

# A Jump-Diffusion Model for Option Pricing

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**B**rownian motion and normal distribution have been widely used in the Black–Scholes option-pricing framework to model the return of assets. However, two puzzles emerge from many empirical investigations: the leptokurtic feature that the return distribution of assets may have a higher peak and two (asymmetric) heavier tails than those of the normal distribution, and an empirical phenomenon called “volatility smile” in option markets. To incorporate both of them and to strike a balance between reality and tractability, this paper proposes, for the purpose of option pricing, a double exponential jump-diffusion model. In particular, the model is simple enough to produce analytical solutions for a variety of option-pricing problems, including call and put options, interest rate derivatives, and path-dependent options. Equilibrium analysis and a psychological interpretation of the model are also presented.

*(Contingent Claims; High Peak; Heavy Tails; Interest Rate Models; Rational Expectations; Overreaction and Underreaction)*

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## 1. Introduction

Despite the success of the Black–Scholes model based on Brownian motion and normal distribution, two empirical phenomena have received much attention recently: (1) the asymmetric leptokurtic features—in other words, the return distribution is skewed to the left, and has a higher peak and two heavier tails than those of the normal distribution, and (2) the volatility smile. More precisely, if the Black–Scholes model is correct, then the implied volatility should be constant. In reality, it is widely recognized that the implied volatility curve resembles a “smile,” meaning it is a convex curve of the strike price.

Many studies have been conducted to modify the Black–Scholes model to explain the two empirical phenomena. To incorporate the asymmetric leptokurtic features in asset pricing, a variety of models have been proposed:<sup>1</sup> (a) chaos theory,

fractal Brownian motion, and stable processes; see, for example, Mandelbrot (1963), Rogers (1997), Samorodnitsky and Taqqu (1994); (b) generalized hyperbolic models, including log  $t$  model and log hyperbolic model; see, for example, Barndorff-Nielsen and Shephard (2001), Blattberg and Gonedes (1974); (c) time-changed Brownian motions; see, for example, Clark (1973), Madan and Seneta (1990), Madan et al. (1998), and Heyde (2000). An immediate problem with these models is that it may be difficult to obtain analytical solutions for option prices. More precisely, they might give some analytical formulae for standard European call and put options, but any analytical solutions for interest rate derivatives and path-dependent options, such as perpetual American options, barrier, and lookback options, are unlikely.

In a parallel development, different models are also proposed to incorporate the “volatility smile” in option pricing. Popular ones include: (a) stochastic volatility and ARCH models; see, for example, Hull and White (1987), Engle (1995), Fouque et al. (2000); (b) constant elasticity model (CEV) model; see,

<sup>1</sup> Although most of the studies focus on the leptokurtic features under the physical measure, it is worth mentioning that the leptokurtic features under the risk-neutral measure(s) lead to the “volatility smiles” in option prices.

for example, Cox and Ross (1976), and Davydov and Linetsky (2001); (c) normal jump models proposed by Merton (1976); (d) affine stochastic-volatility and affine jump-diffusion models; see, for example, Heston (1993), and Duffie et al. (2000); (e) models based on Lévy processes; see, for example, Geman et al. (2001) and references therein; (f) a numerical procedure called “implied binomial trees”; see, for example, Derman and Kani (1994) and Dupire (1994). Aside from the problem that it might not be easy to find analytical solutions for option pricing, especially for path-dependent options (such as perpetual American options, barrier, and lookback options), some of these models may not produce the asymmetric leptokurtic feature (see §2.3).

The current paper proposes a new model with the following properties: (a) It offers an explanation for two empirical phenomena—the asymmetric leptokurtic feature, and the volatility smile (see §§3 and 5.3). (b) It leads to analytical solutions to many option-pricing problems, including European call and put options (see §5); interest rate derivatives, such as swaptions, caps, floors, and bond options (see §5.3 and Glasserman and Kou 1999); path-dependent options, such as perpetual American options, barrier, and lookback options (see §2.3 and Kou and Wang 2000, 2001). (c) It can be embedded into a rational expectations equilibrium framework (see §4). (d) It has a psychological interpretation (see §2.2).

The model is very simple. The logarithm of the asset price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponentially distributed. Because of its simplicity, the parameters in the model can be easily interpreted, and the analytical solutions for option pricing can be obtained. The explicit calculation is made possible partly because of the memoryless property of the double exponential distribution.

The paper is organized as follows. In §2, the model is proposed, is evaluated by four criteria, and is compared with other alternative models. Section 3 studies the leptokurtic feature. A rational expectations equilibrium justification of the model is given in §4. Some preliminary results, including the Hh functions, are given in §5.1. Formulae for option-pricing problems, including options on futures, are provided in §5.2.

The “volatility smiles” phenomenon is illustrated in §5.3. The final section discusses some limitations of the model.

## 2. The Model

### 2.1. The Model Formulation

The following dynamic is proposed to model the asset price,  $S(t)$ , under the physical probability measure  $\mathbf{P}$ :

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + d \left( \sum_{i=1}^{N(t)} (V_i - 1) \right), \quad (1)$$

where  $W(t)$  is a standard Brownian motion,  $N(t)$  is a Poisson process with rate  $\lambda$ , and  $\{V_i\}$  is a sequence of independent identically distributed (i.i.d.) nonnegative random variables such that  $Y = \log(V)$  has an asymmetric double exponential distribution<sup>2,3</sup> with the density

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y < 0\}}, \\ \eta_1 > 1, \eta_2 > 0,$$

where  $p, q \geq 0, p + q = 1$ , represent the probabilities of upward and downward jumps. In other words,

$$\log(V) = Y \stackrel{d}{=} \begin{cases} \xi^+, & \text{with probability } p \\ -\xi^-, & \text{with probability } q \end{cases}, \quad (2)$$

where  $\xi^+$  and  $\xi^-$  are exponential random variables with means  $1/\eta_1$  and  $1/\eta_2$ , respectively, and the notation  $\stackrel{d}{=}$  means equal in distribution. In the model, all sources of randomness,  $N(t)$ ,  $W(t)$ , and  $Y$ s, are assumed to be independent, although this can be relaxed, as will be suggested in §2.2. For notational simplicity and in order to get analytical solutions for various option-pricing problems, the drift  $\mu$  and the volatility  $\sigma$  are assumed to be constants, and the Brownian motion and jumps are assumed to be

<sup>2</sup> If  $\eta_1 = \eta_2$  and  $p = 1/2$ , then the double exponential distribution is also called “the first law of Laplace” (proposed by Laplace in 1774), while the “second law of Laplace” is the normal density.

