}_+ = \tilde{K}_-$ and $\delta(\cdot) \leq \tilde{\delta}(\cdot)$ on $\tilde{K}_+$ $(= \tilde{K}_-)$; furthermore, if $K_+ \cap K_- \neq \varnothing$, then $\delta(\cdot) = \tilde{\delta}(\cdot)$ on $\tilde{K}_+$ $(= \tilde{K}_-)$.*

PROOF. Fix an arbitrary $x \in \tilde{K}_+$, so that $\delta(x) < \infty$. For $\lambda > 1$ and arbitrary $\pi_+ \in K_+$, $\pi_- \in K_-$, we have

$$x^*(\lambda\pi_+) + x^*((1-\lambda)\pi_-) = x^*(\lambda\pi_+ + (1-\lambda)\pi_-) \geq \inf_{\pi \in K_+}(x^*\pi) = -\delta(x).$$

Therefore, taking infima and recalling the positive homogeneity properties of $\delta(\cdot)$ and $-\tilde{\delta}(\cdot)$, we get

$$-\lambda\delta(x) + (\lambda - 1)\tilde{\delta}(x) \geq -\delta(x) > -\infty.$$

It follows that $\tilde{\delta}(x) > -\infty$ (thus $\tilde{K}_+ \subseteq \tilde{K}_-$) and in fact $\delta(x) \leq \tilde{\delta}(x)$.

Now fix an arbitrary $x \in \tilde{K}_-$, so that $-\tilde{\delta}(x) < \infty$. For $\lambda < 0$ and arbitrary $\pi_+ \in K_+$ and $\pi_- \in K_-$ we have

$$x^*(\lambda\pi_+) + x^*((1-\lambda)\pi_-) = x^*(\lambda\pi_+ + (1-\lambda)\pi_-) \le \sup_{\pi \in K_-}(x^*\pi) = -\tilde{\delta}(x).$$

Therefore, again by taking suprema and using the same homotheticity properties, we obtain

$$-\lambda\delta(x) - (1-\lambda)\tilde{\delta}(x) \le -\tilde{\delta}(x) < \infty.$$

It follows that $\delta(x) < \infty$ (whence $\tilde{K}_+ \supseteq \tilde{K}_-$) and again $\delta(x) \le \tilde{\delta}(x)$.

The inequality $\tilde{\delta}(x) \le \delta(x)$ on $\mathscr{R}^d$ follows directly from $K_+ \cap K_- \neq \varnothing$. □

REMARK 7.2. If the two closed convex sets K_+ and K_- satisfy the conditions (7.12) and $K_+ \cap K_- \neq \varnothing$, then *the endpoints of the arbitrage-free interval* $[h_{\text{low}}, h_{\text{up}}]$ *are characterized solely in terms of the set* K_+ (recall Theorem 6.1 and the notation of Section 6).

7.3. *A representation for convex constraints.* The following result will be used to obtain the representation (7.25) for the fair price $\hat{p}(x)$. It was established by Davis (1994), but we provide here an alternative argument, based on our Definitions 7.3 and 7.2 for the fair price.

THEOREM 7.2. *Under Assumption* 7.1, *the fair price* $\hat{p}(x)$ *is uniquely determined by*

$$\hat{p}(x) = \frac{\mathsf{E}[U'(X^{x,\pi^*,C^*}(T))B(T)]}{V'(x)} \qquad \forall\, x > 0. \tag{7.15}$$

PROOF. We shall use the inequalities

$$U(x) + (y-x)U'(x) \ge U(y) \ge U(x) + (y-x)U'(y), \qquad \forall\, 0 < x < y < \infty, \tag{7.16}$$

which is a simple consequence of concavity. With the notation of (7.2), we have from the second inequality in (7.16), for $x > \delta > 0$, $p > 0$,

$$\begin{aligned} W(\delta, p, x) &\ge \mathsf{E}\Big[U\Big(X_*^{x-\delta}(T) + \frac{\delta}{p}B(T)\Big)\Big] \\ &\ge \mathsf{E}[U(X_*^{x-\delta}(T))] + \frac{\delta}{p}\mathsf{E}\Big[U'\Big(X_*^{x-\delta}(T) + \frac{\delta}{p}B(T)\Big)B(T)\Big]. \end{aligned}$$

Since $x \mapsto X_*^x(T)$ is nondecreasing, we get

$$W(\delta, p, x) \ge V(x-\delta) + \frac{\delta}{p}\mathsf{E}\Big[U'\Big(X_*^x(T) + \frac{\delta}{p}B(T)\Big)B(T)\Big]. \tag{7.17}$$

Thus, from Fatou's lemma,

$$\begin{aligned}\liminf_{\delta\downarrow 0}&\frac{W(\delta,p,x)-W(0,p,x)}{\delta}\\ &\geq \lim_{\delta\downarrow 0}\frac{V(x-\delta)-V(x)}{\delta}+\frac{1}{p}\liminf_{\delta\downarrow 0}\mathsf{E}\Big[U'\Big(X_*^x(T)+\frac{\delta}{p}B(T)\Big)B(T)\Big]\\ &\geq -V'(x)+\frac{1}{p}\mathsf{E}[U'(X_*^x(T))B(T)].\end{aligned}$$

On the other hand, with $\delta < 0$, $p > 0$, we have, from the first inequality in (7.16), that (7.17) is valid again [with the interpretation $U'(x) \equiv U'(0+) \equiv \infty$ for $x < 0$], and thus by the monotone convergence theorem,

$$\begin{aligned}\limsup_{\delta\uparrow 0}&\frac{W(\delta,p,x)-W(0,p,x)}{\delta}\\ &\leq \lim_{\delta\uparrow 0}\frac{V(x-\delta)-V(x)}{\delta}+\frac{1}{p}\lim_{\delta\uparrow 0}\mathsf{E}\Big[U'\Big(X_*^x(T)+\frac{\delta}{p}B(T)\Big)B(T)\Big]\\ &\leq -V'(x)+\frac{1}{p}\mathsf{E}[U'(X_*^x(T))B(T)].\end{aligned}$$

Therefore, for all $x > 0$ and $p > 0$,

$$\begin{aligned}\limsup_{\delta\uparrow 0}\frac{W(\delta,p,x)-W(0,p,x)}{\delta}&\leq -V'(x)+\frac{1}{p}\mathsf{E}[U'(X_*^x(T))B(T)]\\ &\leq \liminf_{\delta\downarrow 0}\frac{W(\delta,p,x)-W(0,p,x)}{\delta}.\end{aligned}\tag{7.18}$$

Let ϕ denote an arbitrary function as in Definition 7.2; then (7.18) yields

$$\frac{\partial\phi}{\partial\delta}(0,p,x)=-V'(x)+\frac{1}{p}\mathsf{E}[U'(X_*^x(T))B(T)],$$

from which it is easy to check that $\hat{p}(x)$ defined in (7.15) is the *unique* weak solution of (7.4) in the sense of Definition 7.2. □

To give an explicit form of the fair price for convex constraints, we need a result from Cvitanić and Karatzas (1992) along with some additional notation and assumptions. For each $\nu \in \mathcal{D}$, introduce the (continuous, strictly decreasing) function

$$\mathcal{J}_\nu(y) \triangleq \mathsf{E}^\nu[\gamma_\nu(T)I(y\gamma_\nu(T)Z_\nu(T))], \qquad 0 < y < \infty,$$

along with its inverse

$$\mathcal{Y}_\nu(x) \triangleq \mathcal{J}_\nu^{-1}(x), \qquad 0 < x < \infty.$$

Furthermore, let us impose, in addition to the requirements of Definition 7.1, the following conditions on the utility function U:

$$U(\infty) \triangleq \lim_{x\to\infty} U(x) = \infty, \tag{7.19}$$

(7.20) $$U(0) > -\infty \quad \text{or} \quad U(x) = \log x,$$

(7.21) $$x \mapsto xU'(x) \text{ is nondecreasing on } (0, \infty)$$

and

(7.22) for some $\alpha \in (0,1)$, $\gamma \in (1, \infty)$ we have $\alpha U'(x) \geq U'(\gamma x)\ \forall\, x \in (0, \infty)$.

The following result of Cvitanić and Karatzas (1992) describes the terminal wealth corresponding to the optimal pair $(\pi^*, 0) \in \mathscr{A}_+(x)$ for the constrained portfolio optimization problem of (7.1) under conditions (7.19)–(7.22) and shows that these guarantee the validity of Assumption 7.1.