<sup>3</sup> Ramezani and Zeng (1999) independently propose the same jump-diffusion model from an econometric viewpoint as a way of improving the empirical fit of Merton’s normal jump-diffusion model to stock price data.

one dimensional. These assumptions, however, can be easily dropped to develop a general theory.

Solving the stochastic differential equation (1) gives the dynamics of the asset price:

$$S(t) = S(0) \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right\} \prod_{i=1}^{N(t)} V_i. \quad (3)$$

Note that  $E(Y) = \frac{p}{\eta_1} - \frac{q}{\eta_2}$ ,  $\text{Var}(Y) = pq \left( \frac{1}{\eta_1} + \frac{1}{\eta_2} \right)^2 + \left( \frac{p}{\eta_1^2} + \frac{q}{\eta_2^2} \right)$ , and

$$\begin{aligned} E(V) &= E(e^Y) \\ &= q \frac{\eta_2}{\eta_2 + 1} + p \frac{\eta_1}{\eta_1 - 1}, \quad \eta_1 > 1, \quad \eta_2 > 0. \end{aligned} \quad (4)$$

The requirement  $\eta_1 > 1$  is needed to ensure that  $E(V) < \infty$  and  $E(S(t)) < \infty$ ; it essentially means that the average upward jump cannot exceed 100%, which is quite reasonable.

There are two interesting properties of the double exponential distribution that are crucial for the model. First, it has the leptokurtic feature; see Johnson et al. (1995). As will be shown in §3, the leptokurtic feature of the jump size distribution is inherited by the return distribution. Secondly, a unique feature (also inherited from the exponential distribution) of the double exponential distribution is the memoryless property. This special property explains why the closed-form solutions for various option-pricing problems, including barrier, lookback, and perpetual American options, are feasible under the double exponential jump-diffusion model while it seems impossible for many other models, including the normal jump-diffusion model (Merton 1976); see §2.3 for details.

## 2.2. Evaluating the Model

Because essentially all models are “wrong” and rough approximations of reality, instead of arguing the “correctness” of the proposed model I shall evaluate and justify the double exponential jump-diffusion model by four criteria.

1. A model must be internally self-consistent. In the finance context, it means that a model must be arbitrage-free and can be embedded in an equilibrium setting. Note that some of the alternative models may

have arbitrage opportunities, and thus are not self-consistent (to give an example, it is shown by Rogers 1997 that models using fractal Brownian motion may lead to arbitrage opportunities). The double exponential jump-diffusion model can be embedded in a rational expectations equilibrium setting; see §4.

2. A model should be able to capture some important empirical phenomena. The double exponential jump-diffusion model is able to reproduce the leptokurtic feature of the return distribution (see §3) and the “volatility smile” observed in option prices (see §5.3). In addition, the empirical tests performed in Ramezani and Zeng (1999) suggest that the double exponential jump-diffusion model fits stock data better than the normal jump-diffusion model. Andersen et al. (1999) demonstrate empirically that, for the S&P 500 data from 1980–1996, the normal jump-diffusion model has a much higher  $p$ -value (0.0152) than those of the stochastic volatility model (0.0008) and the Black–Scholes model ( $< 10^{-5}$ ). Therefore, the combination of results in the two papers gives some empirical support of the double exponential jump-diffusion model.<sup>4</sup>

3. A model must be simple enough to be amenable to computation. Like the Black–Scholes model, the double exponential jump-diffusion model not only yields closed-form solutions for standard call and put options (see §5), but also leads to a variety of closed-form solutions for path-dependent options, such as barrier options, lookback options, and perpetual American options (see §2.3 and Kou and Wang 2000, 2001), as well as interest rate derivatives (see §5.3 and Glasserman and Kou 1999).

4. A model must have some (economical, physical, psychological, etc.) interpretation. One motivation for the double exponential jump-diffusion model comes from behavioral finance. It has been suggested from extensive empirical studies that markets tend to have

<sup>4</sup> However, we should emphasize that empirical tests should not be used as the only criterion to judge a model good or bad. Empirical tests tend to favor models with more parameters. However, models with many parameters tend to make calibration more difficult (the calibration may involve high-dimensional numerical optimization with many local optima), and tend to have less tractability. This is a part of the reason why practitioners still like the simplicity of the Black–Scholes model.

both overreaction and underreaction to various good or bad news (see, for example, Fama 1998 and Barberis et al. 1998, and references therein). One may interpret the jump part of the model as the market response to outside news. More precisely, in the absence of outside news the asset price simply follows a geometric Brownian motion. Good or bad news arrives according to a Poisson process, and the asset price changes in response according to the jump size distribution.<sup>5</sup> Because the double exponential distribution<sup>5</sup> has both a high peak and heavy tails, it can be used to model both the overreaction (attributed to the heavy tails) and underreaction (attributed to the high peak) to outside news. Therefore, the double exponential jump-diffusion model can be interpreted as an attempt to build a simple model, within the traditional random walk and efficient market framework, to incorporate investors' sentiment.

Incidentally, as a by-product, the model also suggests that the fact of markets having both overreaction and underreaction to outside news can lead to the leptokurtic feature of asset return distribution.

### 2.3. Comparison with Other Models

There are many alternative models that can satisfy at least some of the four criteria listed above. A main attraction of the double exponential jump-diffusion model is its *simplicity, particularly its analytical tractability for path-dependent options and interest rate derivatives*. Unlike the original Black-Scholes model, many alternative models can only compute prices for standard call and put options, and analytical solutions for other equity derivatives (such as path-dependent options) and some most liquid interest rate derivatives (such as swaption, caps, and floors) are unlikely. Even numerical methods for interest rate derivatives and path-dependent options are not easy, as the convergence rates of binomial trees and Monte Carlo simulation for path-dependent options are typically much slower than those for call and put options (for a survey, see Boyle et al. 1997).

<sup>5</sup> Interestingly enough, the double exponential distribution has been widely used in mathematical psychology literature, particularly in vision cognitive studies; see, for example, the list of papers on the web page of David Mumford at the computer vision group, Brown University.

This makes it harder to persuade practitioners to switch from the Black-Scholes model to more realistic alternative models. The double exponential jump-diffusion model attempts to improve the empirical implications of the Black-Scholes model while still retaining its analytical tractability. Below is a more detailed comparison between the proposed model and some popular alternative models.