THEOREM 7.3. *Suppose that the constraint set K_+ is closed, convex and satisfies Assumptions* 6.1 *and* 6.2. *Assume also that conditions* (7.19)–(7.22) *hold. Then, for every $x > 0$, there exists a $\hat{\nu} = \hat{\nu}_x \in \mathscr{D}$ and a pair $(\pi^*, 0) \in \mathscr{A}_+(x)$ with corresponding terminal wealth*

(7.23) $$X^{x, \pi^*, 0}(T) = I(\mathscr{Y}_{\hat{\nu}}(x)\gamma_{\hat{\nu}}(T)Z_{\hat{\nu}}(T)) \quad a.s.$$

This pair attains the supremum $V(x)$ of (7.2), *that is, is optimal for the problem of* (7.2). *The value function $V(\cdot)$ is continuously differentiable, and its derivative can be represented as*

(7.24) $$V'(x) = \mathscr{Y}_{\hat{\nu}}(x) > 0 \qquad \forall\, x > 0.$$

The process $\hat{\nu}(\cdot) \in \mathscr{D}$ is optimal in a dual (minimization) stochastic control problem, whence the adjectives "minimal," "dual-optimal" or "least-favorable." Now Theorem 7.3 leads directly to a representation for $\hat{p}(x)$ in the market with convex constraints.

THEOREM 7.4. *We have, for all $x > 0$,*

(7.25) $$\hat{p}(x) = \mathsf{E}^{\hat{\nu}}[\gamma_{\hat{\nu}}(T)B(T)].$$

PROOF. We have for any given $x > 0$,

$$\begin{aligned}\mathsf{E}[U'(X_*^x(T))B(T)] &= \mathsf{E}[U'(X^{x,\pi^*,0}(T))B(T)]\\ &= \mathsf{E}[U'(I(\mathscr{Y}_{\hat{\nu}}(x)\gamma_{\hat{\nu}}(T)Z_{\hat{\nu}}(T)))B(T)] \quad \text{[by (7.23)]}\\ &= \mathsf{E}[\mathscr{Y}_{\hat{\nu}}(x)\gamma_{\hat{\nu}}(T)Z_{\hat{\nu}}(T)B(T)] \quad \text{[using } U'(I(x)) = x]\\ &= V'(x)\mathsf{E}[\gamma_{\hat{\nu}}(T)Z_{\hat{\nu}}(T)B(T)] \quad \text{[by (7.24)]}\\ &= V'(x)\mathsf{E}^{\hat{\nu}}[\gamma_{\hat{\nu}}(T)B(T)].\end{aligned}$$

We can now apply Theorem 7.2 to get (7.25). □

REMARK 7.3. Combining the representation for $\hat{p}(x)$ in Theorem 7.4, the representations for h_{low} and h_{up} in Theorem 6.1, and Proposition 7.2, we recover (7.5): namely, if the two closed convex sets K_+ and K_- satisfy the condition (7.12), then $\hat{p}(x) \in [h_{\text{low}}, h_{\text{up}}]$ for all $x > 0$.

It follows from Theorem 7.4 that $\hat{p}(x)$ is the Black–Scholes price $\mathsf{E}^{\hat{\nu}}[\gamma_{\hat{\nu}}(T)B(T)]$ of $B(T)$ in an auxiliary *unconstrained* market $\mathscr{M}_{\hat{\nu}}$ with interest rate $r(\cdot)+\delta(\hat{\nu}(\cdot))$, appreciation rate vector $b(\cdot)+\hat{\nu}(\cdot)+\delta(\hat{\nu}(\cdot))\mathbf{1}$ and volatility matrix $\sigma(\cdot)$, corresponding to the "minimal" ("dual-optimal") process $\hat{\nu}(\cdot)=\hat{\nu}_x(\cdot)$ of Theorems 7.3 and 7.4. Here are some examples from Cvitanić and Karatzas (1992) in which this process can be computed explicitly.

EXAMPLE 7.1. *Logarithmic utility function, general random adapted coefficients.* If $U(x)=\log x$, then it is shown in Cvitanić and Karatzas (1992), page 790, that $\hat{\nu}(\cdot)$ is given by

$$\hat{\nu}(t)=\arg\min_{y\in\tilde{K}_+}[2\delta(y)+\|\theta(t)+\sigma^{-1}(t)y\|^2],\qquad 0\le t\le T. \tag{7.26}$$

Thus, $\hat{\nu}(\cdot)$ (as well as $\hat{p}$) does not depend on the initial wealth $x\in(0,\infty)$.

In particular, if K_+ is a *cone* [thus $\tilde{\delta}(\cdot)\equiv 0$ on $\tilde{K}_+$], the expression of (7.26) becomes

$$\hat{\nu}(t)=\arg\min_{y\in\tilde{K}_+}\|\theta(t)+\sigma^{-1}(t)y\|^2,\qquad 0\le t\le T; \tag{7.26$'$}$$

this $\hat{\nu}(\cdot)$ also minimizes the *relative entropy*

$$\begin{aligned}H(\mathsf{P}|\mathsf{P}^{\nu})&\triangleq \mathsf{E}\left(\log\frac{d\mathsf{P}}{d\mathsf{P}^{\nu}}\right)=\mathsf{E}(-\log Z_\nu(T))\\&=\mathsf{E}\left[\int_0^T\theta_\nu^*(t)\,dW(t)+\frac{1}{2}\int_0^T\|\theta_\nu(t)\|^2\,dt\right]\\&=\frac{1}{2}\mathsf{E}\int_0^T\|\theta(t)+\sigma^{-1}(t)\nu(t)\|^2\,dt\end{aligned}$$

over $\nu\in\mathscr{D}$, answering a question of John van der Hoek.

Now consider the special case $K_+=\{\pi\in\mathscr{R}^d;\ \pi_i=0,\ \forall\, i=1,\dots,m\}$ of an *incomplete market* as in Example 6.1(v) for some $m=1,\dots,d-1$, $d\ge 2$. Then (7.26$'$) becomes

$$\hat{\nu}(t)=\begin{bmatrix} r(t)\mathbf{1}_m-\hat{b}(t)\\ \mathbf{0}_n\end{bmatrix},\qquad 0\le t\le T,$$

where $\hat{b}(t)=(b_1(t),\dots,b_m(t))^*$ and $n=d-m$; see Karatzas, Lehoczky, Shreve and Xu [(1991), page 721] and Cvitanić and Karatzas (1992), pages 797–798 [as well as Hofmann, Platen and Schweizer (1992), who show that $\mathsf{P}^{\hat{\nu}}$, the "least-favorable" equivalent martingale measure of Karatzas, Lehoczky, Shreve and Xu (1991), coincides in this case with the "minimal equivalent martingale measure" in the sense of Föllmer and Schweizer (1991)].

EXAMPLE 7.2. *Deterministic coefficients, utility function of power-type.* Suppose that the coefficients $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$ of the market $\mathscr{M}$ in (2.1) and (2.2) are nonrandom (deterministic) functions and that the utility function $U(\cdot)$ is of the so-called "power-type"

$$(7.27)\qquad U_\alpha(x) \triangleq \begin{cases} x^\alpha/\alpha, & 0<\alpha<1, \\ \log x = \lim_{\beta\downarrow 0} x^\beta/\beta, & \alpha = 0, \end{cases} \qquad 0<x<\infty.$$

Then it is shown in Cvitanić and Karatzas (1992), page 802, that

$$(7.28)\qquad \hat{\nu}(t) = \arg\min_{y\in\tilde{K}_+}[2(1-\alpha)\delta(y) + ||\theta(t)+\sigma^{-1}(t)y||^2], \qquad 0\le t\le T,$$

is again independent of the initial wealth; the same is thus true of $\hat{p}$.

EXAMPLE 7.3. *Deterministic coefficients, cone constraints.* Suppose again that $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$ are deterministic and that the constraint set K_+ is a (closed, convex) *cone* in $\mathscr{R}^d$ [as in Examples 6.1(i), (ii) and (iv)–(vi)], so that $\tilde{\delta}(\cdot) \equiv 0$ on $\tilde{K}_+$. Then it is shown in Cvitanić and Karatzas (1992), page 801, that, under certain mild conditions on the utility function $U(\cdot)$, the function

$$(7.29)\qquad \hat{\nu}(t) = \arg\min_{y\in\tilde{K}_+} ||\theta(t)+\sigma^{-1}(t)y||^2, \qquad 0\le t\le T,$$

is *independent*, not only of the initial wealth $x>0$, but also of the *utility function* $U(\cdot)$; these same properties are inherited by $\hat{p}$ as well. Notice that $\hat{\nu}(\cdot)$ of (7.29) minimizes not only the relative entropy $H(\mathsf{P}|\mathsf{P}^\nu)$ as in Example 7.1, but also the relative entropy

$$\begin{aligned} H(\mathsf{P}^\nu|\mathsf{P}) &\triangleq \mathsf{E}^\nu\left(\log\frac{d\mathsf{P}^\nu}{d\mathsf{P}}\right) = \mathsf{E}^\nu\left[-\int_0^T \theta_\nu^*(t)\,dW(t) - \frac{1}{2}\int_0^T ||\theta_\nu(t)||^2\,dt\right] \\ &= \mathsf{E}^\nu\left[-\int_0^T \theta_\nu^*(t)\,dW_\nu(t) + \frac{1}{2}\int_0^T ||\theta_\nu(t)||^2\,dt\right] \\ &= \frac{1}{2}\mathsf{E}^\nu\left[\int_0^T ||\theta(t)+\sigma^{-1}(t)\nu(t)||^2\,dt\right] \end{aligned}$$

over $\nu \in \mathscr{D}$.