(1) The CEV Model. Like the double exponential jump-diffusion model, analytical solutions for path-dependent options (see Davydov and Linetsky 2001) and interest rate derivatives (e.g., Cox et al. 1985 and Andersen and Andersen 2000) are available under the CEV model. However, the CEV model does not have the leptokurtic feature. More precisely, the return distribution in the CEV model has a thinner right tail than that of the normal distribution. This undesirable feature also has a consequence in terms of the implied volatility in option pricing. Under the CEV model the implied volatility can only be a monotone function of the strike price. Therefore, if the implied volatility is a convex function (but not necessarily a decreasing function), as frequently observed in option markets, the CEV model is unable to reproduce the implied volatility curve.

(2) The Normal Jump-Diffusion Model. Merton (1976) was the first to consider a jump-diffusion model similar to (1) and (3). In Merton's paper  $Y_t$ s are normally distributed. Both the double exponential and normal jump-diffusion models can lead to the leptokurtic feature (although the kurtosis from the double exponential jump-diffusion model is significantly more pronounced), implied volatility smile, and analytical solutions for call and put options, and interest rate derivatives (such as caps, floors, and swaptions; see Glasserman and Kou 1999). The main difference between the double exponential jump-diffusion model and the normal jump-diffusion model is the analytical tractability for the path-dependent options.

Here I provide some intuition to understand why the double exponential jump-diffusion model can lead to closed-form solutions for path-dependent options, while the normal jump-diffusion model cannot. To price perpetual American options, barrier options,

and lookback options for general jump-diffusion processes, it is crucial to study the first passage time of a jump-diffusion process to a flat boundary. When a jump-diffusion process crosses a boundary sometimes it hits the boundary exactly and sometimes it incurs an "overshoot" over the boundary.

The overshoot presents several problems for option pricing. First, one needs to get the exact distribution of the overshoot. It is well known from stochastic renewal theory that this is only possible if the jump size  $Y$  has an exponential-type distribution, thanks to the special memoryless property of the exponential distribution. Secondly, one needs to know the dependent structure between the overshoot and the first passage time. The two random variables are conditionally independent, given that the overshoot is bigger than 0, if the jump size  $Y$  has an exponential-type distribution, thanks to the memoryless property. This conditionally independent structure seems to be very special to the exponential-type distribution and does not hold for other distributions, such as the normal distribution.

Consequently, analytical solutions for the perpetual American, lookback, and barrier options can be derived for the double exponential jump-diffusion model. However, it seems impossible to get similar results for other jump-diffusion processes, including the normal jump-diffusion model.

(3) Models Based on  $t$ -Distribution. The  $t$ -distribution is widely used in empirical studies of asset pricing. One problem with  $t$ -distribution (or other distributions with power-type tails) as a return distribution is that it cannot be used in models with continuous compounding. More precisely, suppose that at time 0 the daily return distribution  $X$  has a power-type right tail. Then in models with continuous compounding, the asset price tomorrow  $A(\Delta t)$  is given by  $A(\Delta t) = A(0)e^X$ . Since  $X$  has a power-type right tail, it is clear that  $E(e^X) = \infty$ . Consequently,

$$E(A(\Delta t)) = E(A(0)e^X) = A(0)E(e^X) = \infty.$$

In other words, the asset price tomorrow has an infinite expectation. This paradox holds for  $t$ -distribution with any degrees of freedom, as long as one considers models with continuous compounding. Furthermore,

if the risk-neutral return also has a power-type right tail, then the call option price is also infinite:

$$E^*((A(\Delta t) - K)^+) \geq E^*(A(\Delta t) - K) = \infty.$$

Therefore, the only relevant models with  $t$ -distributed returns are models with discretely compounded returns. However, in models with discrete compounding, closed-form solutions are in general impossible.<sup>6</sup>

(4) Stochastic Volatility Models. The double exponential jump-diffusion model and the stochastic volatility model complement each other: The stochastic volatility model can incorporate dependent structure better, while the double exponential jump-diffusion model has better analytical tractability, especially for path-dependent options and complex interest rate derivatives. One empirical phenomenon worth-mentioning is that the daily return distribution tends to have more kurtosis than the distribution of monthly returns. As Das and Foresi (1996) point out, this is consistent with models with jumps, but inconsistent with stochastic volatility models. More precisely, in stochastic volatility models (or essentially any models in a pure diffusion setting) the kurtosis decreases as the sampling frequency increases, while in jump models the instantaneous jumps are independent of the sampling frequency.

(5) Affine Jump-Diffusion Models. Duffie et al. (2000) propose a very general class of affine jump-diffusion models which can incorporate jumps, stochastic volatility, and jumps in volatility. Both normal and double exponential jump-diffusion models can be viewed as special cases of their model. However, because of the special features of the exponential distribution, the double exponential jump-diffusion model leads to analytical solutions for path-dependent options, which are difficult for other affine jump-diffusion models (even numerical methods are not easy). Furthermore, the double exponential model is simpler than general affine

<sup>6</sup> Another interesting point worth mentioning is that, for a sample size of 5,000 (20-years-daily data), it may be very difficult to distinguish empirically the double exponential distribution from the power-type distributions, such as  $t$ -distribution (although it is quite easy to detect the differences between them and the normal density).

jump-diffusion models: It has fewer parameters that makes calibration easier. The double exponential jump-diffusion model attempts to strike a balance between reality and tractability.

(6) Models Based on Lévy Processes (the processes with independent and stationary increments). Although the double exponential jump-diffusion model is a special case of Lévy processes, because of the special features of the exponential distribution it has analytical tractability for path-dependent options and interest rate derivatives, which are difficult for other Lévy processes.

### 3. Leptokurtic Feature

Using (3), the return over a time interval  $\Delta t$  is given by:

$$\begin{aligned} \frac{\Delta S(t)}{S(t)} &= \frac{S(t + \Delta t)}{S(t)} - 1 \\ &= \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (W(t + \Delta t) - W(t)) \right. \\ &\quad \left. + \sum_{i=N(t)+1}^{N(t+\Delta t)} Y_i \right\} - 1, \end{aligned}$$

where the summation over an empty set is taken to be zero. If the time interval  $\Delta t$  is small, as in the case of daily observations, the return can be approximated in distribution, ignoring the terms with orders higher than  $\Delta t$  and using the expansion  $e^x \approx 1 + x + x^2/2$ , by

$$\frac{\Delta S(t)}{S(t)} \approx \mu \Delta t + \sigma Z \sqrt{\Delta t} + B \cdot Y, \tag{5}$$

where  $Z$  and  $B$  are standard normal and Bernoulli random variables, respectively, with  $P(B = 1) = \lambda \Delta t$  and  $P(B = 0) = 1 - \lambda \Delta t$ , and  $Y$  is given by (2).

The density<sup>7</sup>  $g$  of the right-hand side of (5), being an approximation for the return  $\Delta S(t)/S(t)$ , is plotted

<sup>7</sup>The density

$$\begin{aligned} g(x) &= \frac{1 - \lambda \Delta t}{\sigma \sqrt{\Delta t}} \varphi \left( \frac{x - \mu \Delta t}{\sigma \sqrt{\Delta t}} \right) \\ &\quad + \lambda \Delta t \left\{ p \eta_1 e^{(\sigma^2 \eta_1^2 \Delta t)/2} e^{-(x - \mu \Delta t) \eta_1} \Phi \left( \frac{x - \mu \Delta t - \sigma^2 \eta_1 \Delta t}{\sigma \sqrt{\Delta t}} \right) \right. \\ &\quad \left. + q \eta_2 e^{(\sigma^2 \eta_2^2 \Delta t)/2} e^{(x - \mu \Delta t) \eta_2} \right. \\ &\quad \left. \times \Phi \left( - \frac{x - \mu \Delta t + \sigma^2 \eta_2 \Delta t}{\sigma \sqrt{\Delta t}} \right) \right\}, \end{aligned}$$

in Figure 1 along with the normal density with the same mean and variance. The parameters are  $\Delta t = 1 \text{ day} = 1/250 \text{ year}$ ,  $\sigma = 20\%$  per year,  $\mu = 15\%$  per year,  $\lambda = 10$  per year,  $p = 0.30$ ,  $1/\eta_1 = 2\%$ , and  $1/\eta_2 = 4\%$ . In this case,  $E(Y) = -2.2\%$ , and  $SD(Y) = 4.47\%$ . In other words, there are about 10 jumps per year with the average jump size  $-2.2\%$ , and the jump volatility  $4.47\%$ . The jump parameters used here seem to be quite reasonable, if not conservative, for the U.S. stocks.

The leptokurtic feature is quite evident. The peak of the density  $g$  is about 31, whereas that of the normal density is about 25. The density  $g$  has heavier tails than the normal density, especially for the left tail, which could reach well below  $-10\%$ , while the normal density is basically confined within  $-6\%$ . Additional numerical plots suggest that the feature of having a higher peak and heavier tails becomes more pronounced if either  $1/\eta_i$  (the jump size expectations) or  $\lambda$  (the jump rate) increases.

### 4. Equilibrium for General Jump-Diffusion Models

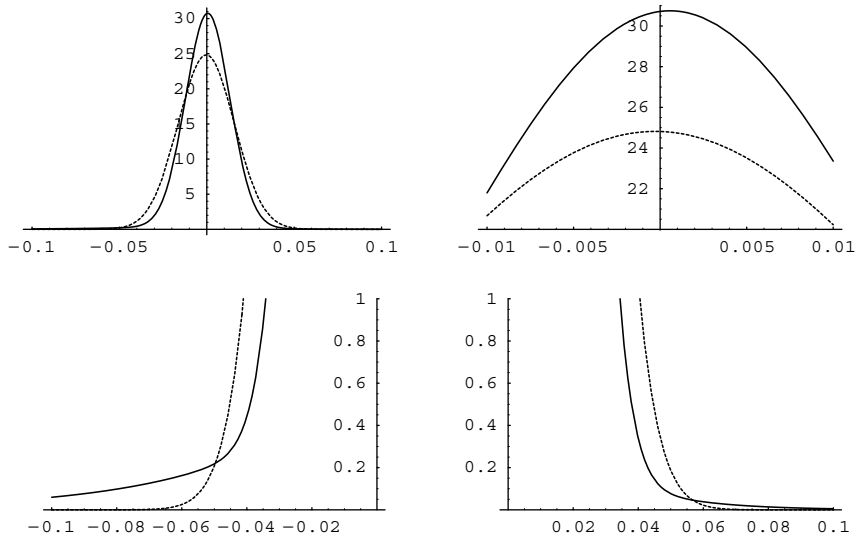
Consider a typical rational expectations economy (Lucas 1978) in which a representative investor tries to solve a utility maximization problem  $\max_c E[\int_0^\infty U(c(t), t) dt]$ , where  $U(c(t), t)$  is the utility function of the consumption process  $c(t)$ . There is an exogenous endowment process, denoted by  $\delta(t)$ , available to the investor. Also given to the investor is an opportunity to invest in a security (with a finite liquidation date  $T_0$ , although  $T_0$  can be very large) which pays no dividends. If  $\delta(t)$  is Markovian, it can be shown (see, for example, pp. 484–485 in Stokey and Lucas 1989) that, under mild conditions, the rational expectations equilibrium price (also called the

with

$$\begin{aligned} E_g(G) &= \mu \Delta t + \lambda \left( \frac{p}{\eta_1} - \frac{q}{\eta_2} \right) \Delta t, \\ \text{Var}_g(G) &= \sigma^2 \Delta t + \left\{ pq \left( \frac{1}{\eta_1} + \frac{1}{\eta_2} \right)^2 + \left( \frac{p}{\eta_1^2} + \frac{q}{\eta_2^2} \right) \right\} \lambda \Delta t \\ &\quad + \left( \frac{p}{\eta_1} - \frac{q}{\eta_2} \right)^2 \lambda \Delta t (1 - \lambda \Delta t), \end{aligned}$$

where  $\varphi(\cdot)$  is the standard normal density function.

**Figure 1**



*Notes.* The first panel compares the overall shapes of the density  $g$  and the normal density with the same mean and variance, the second one details the shapes around the peak area, and the last two show the left and right tails. The dotted line is used for the normal density, and the solid line is used for the model.

“shadow” price) of the security,  $p(t)$ , must satisfy the Euler equation

$$p(t) = \frac{\mathbf{E}(U_c(\delta(T), T)p(T) | \mathcal{F}_t)}{U_c(\delta(t), t)}, \quad \forall T \in [t, T_0], \quad (6)$$

where  $U_c$  is the partial derivative of  $U$  with respect to  $c$ . At this price  $p(t)$ , the investor will never change his/her current holdings to invest in (either long or short) the security, even though he/she is given the opportunity to do so. Instead, in equilibrium the investor finds it optimal to just consume the exogenous endowment; i.e.,  $c(t) = \delta(t)$  for all  $t \geq 0$ .