In any of Examples 7.1–7.3 and with *deterministic* market coefficients $[r(\cdot), b(\cdot), \sigma(\cdot)]$, the process of (7.26), (7.28) or (7.29) is again a *nonrandom* (deterministic) function $\hat{\nu}$: $[0,T]\mapsto \tilde{K}_+$. Suppose, furthermore, that

(7.30) $B(T) = \varphi(P(T))$, where $P(\cdot) = (P_1(\cdot),\dots,P_d(\cdot))^*$ is the vector of stock price processes and $\varphi(p)$: $(0,\infty)^d \mapsto [0,\infty)$ is a continuous function that satisfies polynomial growth conditions in both $||p||$ and $1/||p||$.

Then from (7.25), (7.30), (6.6) and the Feynman–Kac theorem [cf. Karatzas and Shreve (1991), page 366], we see that the fair price for $B(T)$ is given by

$$(7.31)\qquad \hat{p} = \exp\left(-\int_0^T (r(s)+\delta(\hat{\nu}(s)))\,ds\right)Q(0,P(0)).$$

Here $Q(t, p)$: $[0, T] \times (0, \infty)^d \mapsto [0, \infty)$ is the solution of the Cauchy problem for the linear parabolic equation

$$\begin{aligned}(7.32)\qquad &\frac{\partial Q}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(t) p_i p_j \frac{\partial^2 Q}{\partial p_i \partial q_j} \\ &+ \sum_{i=1}^{d}(r(t) - \hat{\nu}_i(t)) p_i \frac{\partial Q}{\partial p_i} = 0, \qquad 0 \le t < T,\end{aligned}$$

subject to the terminal condition

$$(7.33)\qquad Q(T, p) = \varphi(p), \qquad p = (p_1, \ldots, p_d)^* \in (0, \infty)^d,$$

where we recall that the matrix $a(t) = (a_{ij}(t)) = \sigma(t)\sigma^*(t)$ is as in (2.3). The Cauchy problem of (7.32) and (7.33) has a unique classical solution, subject to mild regularity conditions on the coefficients and on the terminal condition φ; see Friedman [(1964), Chapter 1].

REMARK 7.4. In the case of *constant* coefficients $[r(\cdot) = r,\ b(\cdot) = b,\ \sigma(\cdot) = \sigma]$, the formulae (7.31)–(7.33) become

$$(7.34)\qquad \hat{p} = \exp(-(r + \delta(\hat{\nu}))T) Q(0, P(0)),$$

$$(7.35)\qquad Q(T - t, p) = \begin{cases} (2\pi t)^{-d/2} \int_{\mathscr{R}^d} \varphi(h(t, p, \sigma z)) \exp(-\|z\|^2/2t)\, dz, & 0 < t \le T,\ p \in (0, \infty)^d, \\ \varphi(p), & t = 0,\ p \in (0, \infty)^d, \end{cases}$$

where $\hat{\nu} = \arg\min_{y \in \tilde{K}_+}[2(1 - \alpha)\delta(y) + \|\sigma^{-1}(b - r + y)\|^2]$ of (7.28) is now a constant vector in $\tilde{K}_+$ and the function h: $[0, T] \times (0, \infty)^d \times \mathscr{R}^d \mapsto (0, \infty)^d$ is given by

$$(7.36)\qquad h_i(t, p, y) \triangleq p_i \exp\left[(r - \hat{\nu}_i - \tfrac{1}{2}a_{ii})t + y_i\right], \qquad i = 1, \ldots, d.$$

The Gaussian computation of (7.35) takes a very explicit form in the special case of a European call option on the first stock, where $\varphi(p) = (p_1 - q)^+$, $0 < p_1 < \infty$, for some exercise price $q > 0$ in a market with constant coefficients (r, b, σ). Then (7.34) becomes

$$(7.37)\qquad \hat{p} = \exp(-(\hat{\nu}_1 + \delta(\hat{\nu}_1))T) u_0(r - \hat{\nu}_1, q; P_1(0)),$$

where

(7.38) $u_0(r, q; p)$ is the Black–Scholes price of (4.4) and (4.5) with interest rate r, exercise price q and $P_1(0) = p$.

EXAMPLE 7.4. *"Look-back" option $B(T) = \max_{0 \le t \le T} P_1(t)$ with constant coefficients, $d = 1$ and $U(\cdot) = U_\alpha(\cdot)$ as in* (7.27). Again $\hat{\nu} = \arg\min_{y \in \tilde{K}_+}[2(1 -$

$\alpha)\delta(y) + ||\sigma^{-1}(b - r + y)||^2]$ is constant and (7.25) becomes

$$\hat{p} = P_1(0)\exp(-(r + \delta(\hat{\nu}))T)\int_0^\infty f(T, \xi; \hat{\rho})\exp(\sigma\xi)\,d\xi$$

in the notation of (4.7) with $\hat{\rho} \triangleq (r - \hat{\nu})/\sigma - \sigma/2$.

8. European call option in a market with constant coefficients. In this section, we use the general results of previous sections to study in detail the three prices h_{low}, h_{up} and $\hat{p}$ for a European call option on the first stock $B(T) = (P_1(T) - q)^+$ in a market with constant coefficients, that is, when the coefficient $b(\cdot) \equiv b = (b_1, b_2, \ldots, b_d)^*$, $r(\cdot) \equiv r$ and $\sigma(\cdot) \equiv \sigma = (\sigma_{ij})$ in (2.2) and (2.1) are all constants. All our examples involve closed, convex sets K_+ and K_- as in Section 6.

8.1. *Lower and upper arbitrage prices.*

EXAMPLE 8.1. *Constraints on borrowing* [Example 6.1(viii) with $K_+ = (-\infty, k]$, $K_- = [l, \infty)$ for some $k \geq 1$ and $l \leq 1$]. It is easy to see from (4.6) that the Black–Scholes price u_0 belongs to $\mathscr{L}$. Thus

$$h_{\text{low}} = u_0 \tag{8.1}$$

by Theorem 5.1. On the other hand, we claim that

$$h_{\text{up}} \leq \mathsf{E}^0\left[\gamma_0(T)\left(\frac{k-1}{k}P_1(T) - q\right)^+\right] + \frac{1}{k}P_1(0) =: a_k. \tag{8.2}$$

PROOF OF (8.2). By the definition of h_{up} it is enough to show that we can find for a_k an admissible pair $(\tilde{\pi}, \tilde{C}) \in \mathscr{A}(a_k)$, such that $\tilde{\pi}(\cdot) \leq k$ and $X^{a_k, \tilde{\pi}, \tilde{C}}(\cdot) \geq 0$, $X^{a_k, \tilde{\pi}, \tilde{C}}(T) \geq (P_1(T) - q)^+$ almost surely. Actually, we can take $\tilde{C} \equiv 0$.

Define for $0 \leq t \leq T$,

$$\begin{aligned} X^{(1)}(t) &\triangleq \frac{1}{\gamma_0(t)}\mathsf{E}^0\left[\gamma_0(T)\left(\frac{k-1}{k}P_1(T) - q\right)^+ \middle| \mathscr{F}_t\right] + \frac{1}{k}P_1(t) \\ &= \tilde{U}^{(1)}(T - t, P_1(t)) + \frac{1}{k}P_1(t), \qquad 0 \leq t \leq T, \end{aligned} \tag{8.3}$$

where

$$\tilde{U}^{(1)}(t, x) \triangleq \mathsf{E}^0\left[e^{-rt}\left(\frac{k-1}{k}P_1(t) - q\right)^+ \middle| P_1(0) = x\right], \qquad 0 \leq t \leq T,\ 0 < x < \infty. \tag{8.4}$$

It is clear from (8.3) that

$$X^{(1)}(0) = a_k,$$

$$X^{(1)}(T) = \left(\frac{k-1}{k}P_1(T) - q\right)^+ + \frac{1}{k}P_1(T) \geq (P_1(T) - q)^+ = B(T).$$

Using the function $\tilde{U}^{(1)}(t,x)$ of (8.4), we can define

$$\pi^{(1)}(t)=\left(\frac{\partial\tilde{U}^{(1)}(T-t,P_1(t))}{\partial x}P_1(t)+\frac{1}{k}P_1(t)\right)\Big/\left(\tilde{U}^{(1)}(T-t,P_1(t))+\frac{1}{k}P_1(t)\right).$$

We shall show that

$$X^{(1)}(\cdot)=X^{a_k,\pi^{(1)},0}(\cdot)\quad\text{and}\quad \pi^{(1)}(\cdot)\le k.$$

Notice, by the Feynman–Kac formula (cf. Karatzas and Shreve (1991), page 366) and (8.4), that the function $\tilde{U}^{(1)}(t,x)$ satisfies the Cauchy problem

$$\begin{aligned}&\frac{\partial\tilde{U}^{(1)}}{\partial t}+r\tilde{U}^{(1)}=\frac{1}{2}\sigma^2x^2\frac{\partial^2\tilde{U}^{(1)}}{\partial x^2}+rx\frac{\partial\tilde{U}^{(1)}}{\partial x},\\ &\tilde{U}^{(1)}(0,x)=\left(\frac{k-1}{k}x-q\right)^+.\end{aligned}\tag{8.5}$$

From (8.5), (3.9) in the form $dP_1(t)=rP_1(t)\,dt+\sigma P_1(t)\,dW^0(t)$ and Itô's rule, we obtain

$$\begin{aligned}
&d\tilde{U}^{(1)}(T-t,P_1(t))\\
&\quad=-\frac{\partial\tilde{U}^{(1)}}{\partial t}\,dt+\frac{\partial\tilde{U}^{(1)}}{\partial x}\,dP_1(t)+\frac{1}{2}\frac{\partial^2\tilde{U}^{(1)}}{\partial x^2}\,d\langle P_1(t)\rangle\\
&\quad=-\left(\frac{1}{2}\sigma^2P_1^2(t)\frac{\partial^2\tilde{U}^{(1)}}{\partial x^2}+rP_1(t)\frac{\partial\tilde{U}^{(1)}}{\partial x}-r\tilde{U}^{(1)}\right)dt\\
&\qquad+\frac{\partial\tilde{U}^{(1)}}{\partial x}\left(rP_1(t)\,dt+\sigma P_1(t)\,dW^0(t)\right)+\frac{1}{2}\frac{\partial^2\tilde{U}^{(1)}}{\partial x^2}\sigma^2P_1^2(t)\,dt\\
&\quad=r\tilde{U}^{(1)}(T-t,P_1(t))\,dt+\frac{\partial\tilde{U}^{(1)}(T-t,P_1(t))}{\partial x}\sigma P_1(t)\,dW^0(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
dX^{(1)}(t)&=d\tilde{U}^{(1)}(T-t,P_1(t))+\frac{1}{k}\,dP_1(t)\\
&=r\tilde{U}^{(1)}(T-t,P_1(t))\,dt+\frac{\partial\tilde{U}^{(1)}(T-t,P_1(t))}{\partial x}\sigma P_1(t)\,dW^0(t)\\
&\quad+\frac{1}{k}rP_1(t)\,dt+\frac{1}{k}\sigma P_1(t)\,dW^0(t)\\
&=rX^{(1)}(t)\,dt+\left(\frac{\partial\tilde{U}^{(1)}(T-t,P_1(t))}{\partial x}P_1(t)\sigma\,dW^0(t)\right.\\
&\qquad\qquad\left.+\frac{1}{k}\sigma P_1(t)\,dW^0(t)\right)\\
&=rX^{(1)}(t)\,dt+X^{(1)}(t)\tilde{\pi}(t)\sigma\,dW^0(t).
\end{aligned}$$

Thus $X^{(1)}(\cdot)$ satisfies (3.2) with $C \equiv 0$, whence the pair $(\tilde{\pi}, 0)$ is indeed the one we needed, except we have to verify $\tilde{\pi}(\cdot) \le k$. This can be checked easily from (8.4) and the inequality

$$x\left(\varphi'(x) + \frac{1}{k}\right) \le k\left(\varphi(x) + \frac{1}{k}x\right),$$

where $\varphi(x) = (x(k-1)/k - q)^+$. □

REMARK 8.1. The case $k = 1$ corresponds to the "no-borrowing" constraints and is discussed in Cvitanić and Karatzas (1993), where it is also shown that $h_{\text{up}} = a_1 = P_1(0)$ (for $k = 1$). In addition, these authors show that the consumption process C corresponding to the hedging strategy can be taken as $C(t) = 0$ for $0 \le t < T$ and $C(T) = \min(P_1(T), q) > 0$ at time $t = T$.

REMARK 8.2. If $k > 1$, then we can rewrite a_k as

$$\frac{k-1}{k}u_0(r, qk/(k-1); P_1(0)) + \frac{1}{k}P_1(0).$$

Furthermore, if k increases (in other words, as the constraint becomes weaker), it is readily seen that the upper bound a_k converges to the Black–Scholes price u_0:

$$\begin{aligned} h_{\text{up}} &\to u_0 \quad \text{as } k \to \infty \\ &= u_0(r, q; P_1(0)) = \text{Black–Scholes price.} \end{aligned}$$

EXAMPLE 8.2. *Constraints on short-selling* [Example 6.1(iii) with $d = 1$, $K_+ = [-k, \infty)$, $K_- = (-\infty, l]$ for some $k \ge 0$ and $l > 1$]. It is easy to see from (4.6) that $u_0 \in \mathscr{U}$, whence

$$h_{\text{up}} = u_0. \tag{8.6}$$

We claim that in this case,

$$h_{\text{low}} \ge \mathsf{E}^0[\gamma_0(T)(P_1(T) - q)\mathbf{1}_{[P_1(T) \ge ql/(l-1)]}] =: \rho_l. \tag{8.7}$$

PROOF OF (8.7). Clearly, it is enough to show that $\rho_l \in \mathscr{L}$. Define the process

$$\begin{aligned} X^{(2)}(t) &\triangleq -\frac{1}{\gamma_0(t)}\mathsf{E}^0[\gamma_0(T)(P_1(T) - q)\mathbf{1}_{[P_1(T) \ge ql/(l-1)]}|\mathscr{F}_t] \\ &= -\tilde{U}^{(2)}(T - t, P_1(t)), \qquad 0 \le t \le T, \end{aligned} \tag{8.8}$$

where

$$\tilde{U}^{(2)}(t,x) \triangleq \mathsf{E}^0[e^{-rt}(P_1(t)-q)\mathbf{1}_{[P_1(t)\geq ql/(l-1)]}\,|P_1(0)=x], \qquad 0\leq t\leq T,\ 0<x<\infty. \tag{8.9}$$

Then we have, at time $t=0$,

$$X^{(2)}(0) = -\mathsf{E}^0[\gamma_0(T)(P_1(T)-q)\mathbf{1}_{[P_1(T)\geq lq/(l-1)]}] = -\rho_l < 0$$

and, at time $t=T$,

$$X^{(2)}(T) = -(P_1(T)-q)\mathbf{1}_{[P_1(T)\geq ql/(l-1)]} \geq -(P_1(T)-q)^+.$$

On the other hand, (8.9) gives $0 \leq -\tilde{U}^{(2)}(t,x) \leq \mathsf{E}^0[e^{-rt}P_1(t)|P_1(0)=x] = x$, so that from (8.8) the positive process $-X^{(2)}(\cdot)$ is dominated by the P-integrable random variable $\max_{0\leq t\leq T} P_1(t)$. Hence, it is enough to find a pair $(\pi^{(2)}, C^{(2)}) \in \mathscr{A}(-\rho_l)$ with $X^{(2)}(\cdot) = X^{-\rho_l,\pi^{(2)},C^{(2)}}(\cdot)$. Introduce

$$\pi^{(2)}(t) = \left(\frac{\partial \tilde{U}^{(2)}(T-t,P_1(t))}{\partial x}P_1(t)\right)\Big/(\tilde{U}^{(2)}(T-t,P_1(t))), \qquad 0\leq t\leq T.$$