In this section I shall derive explicitly the implications of the Euler equation (6) when the endowment process  $\delta(t)$  follows a general jump-diffusion process under the physical measure  $\mathbf{P}$ :

$$\frac{d\delta(t)}{\delta(t-)} = \mu_1 dt + \sigma_1 dW_1(t) + d \left[ \sum_{i=1}^{N(t)} (\tilde{V}_i - 1) \right], \quad (7)$$

where the  $\tilde{V}_i \geq 0$  are any independent identically distributed, nonnegative random variables. In addition, all three sources of randomness, the Poisson process  $N(t)$ , the standard Brownian motion  $W_1(t)$ , and the jump sizes  $\tilde{V}_i$ , are assumed to be independent.

Although it is intuitively clear that, generally speaking, the asset price  $p(t)$  should follow a similar jump-diffusion process as that of the dividend process  $\delta(t)$ , a careful study of the connection between the two is needed. This is because  $p(t)$  and  $\delta(t)$  may not have similar jump dynamics (see the remark after Corollary 1). Furthermore, deriving explicitly the change of parameters from  $\delta(t)$  to  $p(t)$  also provides some valuable information about the risk premiums embedded in jump-diffusion models.

The work in this section builds upon and extends the previous work by Naik and Lee (1990), in which the special case that  $\tilde{V}_i$  has a lognormal distribution is investigated. Another difference is that Naik and Lee (1990) require that the asset pays continuous dividends and there is no outside endowment process, while here the asset pays no dividends and there is an outside endowment process. Consequently, the pricing formulae are different even in the case of lognormal jumps.

For simplicity, as in Naik and Lee (1990), I shall only consider the utility function of the special forms  $U(c, t) = e^{-\theta t} \frac{c^\alpha}{\alpha}$  if  $0 < \alpha < 1$ , and  $U(c, t) = e^{-\theta t} \log(c)$  if  $\alpha = 0$ , where  $\theta > 0$  (although most of the results below

hold for more general utility functions). Under these types of utility functions, the rational expectations equilibrium price of (6) becomes

$$p(t) = \frac{\mathbf{E}(e^{-\theta T} (\delta(T))^{\alpha-1} p(T) \mid \mathcal{F}_t)}{e^{-\theta t} (\delta(t))^{\alpha-1}}. \quad (8)$$

ASSUMPTION. The discount rate  $\theta$  should be large enough so that

$$\theta > -(1-\alpha)\mu_1 + \frac{1}{2}\sigma_1^2(1-\alpha)(2-\alpha) + \lambda \zeta_1^{\alpha-1},$$

where the notation  $\zeta_1^{(a)}$  means  $\zeta_1^{(a)} := \mathbf{E}[(\tilde{V})^a - 1]$ .

As will be seen in Proposition 1, this assumption guarantees that in equilibrium the term structure of interest rates is positive.

PROPOSITION 1. Suppose  $\zeta_1^{\alpha-1} < \infty$ . (1) Letting  $B(t, T)$  be the price of a zero coupon bond with maturity  $T$ , the yield  $r := -(1/(T-t))\log(B(t, T))$  is a constant independent of  $T$ ,

$$r = \theta + (1-\alpha)\mu_1 - \frac{1}{2}\sigma_1^2(1-\alpha)(2-\alpha) - \lambda \zeta_1^{\alpha-1} > 0. \quad (9)$$

(2) Let  $Z(t) := e^{rt} U_c(\delta(t), t) = e^{(r-\theta)t} (\delta(t))^{\alpha-1}$ . Then  $Z(t)$  is a martingale under  $\mathbf{P}$ ,

$$\frac{dZ(t)}{Z(t-)} = -\lambda \zeta_1^{\alpha-1} dt + \sigma_1(\alpha-1) dW_1(t) + d\left[\sum_{i=1}^{N(t)} (\tilde{V}_i^{\alpha-1} - 1)\right]. \quad (10)$$

Using  $Z(t)$ , one can define a new probability measure  $\mathbf{P}^*$ :  $d\mathbf{P}^*/d\mathbf{P} := Z(t)/Z(0)$ . Under  $\mathbf{P}^*$ , the Euler Equation (8) holds if and only if the asset price satisfies

$$S(t) = e^{-r(T-t)} \mathbf{E}^*(S(T) \mid \mathcal{F}_t), \quad \forall T \in [t, T_0]. \quad (11)$$

Furthermore, the rational expectations equilibrium price of a (possibly path-dependent) European option, with the payoff  $\psi_S(T)$  at the maturity  $T$ , is given by

$$\psi_S(t) = e^{-r(T-t)} \mathbf{E}^*(\psi_S(T) \mid \mathcal{F}_t), \quad \forall t \in [0, T]. \quad (12)$$

PROOF. See Appendix A.  $\square$

Given the endowment process  $\delta(t)$ , it must be decided what stochastic processes are suitable for the asset price  $S(t)$  to satisfy the equilibrium requirement (8) or (11). I now postulate a special jump-diffusion form for  $S(t)$ ,

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma \{ \rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t) \} + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right), \quad V_i = \tilde{V}_i^\beta, \quad (13)$$

where  $W_2(t)$  is a Brownian motion independent of  $W_1(t)$ . In other words, the same Poisson process affects both the endowment  $\delta(t)$  and the asset price  $S(t)$ , and the jump sizes are related through a power function, where the power  $\beta \in (-\infty, \infty)$  is an arbitrary constant. The diffusion coefficients and the Brownian motion part of  $\delta(t)$  and  $S(t)$ , though, are totally different. It remains to determine what constraints should be imposed on this model so that the jump-diffusion model can be embedded in the rational expectations equilibrium requirement (8) or (11).