Again by the Feynman–Kac formula, the function $\tilde{U}^{(2)}(t,x)$ of (8.9) satisfies the Cauchy problem

$$\frac{\partial \tilde{U}^{(2)}}{\partial t} + r\tilde{U}^{(2)} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \tilde{U}^{(2)}}{\partial x^2} + rx\frac{\partial \tilde{U}^{(2)}}{\partial x}, \qquad \tilde{U}^{(2)}(0,x) = (x-q)\mathbf{1}_{[x\geq ql/(l-1)]}$$

and from Itô's rule,

$$\begin{aligned}
dX^{(2)}(t) &= -\left[-\frac{\partial \tilde{U}^{(2)}}{\partial t}\,dt + \frac{\partial \tilde{U}^{(2)}}{\partial x}\,dP_1(t) + \frac{1}{2}\frac{\partial^2 \tilde{U}^{(2)}}{\partial x^2}d\langle P_1(t)\rangle\right]\\
&= \left(\frac{1}{2}\sigma^2 P_1^2(t)\frac{\partial^2 \tilde{U}^{(2)}}{\partial x^2} + rP_1(t)\frac{\partial \tilde{U}^{(2)}}{\partial x} - r\tilde{U}^{(2)}\right)dt\\
&\quad - \frac{\partial \tilde{U}^{(2)}}{\partial x}(rP_1(t)\,dt + \sigma P_1(t)\,dW^0(t)) - \frac{1}{2}\frac{\partial^2 \tilde{U}^{(2)}}{\partial x^2}\sigma^2 P_1^2(t)\,dt\\
&= -r\tilde{U}^{(2)}(T-t,P_1(t))\,dt - \frac{\partial \tilde{U}^{(2)}}{\partial x}\sigma P_1(t)\,dW^0(t)\\
&= rX^{(2)}(t)\,dt + X^{(2)}(t)\pi^{(2)}(t)\sigma\,dW^0(t).
\end{aligned}$$

Hence the wealth equation (3.2) is satisfied with $C = C^{(2)} \equiv 0$. To check that $\pi^{(2)}(\cdot) \leq l$, we need only verify that

$$\left(\frac{\partial \tilde{U}^{(2)}(T-t,P_1(t))}{\partial x}P_1(t)\right)\Big/(\tilde{U}^{(2)}(T-t,P_1(t))) \leq l.$$

This bound is not hard to derive, from (8.9) and the inequality

$$\varphi'(x)x \le l\varphi(x), \quad \text{where } \varphi(x) = (x-q)\mathbf{1}_{[x \ge ql/(l-1)]}.$$

The proof is now complete. □

REMARK 8.3. Notice that we have from (2.2), $P_1(t) = \exp((r - \sigma^2/2)t + \sigma N(t))p$, where $p = P_1(0) > 0$ and $N(\cdot)$ is standard Brownian motion under the probability measure P^0. Therefore, ρ_l can be rewritten as

$$\begin{aligned}\rho_l &= u_0\Big(r, \frac{ql}{l-1}; P_1(0)\Big) + \mathsf{E}^0\Big[\gamma_0(T)\frac{P_1(T)-ql}{l-1}\mathbf{1}_{[P_1(T)\ge ql/(l-1)]}\Big]\\ &= u_0\Big(r, \frac{ql}{l-1}; P_1(0)\Big) + \frac{q\gamma_0(T)}{l-1}\mathsf{P}^0\Big(P_1(T) \ge \frac{ql}{l-1}\Big)\\ &= u_0\Big(r, \frac{ql}{l-1}; P_1(0)\Big) + \frac{qe^{-rT}}{l-1}\mathsf{P}^0\Big(\Big(r - \frac{\sigma^2}{2}\Big)T + \sigma N(T) \ge \log\Big(\frac{ql}{P_1(0)(l-1)}\Big)\Big)\end{aligned}$$

in the notation of (7.38). Invoking the normal distribution, we arrive after a bit of algebra at

$$\begin{aligned}\rho_l = {}& u_0\Big(r, \frac{lq}{l-1}; P_1(0)\Big)\\ &+ \frac{qe^{-rT}}{l-1}\Big\{1 - \Phi\Big(\frac{1}{\sigma\sqrt{T}}\log\Big(\frac{ql}{P_1(0)(l-1)} - (r - (\sigma^2/2))\sqrt{T}\Big)\Big)\Big\}.\end{aligned}$$

As before, if the constraints become weaker and weaker (i.e., $l \to \infty$), then ρ_l converges to the Black–Scholes price u_0:

$$h_{\text{low}} \to u_0 \quad \text{as } l \to \infty.$$

REMARK 8.4. If we consider the no short-selling case of Example 6.1(ii) (or equivalently, Example 8.2 with $k = l = 0$), then instead of the inequality (8.7), we can actually prove

$$h_{\text{low}} = 0.$$

Indeed, we have from (6.6) that

$$\exp\Big(\int_0^t \nu_1(s)\,ds\Big)\gamma_0(t)P_1(t) = P_1(0)\exp\Big[\int_0^t \sigma(s)\,dW_\nu(s) - \tfrac{1}{2}\int_0^t \sigma(s)^2\,ds\Big],$$

which is a P^ν-martingale. Thus, if we denote by $\tilde{\mathscr{D}}_d$ the subset of all nonrandom functions ν: $[0, T] \mapsto \tilde{K}_-$ in the set $\tilde{\mathscr{D}}$, we have

$$\mathsf{E}^\nu[\tilde{\gamma}_\nu(T)P_1(T)] = P_1(0)\exp\Big(-\int_0^T (\tilde{\delta}(\nu(s)) + \nu(s))\,ds\Big) \qquad \forall\, \nu \in \tilde{\mathscr{D}}_d. \tag{8.10}$$

By Theorem 6.1, we get the inequalities

$$
\begin{aligned}
0 \le h_{\text{low}} &= \inf_{\nu \in \tilde{\mathscr{D}}} \mathsf{E}^{\nu}[\tilde{\gamma}_{\nu}(T)(P_1(T) - q)^{+}] \\
&\le \inf_{\nu \in \tilde{\mathscr{D}}} \mathsf{E}^{\nu}[\tilde{\gamma}_{\nu}(T) P_1(T)] \\
&\le P_1(0) \inf_{\nu \in \tilde{\mathscr{D}}_d} \exp\left(- \int_0^T (\tilde{\delta}(\nu(s)) + \nu(s))\, ds \right) \\
&= P_1(0) \inf_{\nu \in \tilde{\mathscr{D}}_d} \exp\left(- \int_0^T \nu(s)\, ds \right) = 0
\end{aligned}
$$

as we can let ν tend to ∞. Thus we conclude that $h_{\text{low}} = 0$ in the no short-selling case.

EXAMPLE 8.3. *Constraints on borrowing* [Example 6.1(viii) with $d = 1$ and $K_{+} = (-\infty, k]$, $K_{-} = [l, \infty)$ for some $k \ge 1$, $l \ge k$, $l > 1$]. Here again the upper bound $h_{\text{up}} \le a_k$ on the upper arbitrage price holds, as in (8.2) of Example 8.1. Now, however, $h_{\text{low}} = 0$, so that the complete picture is

$$
0 = h_{\text{low}} < u_0 < h_{\text{up}} \le a_k < \infty.
$$

Indeed, we have here $\tilde{K}_{\pm} = (-\infty, 0]$ and $\delta(x) = -kx$, $\tilde{\delta}(x) = -lx$ on $\tilde{K}_{\pm}$, so that for deterministic $\nu(\cdot)$,

$$
\begin{aligned}
\mathsf{E}^{\nu}[\tilde{\gamma}_{\nu}(T) P_1(T)] &= P_1(0) \exp\left(- \int_0^T (\tilde{\delta}(\nu(s)) + \nu(s))\, ds \right) \\
&= P_1(0) \exp\left((l - 1) \int_0^T \nu(s)\, ds \right)
\end{aligned}
$$

as in (8.10), and we obtain $h_{\text{low}} = 0$ much as in Remark 8.4, except that now we let $\nu(\cdot)$ tend to $-\infty$.

EXAMPLE 8.4. *Constraints on short-selling* [Example 6.1(iii) with $d = 1$ and $K_{+} = [-k, \infty)$, $K_{-} = (-\infty, -k]$ for some $k \ge 0$]. In this case,

$$
0 = h_{\text{low}} < u_0 = h_{\text{up}} < \infty.
$$

Indeed, $h_{\text{up}} = u_0$ follows as in (8.6) of Example 8.2. As for $h_{\text{low}} = 0$, observe that now we have $\tilde{K}_{\pm} = [0, \infty)$, $\delta(x) = \tilde{\delta}(x) = kx$ on $\tilde{K}_{\pm}$ and (8.10) becomes

$$
\mathsf{E}^{\nu}[\tilde{\gamma}_{\nu}(T) P_1(T)] = P_1(0) \exp\left(-(1 + k) \int_0^T \nu(s)\, ds \right)
$$

for deterministic $\nu(\cdot)$. We conclude $h_{\text{low}} = 0$ by letting $\nu(\cdot)$ become very large as in Remark 8.4.

EXAMPLE 8.5. *Incomplete market cases.* (a) Only the first m stocks can be traded, with $1 \le m \le d - 1$, $d \ge 2$ as in Example 6.1(iv). Then by the explicit formula in (4.6), we have that $h_{\text{up}} = h_{\text{low}} = u_0$.