THEOREM 1. Suppose  $\zeta_1^{\alpha+\beta-1} < \infty$  and  $\zeta_1^{\alpha-1} < \infty$ . The model (13) satisfies the equilibrium requirement (11) if and only if

$$\begin{aligned} \mu &= r + \sigma_1 \sigma \rho (1-\alpha) - \lambda (\zeta_1^{\alpha+\beta-1} - \zeta_1^{\alpha-1}) \\ &= \theta + (1-\alpha) \left\{ \mu_1 - \frac{1}{2} \sigma_1^2 (2-\alpha) + \sigma_1 \sigma \rho \right\} \\ &\quad - \lambda \zeta_1^{\alpha+\beta-1}. \end{aligned} \quad (14)$$

If (14) is satisfied, then under  $\mathbf{P}^*$ ,

$$\begin{aligned} \frac{dS(t)}{S(t-)} &= r dt - \lambda^* \mathbf{E}^*(\tilde{V}_i^\beta - 1) dt \\ &\quad + \sigma dW^*(t) + d\left[\sum_{i=1}^{N(t)} (\tilde{V}_i^\beta - 1)\right]. \end{aligned} \quad (15)$$

Here, under  $\mathbf{P}^*$ ,  $W^*(t)$  is a new Brownian motion,  $N(t)$  is a new Poisson process with jump rate  $\lambda^* = \lambda \mathbf{E}(\tilde{V}_i^{\alpha-1}) = \lambda (\zeta_1^{\alpha-1} + 1)$ , and  $\{\tilde{V}_i\}$  are independent identically distributed random variables with a new density under  $\mathbf{P}^*$ :

$$f_{\tilde{V}}^*(x) = \frac{1}{\zeta_1^{\alpha-1} + 1} x^{\alpha-1} f_{\tilde{V}}(x). \quad (16)$$

PROOF. See Appendix A.  $\square$

The following corollary gives a condition under which all three dynamics,  $\delta(t)$  and  $S(t)$  under  $\mathbf{P}$  and  $S(t)$  under  $\mathbf{P}^*$ , have the same jump-diffusion form, which is very convenient for analytical calculation.

**COROLLARY 1.** *Suppose the family  $\mathcal{V}$  of distributions of the jump size  $\tilde{V}$  for the endowment process  $\delta(t)$  satisfies that, for any real numbers  $a \in [0, 1)$  and  $b \in (-\infty, \infty)$ ,*

$$\tilde{V}^b \in \mathcal{V} \text{ and } \text{const} \cdot x^{a-1} f_{\tilde{V}}(x) \in \mathcal{V}, \quad (17)$$

where the normalizing constant, *const*, is  $\{\zeta_1^{(a-1)} + 1\}^{-1}$  (provided that  $\zeta_1^{(a-1)} < \infty$ ). Then the jump sizes for the asset price  $S(t)$  under  $\mathbf{P}$  and the jump sizes for  $S(t)$  under the rational expectations risk-neutral measure  $\mathbf{P}^*$  all belong to the same family  $\mathcal{V}$ .

PROOF. Immediately follows from (7), (13), and (16).  $\square$

Condition (17) essentially requires that the jump size distribution belongs to the exponential family. It is satisfied if  $\log(V)$  has a normal distribution or a double exponential distribution. However, the log power-type distributions, such as log  $t$ -distribution, do not satisfy (17).

## 5. Option Pricing

In this section I will compute the rational expectations equilibrium option-pricing formula (12) explicitly for the European call and put options. For notational simplicity, I will drop  $*$  in the risk-neutral notation, i.e., write  $\eta_1$  instead of  $\eta_1^*$ , etc. To compute (12), one has to study the distribution of the sum of the double exponential random variables and normal random variables. Fortunately, this distribution can be obtained in closed form in terms of the Hh function, a special function of mathematical physics.

### 5.1. Hh Functions

For every  $n \geq 0$ , the Hh function is a nonincreasing function defined by:

$$\text{Hh}_n(x) = \int_x^\infty \text{Hh}_{n-1}(y) dy = \frac{1}{n!} \int_x^\infty (t-x)^n e^{-t^2/2} dt \geq 0, \quad n = 0, 1, 2, \dots \quad (18)$$

$$\text{Hh}_{-1}(x) = e^{-x^2/2} = \sqrt{2\pi}\varphi(x), \quad \text{Hh}_0(x) = \sqrt{2\pi}\Phi(-x);$$

see Abramowitz and Stegun (1972, p. 691). The Hh function can be viewed as a generalization of the cumulative normal distribution function.

The integral in (18) can be evaluated very fast by many software packages (for example, Mathematica).<sup>8</sup> In addition,

$$\text{Hh}_n(x) = 2^{-n/2} \sqrt{\pi} e^{-x^2/2} \times \left\{ \frac{{}_1F_1(\frac{1}{2}n + \frac{1}{2}, \frac{1}{2}, \frac{1}{2}x^2)}{\sqrt{2}\Gamma(1 + \frac{1}{2}n)} - x \frac{{}_1F_1(\frac{1}{2}n + 1, \frac{3}{2}, \frac{1}{2}x^2)}{\Gamma(\frac{1}{2} + \frac{1}{2}n)} \right\},$$

where  ${}_1F_1$  is the confluent hypergeometric function. A three-term recursion is also available for the Hh function (see pp. 299–300 and p. 691 of Abramowitz and Stegun 1972):

$$n\text{Hh}_n(x) = \text{Hh}_{n-2}(x) - x\text{Hh}_{n-1}(x), \quad n \geq 1. \quad (19)$$

Therefore, one can compute all  $\text{Hh}_n(x)$ ,  $n \geq 1$ , by using the normal density function and normal distribution function. The Hh function is illustrated in Figure 2.

### 5.2. European Call and Put Options

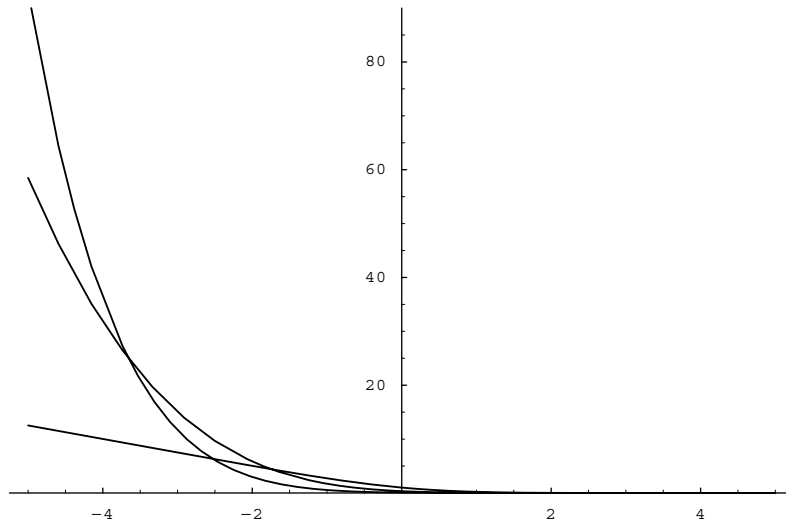
Introduce the following notation: For any given probability  $\mathbf{P}$ , define

$$\Upsilon(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) := \mathbf{P}\{Z(T) \geq a\},$$

where  $Z(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$ ,  $Y$  has a double exponential distribution with density  $f_Y(y) \sim p \cdot \eta_1 e^{-\eta_1 y} \mathbf{1}_{\{y \geq 0\}} + q \cdot \eta_2 e^{y \eta_2} \mathbf{1}_{\{y < 0\}}$ , and  $N(t)$  is a Poisson process with rate  $\lambda$ . The pricing formula of the call option will be expressed in terms of  $\Upsilon$ , which in turn can be derived as a sum of Hh functions. An explicit formula for  $\Upsilon$  will be given in Theorem B.1 in Appendix B.