(b) The first m stocks, $1 \le m \le d-1$, *cannot* be traded as in Example 6.1(v). In this case, it can be shown that $h_{\mathrm{up}} = \infty$ as in Cvitanić and Karatzas (1993). We can show that $h_{\mathrm{low}} = 0$. In fact, observe, once again from (6.6), that

$$\exp\left(\int_0^t \nu_1(s)\,ds\right)\gamma_0(t)P_1(t) = P_1(0)\exp\left[\int_0^t \sigma_1(s)\,dW_\nu(s) - \frac{1}{2}\int_0^t \|\sigma_1(s)\|^2\,ds\right],$$

where $\sigma_1 = (\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1d})^*$. Then the same argument as for the no short-selling case will lead to the desired result.

8.2. *Computation of the fair price.* Let us compute in this subsection the fair price $\hat{p}$ of (7.25) in a few examples, with closed and convex sets $K_\pm$ that satisfy condition (7.12)—so that $\hat{p}$ is in the interval $[h_{\mathrm{low}}, h_{\mathrm{up}}]$ for all these examples.

EXAMPLE 8.6. *Cone constraints.* Let K_+ be a (closed, convex) cone in $\mathscr{R}^d$ and let $K_- = -K_+$. Then from (7.29), (7.37) and the fact that $\delta \equiv 0$ on $\tilde{K}_+$, we have

$$\hat{p} = \exp(-\hat{\nu}_1 T)u_0(r - \hat{\nu}_1, q; P_1(0)) \tag{8.11}$$

in the notation of (7.38), where

$$\hat{\nu} = \arg\min_{y \in \tilde{K}_+} \|\sigma^{-1}(b - r + y)\|^2. \tag{8.12}$$

In particular, $\hat{p}$ does not depend on either the utility function or the initial level of wealth. Here are some particular cases.

(a) *Incomplete market with only the first m stocks available* [Example 6.1(iv)]. Then (8.12) gives $\hat{\nu}_1 = 0$, and (8.11) becomes

$$\hat{p} = u_0(r, q; P_1(0)) = \text{Black–Scholes price}.$$

(b) *Incomplete market with the first m stocks unavailable* [Example 6.1(v)]. We again have from (8.12) that $\hat{\nu}_1 = r - b_1$ (see also Example 7.1), and (8.11) takes the form

$$\hat{p} = \exp(-(r - b_1)T)u_0(b_1, q; P_1(0)).$$

(c) *Prohibition of short-selling* [Example 6.1(ii) with $d = 1$]. Then it can be seen by simple algebra that in this case $\hat{\nu} = (r - b)^+$ in (8.12). Thus (8.11) becomes

$$\hat{p} = \begin{cases} u_0(r, q; P_1(0)), & \text{if } r \le b, \\ e^{-(r-b)T}u_0(b, q; P_1(0)), & \text{if } r > b. \end{cases} \tag{8.13}$$

EXAMPLE 8.7. *Utility function of the power type* (7.27). In this case, (7.28) or (7.26) gives

$$\hat{\nu} = \arg\min_{y \in \tilde{K}_+}[\|\sigma^{-1}(b - r\mathbf{1} + y)\|^2 + 2(1 - \alpha)\delta(y)] \tag{8.14}$$

and $\hat{p}$ is then as in (7.37); for a set K_+ that is not a cone, this $\hat{p}$ depends in general on the utility function through the constant $\alpha \in [0, 1)$. Here are some concrete examples.

(a) *Prohibition of borrowing* [Example 6.1(vii) with $d = 1$]. Then (8.14) gives $\hat{\nu} = -(r - b + (1 - \alpha)\sigma^2)^-$ and thus (7.37) becomes

$$\hat{p} = \begin{cases} u_0((b + (\alpha - 1)\sigma^2), q; P_1(0)), & \text{if } r \le b + (\alpha - 1)\sigma^2, \\ u_0(r, q; P_1(0)), & \text{otherwise.} \end{cases}$$

(b) *Constraints on borrowing* (Example 8.3). Then $\delta(x) = -kx$ on $\tilde{K}_+ = (-\infty, 0]$ for some $k \ge 1$, so (8.14) and (7.37) give $\hat{\nu} = -(r - b + (1 - \alpha)k\sigma^2)^-$ and

$$\hat{p} = \begin{cases} u_0(r, q; P_1(0)), & \text{if } b + k(\alpha - 1)\sigma^2 \le r, \\ \exp(-(k - 1)(b + k(\alpha - 1)\sigma^2 - r)) \\ \qquad \times u_0(b + k(\alpha - 1)\sigma^2, q; P_1(0)), & \text{otherwise.} \end{cases} \tag{8.15}$$

(c) *Constraints on short-selling* (Example 8.4). Then $\delta(x) = kx$ on $\tilde{K}_+ = [0, \infty)$ for some $k \ge 0$ and (8.14) and (7.37) lead, respectively, to $\hat{\nu} = (r - b + k(\alpha - 1)\sigma^2)^+$ and

$$\hat{p} = \begin{cases} u_0(r, q; P_1(0)), & \text{if } r \le b + k(1 - \alpha)\sigma^2, \\ \exp(-(1 + k)(r - b + k(\alpha - 1)\sigma^2)) \\ \qquad \times u_0(b + k(1 - \alpha)\sigma^2, q; P_1(0)), & \text{otherwise.} \end{cases} \tag{8.16}$$

8.3. *Counterexamples.* Finally, let us demonstrate by some examples that the lower bound of (7.5) may fail in the absence of condition (7.12) on the sets $K_\pm$. In all these examples, the set K_+ is convex, so the upper bound of (7.5) must hold; see Remark 7.1.

EXAMPLE 8.1 (Continued with $K_+ = (-\infty, k]$, $K_- = [l, \infty)$ and $k > 1$, $l \le 1$). Here it is easy to check that condition (7.12) fails and with utility function $U_\alpha(\cdot)$, $0 \le \alpha < 1$, as in (7.27), the fair price $\hat{p}$ is given by (8.15) and satisfies

$$\hat{p} \to 0 \quad \text{as } b \to \infty$$

for fixed $(r, k, \alpha, q, \sigma^2, l)$, since $u_0(x, q; P_1(0)) \to P_1(0)$ as $x \to \infty$ [see Cox and Rubinstein (1984), page 216]. However, we know from (8.1) that $h_{\text{low}} \equiv u_0(r, q; P_1(0)) > 0$, whence $h_{\text{low}} > \hat{p} > 0$ for all sufficiently large appreciation rates b.

EXAMPLE 8.2 (Continued). Here $K_+ = [-k, \infty)$, $K_- = (-\infty, l]$ for some $k \ge 0$, $l > 1$. Again, it is verified that condition (7.12) fails and with utility function of the type (7.27), the fair price $\hat{p}$ is given by (8.16) and satisfies

$$\hat{p} \to 0 \quad \text{as } r \to \infty$$

for $(b, k, \alpha, q, \sigma^2)$ fixed. On the other hand, we have from Remark 8.3,

$$\begin{aligned} h_{\text{low}} &\geq \rho_l = u_0\left(r, \frac{lq}{l-1}; p\right) \\ &\quad + \frac{qe^{-rT}}{l-1}\left\{1 - \Phi\left(\frac{1}{\sigma\sqrt{T}}\log\left(\frac{lq}{p(l-1)} - \left(r - \frac{\sigma^2}{2}\right)\sqrt{T}\right)\right)\right\} \\ &\to p \equiv P_1(0) > 0 \quad \text{as } r \to \infty. \end{aligned}$$

Consequently, for all sufficiently large interest rates r, $h_{\text{low}} > \hat{p} > 0$.

EXAMPLE 8.3. Take $d = 1, r > b, K_+ = [0, \infty), K_- = [1, \infty)$ [a combination of Examples 6.1(ii) and (vii)], so that (7.12) fails again. Now $\tilde{K}_+ = [0, \infty)$ and $\tilde{K}_- = (-\infty, 0]$, $h_{\text{low}} = u_0(r, q; P_1(0))$ as in (8.1), and from (8.13),

$$\hat{p} = e^{-(r-b)T} u_0(b, q; P_1(0)) < u_0(r, q; P_1(0)) = h_{\text{low}}.$$

9. Market with higher interest rate for borrowing. We have studied so far the pricing problem for contingent claims in a financial market with the same interest rate for borrowing and for saving. However, the techniques developed in the previous sections can be adapted to a market $\mathscr{M}^*$ with interest rate $R(\cdot)$ for borrowing higher than the bond rate $r(\cdot)$ (saving rate).