<sup>8</sup>A short code (about seven lines) can be downloaded from the author's Web page. Note that the integrand in (18) has a maxima at  $(x + \sqrt{x^2 + 4n})/2$ ; therefore, to numerically compute the integral, one should split the integral more around the maxima.

**Figure 2** The Hh Function for  $n = 1, 3, 5$  with the Steepest Curve for  $n = 5$  and the Flattest Curve for  $n = 1$



**THEOREM 2.** From (12), the price of a European call option<sup>9</sup> is given by

$$\begin{aligned} \psi_c(0) = S(0) \Upsilon \left( r + \frac{1}{2} \sigma^2 - \lambda \zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \right. \\ \left. \log(K/S(0)), T \right) \\ - Ke^{-rT} \cdot \Upsilon \left( r - \frac{1}{2} \sigma^2 - \lambda \zeta, \sigma, \lambda, p, \eta_1, \eta_2; \right. \\ \left. \log(K/S(0)), T \right), \end{aligned} \quad (20)$$

where

$$\begin{aligned} \tilde{p} &= \frac{p}{1+\zeta} \cdot \frac{\eta_1}{\eta_1-1}, \quad \tilde{\eta}_1 = \eta_1 - 1, \\ \tilde{\eta}_2 &= \eta_2 + 1, \quad \tilde{\lambda} = \lambda(\zeta + 1), \quad \zeta = \frac{p\eta_1}{\eta_1-1} + \frac{q\eta_2}{\eta_2+1} - 1. \end{aligned}$$

The price of the corresponding put option,  $\psi_p(0)$ , can be obtained by the put-call parity:

$$\begin{aligned} \psi_p(0) - \psi_c(0) &= e^{-rT} \mathbf{E}^*((K - S(T))^+ - (S(T) - K)^+) \\ &= e^{-rT} \mathbf{E}^*(K - S(T)) = Ke^{-rT} - S(0). \end{aligned}$$

<sup>9</sup> To give a numerical example, if  $\eta_1 = 10$ ,  $\eta_2 = 5$ ,  $\lambda = 1$ ,  $p = 0.4$ ,  $\sigma = 0.16$ ,  $r = 5\%$ ,  $S(0) = 100$ ,  $K = 98$ ,  $T = 0.5$ , then (20) yields the call price 9.14732. Although in the pricing formula  $\Upsilon$  involves infinite series, my experience suggests that numerically only the first 10 to 15 terms in the series are needed for most applications.

The result (20) resembles the Black-Scholes formula for a call option under the geometric Brownian motion model, with  $\Upsilon$  taking the place of  $\Phi$ . The proof of Theorem 2 is similar to that of Theorem 3 in Kou and Wang (2001), hence is omitted.

Now I consider the problem of pricing options on futures contracts. Assume, for now, that the term structure of interest rate is flat, and  $r$  is a constant. Then the futures price,  $F(t, T^*)$ , with delivery date  $T^*$ , is given by  $F(t, T^*) = \mathbf{E}^*(S(T^*) | \mathcal{F}_t) = e^{r(T^*-t)} S(t)$ .

**COROLLARY 2.** The price of the European call option on a futures contract is given by

$$\begin{aligned} \psi_{c,F}(D, F(0, T^*), T) \\ = D \cdot \left\{ F(0, T^*) \Upsilon \left( \frac{1}{2} \sigma^2 - \lambda \zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \right. \right. \\ \left. \left. \log \left( \frac{K}{F(0, T^*)} \right), T \right) - K \Upsilon \left( -\frac{1}{2} \sigma^2 - \lambda \zeta, \sigma, \right. \right. \\ \left. \left. \lambda, p, \eta_1, \eta_2; \log \left( \frac{K}{F(0, T^*)} \right), T \right) \right\}, \end{aligned}$$

where  $D = e^{-rT}$ . The put option can be priced according to the put-call parity:

$$\begin{aligned} \psi_{p,F}(D, F(0, T^*), T) - \psi_{c,F}(D, F(0, T^*), T) \\ = e^{-rT} (K - F(0, T^*)). \end{aligned}$$

PROOF. Since  $(F(T, T^*) - K)^+ = e^{r(T^*-T)}(S(T) - Ke^{-r(T^*-T)})^+$ , we have

$$\begin{aligned}
 & E[e^{-rT}(F(T, T^*) - K)^+] \\
 &= e^{r(T^*-T)} \left\{ S(0) \Upsilon \left( r + \frac{1}{2} \sigma^2 - \lambda \zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \right. \right. \\
 &\quad \left. \left. \log(Ke^{-r(T^*-T)}/S(0)), T \right) \right. \\
 &\quad \left. - Ke^{-r(T^*-T)} e^{-rT} \Upsilon \left( r - \frac{1}{2} \sigma^2 - \lambda \zeta, \sigma, \lambda, p, \right. \right. \\
 &\quad \left. \left. \eta_1, \eta_2; \log(Ke^{-r(T^*-T)}/S(0)), T \right) \right\} \\
 &= e^{-rT} \left\{ F(0, T^*) \Upsilon \left( r + \frac{1}{2} \sigma^2 - \lambda \zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \right. \right. \\
 &\quad \left. \left. \log(K/F(0, T^*)) + rT, T \right) \right. \\
 &\quad \left. - K \Upsilon \left( r - \frac{1}{2} \sigma^2 - \lambda \zeta, \sigma, \lambda, p, \eta_1, \eta_2; \right. \right. \\
 &\quad \left. \left. \log(K/F(0, T^*)) + rT, T \right) \right\},
 \end{aligned}$$

from which the conclusion follows by noting that  $P\{(r + \mu)T + \sigma W(T) + \sum_{i=1}^{N(T)} Y_i \geq a + rT\} = P\{\mu T + \sigma W(T) + \sum_{i=1}^{N(T)} Y_i \geq a\}$ . □

Using the fact that for every  $t \geq 0$ ,  $Z(t)$  converges in distribution to  $\mu t + \sigma W(t)$  as both  $\eta_1 \rightarrow \infty$  and  $\eta_2 \rightarrow \infty$ , one easily gets the following corollary.