We consider in this section an *unconstrained* market $\mathscr{M}^*$ with two different (bounded, $\{\mathscr{F}_t\}$-progressively measurable) interest rate processes $R(\cdot) \geq r(\cdot)$ for borrowing and saving, respectively. In this market $\mathscr{M}^*$, it is not reasonable to borrow money and to invest money in the bond at the same time. Therefore, the relative amount borrowed at time t is equal to $(1 - \sum_{i=1}^d \pi_i(t))^-$. As shown in CK1, the wealth process $X(\cdot) = X^{x,\pi,C}(\cdot)$ corresponding to initial wealth x and a portfolio–consumption pair (π, C) as in Definition 3.5 now satisfies the analogue of the wealth equation (3.2),

$$\begin{aligned} dX(t) &= r(t)X(t)\,dt - dC(t) \\ &\quad + X(t)\left[\pi^*(t)\sigma(t)\,dW_0(t) - (R(t) - r(t))\left(1 - \sum_{i=1}^d \pi_i(t)\right)^- dt\right], \end{aligned} \tag{9.1}$$

whence

$$\begin{aligned} N(t) &\triangleq \gamma_0(t)X(t) + \int_0^t \gamma_0(t)\,dC(t) \\ &\quad + \int_0^t \gamma_0(t)X(t)[R(t) - r(t)]\left(1 - \sum_{i=1}^d \pi_i(t)\right)^- dt, \qquad 0 \leq t \leq T, \end{aligned}$$

is a P^0-local martingale by Itô's rule, in the notation of (2.4)–(2.7) and (3.6).

All the arguments in Section 5 go through under slight modifications. For example, the lower and upper hedging classes are now defined to be

$$\mathscr{L} \triangleq \{x \geq 0\colon \exists\, (\check{\pi}, \check{C}) \in \mathscr{A}(-x), \text{ such that } X^{-x,\check{\pi},\check{C}}(\cdot) \leq 0 \text{ and } X^{-x,\check{\pi},\check{C}}(T) \geq -B(T) \text{ a.s.}\},$$
$$\mathscr{U} \triangleq \{x \geq 0\colon \exists\, (\hat{\pi}, \hat{C}) \in \mathscr{A}(x), \text{ such that } X^{x,\hat{\pi},\hat{C}}(\cdot) \geq 0 \text{ and } X^{x,\hat{\pi},\hat{C}}(T) \geq B(T) \text{ a.s.}\}.$$

The statements of Definition 5.2 and Theorems 5.1–5.3 hold without change.

We set $\delta(\nu(t)) = -\nu_1(t)$, $0 \leq t \leq T$, for $\nu \in \mathscr{D}$, where $\mathscr{D}$ is the class of progressively measurable processes $\nu\colon [0,T] \times \Omega \mapsto \mathscr{R}^d$ with $r - R \leq \nu_1 = \cdots = \nu_d \leq 0$, $l \otimes \mathsf{P}$-a.e. Then with this notation the theory of Section 6 also goes through with only minor changes, such as replacing $\tilde{\delta}$ by δ and $\tilde{\mathscr{D}}$ by $\mathscr{D}$ and so on. In particular, Theorem 6.1 now states that

$$h_{\text{low}} = \inf_{\nu \in \mathscr{D}} \mathsf{E}^{\nu}[\gamma_{\nu}(T)B(T)] \tag{9.2}$$

and that

$$h_{\text{up}} = \sup_{\nu \in \mathscr{D}} \mathsf{E}^{\nu}[\gamma_{\nu}(T)B(T)]. \tag{9.3}$$

The proofs of (9.2) and (9.3) follow the same lines as Theorem 6.1 and Cvitanić and Karatzas (1993), respectively. We sketch the proof of (9.2) here.

SKETCH OF PROOF FOR (9.2). We first repeat the proof of Theorem 6.1, right up to (6.23). There, we change the definition of the consumption process $\check{C}(\cdot)$ to read

$$\check{C}(t) \triangleq \int_0^t \gamma_{\nu}^{-1}(s)\, dA_{\nu}(s) - \int_0^t \check{X}(s)\left[\delta(\nu(s)) + \check{\pi}^*(s)\nu(s) + (R(s) - r(s))\left(1 - \sum_{i=1}^d \check{\pi}_i(s)\right)^-\right] ds.$$

Taking $\nu(t) = \lambda(t) \equiv \lambda_1(t)\mathbf{1}$, where $\lambda_1(t) \triangleq [r(t) - R(t)]\mathbf{1}_{(\sum_{i=1}^d \check{\pi}_i(t) > 1)}$, we get

$$\check{C}(t) = \int_0^t \gamma_{\lambda}^{-1}(s)\, dA_{\lambda}(s)$$

as required. Skip the lines in which $\check{\pi}(t) \in K_-$ is shown and observe that we now have

$$\begin{aligned}
d(-\check{X}(t)\gamma_{\nu}(t)) &= dQ_{\nu}(t) = \psi_{\nu}^*(t)\, dW_{\nu}(t) + dA_{\nu}(t) \\
&= \gamma_{\nu}(t)\Bigg\{-d\check{C}(t) - \check{X}(t)\Bigg[\delta(\nu(t)) + (R(t) - r(t)) \\
&\qquad \times \left(1 - \sum_{i=1}^d \check{\pi}_i(t)\right)^- + \check{\pi}^*(t)\nu(t)\Bigg] dt + \check{X}(t)\check{\pi}^*(t)\sigma(t)\, dW_{\nu}(t)\Bigg\} \\
&= \tilde{\gamma}_{\nu}(t)[-d\check{C}(t) - \check{X}(t)\Psi^{\nu,\check{\pi}}(t)\, dt + \check{X}(t)\check{\pi}^*(t)\sigma(t) dW_{\nu}(t)],
\end{aligned}$$

where

$$\Psi^{\nu,\check{\pi}}(t) \triangleq [R(t) - r(t) + \nu_1(t)]\Big(1 - \sum_{i=1}^{d} \check{\pi}_i(t)\Big)^{-} - \nu_1(t)\Big(1 - \sum_{i=1}^{d} \check{\pi}_i(t)\Big)^{+},$$

$$0 \le t \le T,$$

is a nonnegative process. Now taking $\nu \equiv 0$, we get $\check{X}(\cdot) = X^{-g,\check{\pi},\check{C}}(\cdot)$ by comparing with the new wealth equation (9.1), where g is now the right-hand side of (9.2). The rest of the proof proceeds in a clearly analogous way. □

With the adoption of the new $\delta(\cdot)$ and $\mathscr{D}$, we can define the fair price by analogy with (7.4), and proceed in the same way as we did before, to obtain all theorems in Section 7. In particular, an encouraging phenomenon is that the fair price *always* lies within the arbitrage interval, because Assumption 7.2 is always satisfied in this case.

The argument in Cvitanić and Karatzas (1993) for computing h_{up} in a market $\mathscr{M}^*$ with $d = 1$ and constant coefficients also works for h_{low} after slight adjustments. For example, change "sup" and "max" in (9.8) and (9.9) of Cvitanić and Karatzas (1993) to "inf" and "min," respectively. Then from the Hamilton–Jacobi–Bellman (HJB) equation we can also get

$$h_{\text{low}} = u_0(r, q; P_1(0))$$

for the European call option $B(T) = (P_1(T) - q)^+$. In other words, the lower arbitrage price is exactly the Black–Scholes price with interest rate r, while, as has been shown in Cvitanić and Karatzas (1993), the upper arbitrage price is the Black–Scholes price with interest rate R:

$$h_{\text{up}} = u_0(R, q; P_1(0)).$$

For the fair price within the interval $[h_{\text{low}}, h_{\text{up}}]$, we still use Theorem 7.4 and Remark 7.1 to get the explicit fair price $\hat{p}$ for the constant coefficient market $\mathscr{M}^*$. More precisely, with utility function $U_\alpha(\cdot)$ as in (7.27), it is shown in Cvitanić and Karatzas (1992), page 816, that the $\hat{\nu}$ in Theorem 7.4 is given by

$$\hat{\nu} = \begin{cases} 0, & \text{if } r \ge b_1 + \sigma^2(\alpha - 1), \\ r - b_1 - \sigma^2(\alpha - 1), & \text{if } r \le b_1 + \sigma^2(\alpha - 1) \le R, \\ r - R, & \text{if } b_1 + \sigma^2(\alpha - 1) \ge R. \end{cases}$$

Hence, (7.37) gives the fair price

$$\hat{p} = \begin{cases} u_0(r, q; P_1(0)), & \text{if } r \ge b_1 + \sigma^2(\alpha - 1), \\ u_0(b_1 + \sigma^2(\alpha - 1), q; P_1(0)), & \text{if } r \le b_1 + \sigma^2(\alpha - 1) \le R, \\ u(R, q; P_1(0)), & \text{otherwise.} \end{cases} \tag{9.4}$$

REMARK 9.1. The expression (9.4) coincides with the so-called "minimax price" in Barron and Jensen (1990), defined to be the number $\tilde{p} = \tilde{p}(x)$ for which the function $\delta \mapsto W(\delta, \tilde{p}(x), x)$ of (7.3) is minimized at $\delta = 0$. [Clearly, with $\hat{p}(x)$ as in our Definition 7.3, $\delta \mapsto W(\delta, \hat{p}(x), x)$ is minimized at $\delta = 0$, so we can take $\tilde{p}(x) = \hat{p}(x)$, justifying this "coincidence."]

REMARK 9.2. More generally, with $d \geq 1$, utility function $U_\alpha(\cdot)$ of the type (7.27) and deterministic coefficients [resp., $\alpha = 0$ in (7.27) and general random coefficients], the function (resp., process) $\hat{\nu}(t) = \hat{\nu}_1(t)\mathbf{1}$ is given as

$$\begin{aligned}\hat{\nu}_1(t) &= \operatorname*{arg\,min}_{r(t)-R(t)\leq y\leq 0} [\|\sigma^{-1}(t)(b(t) + (y - r(t))\mathbf{1})\|^2 - 2y] \\ &= \begin{cases} 0, & \xi_\alpha(t) \leq 0, \\ r(t) - R(t), & \xi_\alpha(t) \geq R(t) - r(t), \\ -\xi_\alpha(t), & 0 \leq \xi_\alpha(t) \leq R(t) - r(t), \end{cases}\end{aligned}$$

by analogy with (7.26) and (7.28), where

$$\xi_\alpha(t) = (\alpha - 1 + \theta^*(t)\sigma^{-1}(t)\mathbf{1})/\mathrm{tr}[(\sigma^{-1}(t))^*(\sigma^{-1}(t))].$$

In the special case $B(T) = \varphi(P(T))$ of (7.30) with deterministic coefficients, the computations of (7.31)–(7.36) for the fair price $\hat{p}$ are all still valid.

10. Summary of examples. The results of previous discussions and examples concerning the pricing of a European call option $B(T) = (P_1(T) - q)^+$ in a market with constant coefficients are summarized in Table 1.

In the table, r is the interest rate of the bond (savings account), b_1 is the appreciation rate of the first stock on which the option is written, σ^2 is the stock volatility, $u_0(x) \equiv u_0(x, q; P_1(0))$ is the Black–Scholes price for interest rate x and exercise price q, and $P_1(0)$ is the price for the first stock at time $t = 0$. Finally,

$$\begin{aligned}a_k &= \frac{k-1}{k} u_0\left(r, \frac{qk}{k-1}; P_1(0)\right) + \frac{1}{k} P_1(0) \to u_0(r) \quad \text{as } k \to \infty, \\ \rho_l &= u_0\left(r, \frac{ql}{l-1}; P_1(0)\right) \\ &\quad + \frac{qe^{-rT}}{l-1}\left\{1 - \Phi\left(\frac{1}{\sigma\sqrt{T}} \log\left(\frac{ql}{P_1(0)(l-1)} - \left(r - \frac{\sigma^2}{2}\right)\sqrt{T}\right)\right\} \\ &\quad \to u_0(r) \quad \text{as } l \to \infty, \\ c_k &= \exp(-(1+k)(r - b_1 + (\alpha-1)k\sigma^2))u_0(b_1 + (1-\alpha)k\sigma^2), \\ d_k &= \exp(-(k-1)(b_1 + (\alpha-1)k\sigma^2 - r))u_0(b_1 + (\alpha-1)k\sigma^2), \\ f &= b_1 + (\alpha - 1)\sigma^2.\end{aligned}$$

REMARK 10.1. It should be observed that all the exact values, as well as the bounds, for h_{low} and h_{up} are *independent* of the appreciation rate b of the stock, which is often difficult to estimate. This makes the lower and upper

TABLE 1

	h_{low}	h_{up}	$\hat{p}$
Unconstrained market	$u_0(r)$	$u_0(r)$	$u_0(r)^*$
Incomplete market, with first m stocks available	$u_0(r)$	$u_0(r)$	$u_0(r)^*$
Incomplete market, first m stocks unavailable	0	∞	$\exp(-(r-b_1)T)u_0(b_1)^*$
No short-selling of stocks ($K_+ = [0,\infty)$, $K_- = (-\infty, 0]$)	0	$u_0(r)$	$\begin{cases} u_0(r), & \text{if } r \le b_1 \\ \exp(-(r-b_1)T)u_0(b_1), & \text{if } r > b_1 \end{cases}$ *
No borrowing ($K_+ = (-\infty, 1]$, $K_- = [1,\infty)$)	$u_0(r)$	$P_1(0)$	$\begin{cases} u_0(r), & \text{if } r \ge f,^\dagger \\ u_0(f), & \text{otherwise.} \end{cases}$
Constraints on short-selling ($K_+ = [-k,\infty)$, $K_- = (-\infty, -k]$, $k \ge 0$)	0	$u_0(r)$	$\begin{cases} u_0(r), & \text{if } r \le f,^\dagger \\ c_k, & \text{otherwise.} \end{cases}$
Constraints on borrowing ($K_+ = (-\infty, k]$, $K_- = [l,\infty)$, $l \ge k > 1$)	0	$\le a_k$	$\begin{cases} u_0(r), & \text{if } r \ge b_1 + k(\alpha-1)\sigma^2, \\ d_k, & \text{otherwise.} \end{cases}$ †
Constraints on short-selling ($K_+ = [-k,\infty)$, $K_- = (-\infty, l]$, $k \ge 0$, $l > 1$)	$\ge \rho_l$	$u_0(r)$	not appropriate ($\hat{p} < h_{\text{low}}$)
Constraints on borrowing ($K_+ = (-\infty, k]$, $K_- = [l,\infty)$, $k > 1$, $l \le 1$)	$u_0(r)$	$\le a_k$	not appropriate ($\hat{p} < h_{\text{low}}$)
Market with higher interest rate $R > r$ for borrowing	$u_0(r)$	$u_0(R)$	$\begin{cases} u_0(r), & \text{if } r \ge f,^\dagger \\ u_0(f), & \text{if } r \le f \le R, \\ u_0(R), & \text{if } f \ge R. \end{cases}$

*For arbitrary utility function.
†For utility function $U_\alpha(\cdot)$ of the form (7.27) with $0 \le \alpha < 1$.

arbitrage prices relatively easy to use. In contrast, a main drawback of the fair price is that it *does* depend on b. Heuristically, it may well be that hedging, as it is based on the arbitrage arguments, is a sort of "global" property. On the other hand, Definition 7.3 of the fair price looks like a "local" property, as it involves a derivative; this makes the fair price $\hat{p}$ more likely to depend on the local "drift" b (appreciation rate) of the price process.

11. Discussion.

1. With a little additional care, the method also works for the European option with dividend rate $g(t)$. For example, the analogue for Theorem 6.1 will be

$$h_{\text{low}} = \inf_{\nu \in \tilde{\mathscr{D}}} \mathsf{E}^\nu\left[\tilde{\gamma}_\nu(T)B(T) + \int_0^T \tilde{\gamma}_\nu(s)g(s)\,ds\right]$$

and

$$h_{\text{up}} = \sup_{\nu \in \mathscr{D}} \mathsf{E}^{\nu}\left[\gamma_{\nu}(T)B(T) + \int_0^T \gamma_{\nu}(s)g(s)\,ds\right].$$

2. A similar lower price h_{low} (for "buyers," as opposed to the h_{up} which refers to "sellers") was mentioned by El Karoui and Quenez (1995) in the incomplete market case, but without justification based on considerations of arbitrage.
3. Suppose we want to consider constraints on the number of shares ϕ or on the total amount of money invested in every asset instead of on the vector π of the proportions of wealth invested in assets. Then the general arbitrage arguments in Section 5 still hold. However, we no longer have an easy way to get all the representations of Section 6. For instance, the nice equation (6.24) is changed, as the very helpful term $\delta(\nu(s)) + \nu^*(s)\pi(s)$ disappears.
4. For practical purposes, one may recommend the use of the Black–Scholes price u_0 as a "rough-and-ready" unique price, for a constrained market with the same interest rate for borrowing and saving, when the fair price $\hat{p}$ is difficult to compute. The reasons are:
   (a) The Black–Scholes price u_0 *always* lies within the arbitrage-free interval $[h_{\text{low}}, h_{\text{up}}]$.
   (b) As we saw in the case of constraints on borrowing and short-selling, the arbitrage-free interval will shrink to u_0 as the constraints become weaker and weaker.
   (c) The Black–Scholes price u_0 does not involve the stock appreciation rate b.
   (d) Many numerical procedures, including software, have been developed to calculate u_0.

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