COROLLARY 3. (1) As the jump size gets smaller and smaller, the pricing formulae in Theorem 2 and Corollary 2 degenerate to the Black-Scholes formula and Black's futures option formula. More precisely, as both  $\eta_1 \rightarrow \infty$  and  $\eta_2 \rightarrow \infty$  while all other parameters remain fixed,

$$\psi_c(0) \rightarrow S(0)\Phi(b'_+) - Ke^{-rT}\Phi(b'_-),$$

$$\psi_{c,F}(0) \rightarrow e^{-rT}\{F(0, T^*)\Phi(b'_{+,F}) - K\Phi(b'_{-,F})\},$$

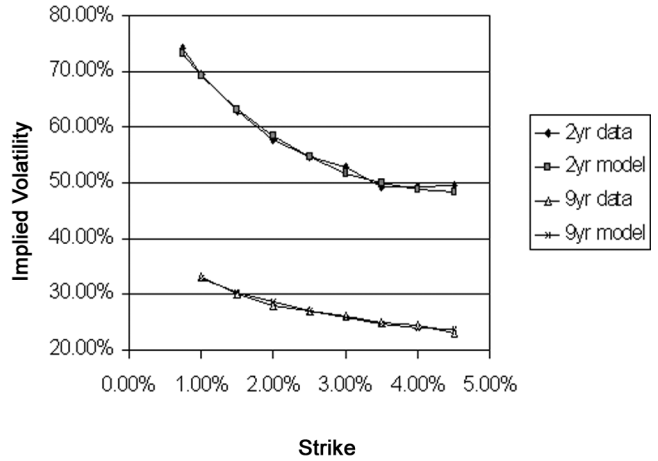
where

$$b'_\pm := \frac{\log(S(0)/K) + (r \pm (\sigma^2/2))T}{\sigma\sqrt{T}},$$

$$b'_{\pm,F} := \frac{\log(F(0, T^*)/K) \pm \sigma^2 T/2}{\sigma\sqrt{T}}.$$

(2) If the jump rate is zero, i.e.,  $\lambda = 0$ , then the pricing formulae again degenerate to the Black-Scholes and Black's futures option formulae, respectively.

Figure 3 Midmarket and Model-Implied Volatilities for Japanese LIBOR Caplets in May 1998



Notes. The parameters used in the fitted model are: for the two-year caplet  $\eta^{(1)} = 3.7, \eta^{(2)} = 1.8, p = 0.04, \lambda = 1.4, \sigma = 0.21$ ; and for the nine-year caplet  $\eta^{(1)} = 2.3, \eta^{(2)} = 1.8, p = 0.09, \lambda = 0.2, \sigma = 0.09$ .

### 5.3. The “Volatility Smile”

To illustrate that the model can produce “implied volatility smile,” I consider a real data set used first in Andersen and Andreasen (2000) for two-year and nine-year caplets in the Japanese LIBOR market as of late May 1998. Figure 3 shows both observed implied volatility curves and calibrated implied volatility curves derived by using the futures option formula in Corollary 2, with the discount parameter  $D$  being the corresponding bond prices and the underlying asset being the LIBOR rate. For details of calibration and the theoretical justification of using the futures option formula for caplets; see Glasserman and Kou (1999).

I should emphasize that this example is not meant to be an empirical test of the model; it only serves as an illustration to show that the model can produce a close fit even to a very sharp volatility skew.

## 6. Limitations of the Model

There are several limitations of the model. First, one disadvantage of the model is that the pricing formulae, although analytical, appear quite complicated. This perhaps is not a major problem because

the Hh function can be computed easily, and what appears to be lengthy to human eyes might make little difference in terms of computer programming, as long as it is a closed-form solution.

Secondly, a more serious criticism is the difficulties with hedging. Due to the jump part, the market is incomplete, and the conventional riskless hedging arguments are not applicable here. However, it should be pointed out that the riskless hedging is really a special property of continuous-time Brownian motion, and it does not hold for most of the alternative models. Even within the Brownian motion framework, the riskless hedging is impossible if one wants to do it in discrete time. For some suggestions about hedging with jump risk, see, for example, Grünewald and Trautmann (1996), Merton (1976), and Naik and Lee (1990).

Finally, just like all models based on Lévy processes, one empirical observation that the double exponential jump-diffusion model cannot incorporate is the possible dependence structure among asset returns (the so-called “volatility clustering effect”), simply because the model assumes independent increments. Here a possible way to incorporate the dependence is to use some other point process,  $\tilde{N}(t)$ , with dependent increments, to replace the Poisson process  $N(t)$ , while still retaining the independence between the Brownian motion, the jump sizes, and  $\tilde{N}(t)$ . The modified model no longer has independent increments, yet is simple enough to produce closed-form solutions, at least for standard call and put options. However, it seems difficult to get analytical solutions for path-dependent options by using the new process  $\tilde{N}(t)$  instead of  $N(t)$ .

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### Appendix A: Derivation of the Rational Expectations

PROOF OF PROPOSITION 1.

(1) Since  $B(T, T) = 1$ , Equation (8) yields

$$B(t, T) = e^{-\theta(T-t)} \frac{\mathbf{E}((\delta(T))^{\alpha-1} | \mathcal{F}_t)}{(\delta(t))^{\alpha-1}}. \quad (A1)$$

Using the facts that

$$\begin{aligned} \left(\frac{\delta(T)}{\delta(t)}\right)^{\alpha-1} &= \exp\left\{(\alpha-1)\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)(T-t) \right. \\ &\quad \left. + \sigma_1(\alpha-1)(W_1(T) - W_1(t))\right\} \prod_{i=N(t)+1}^{N(T)} \tilde{V}_i^{\alpha-1}, \\ \mathbf{E}\left(\prod_{i=N(t)+1}^{N(T)} \tilde{V}_i^{\alpha-1}\right) &= \sum_{j=0}^{\infty} e^{-\lambda(T-t)} \frac{[\lambda(T-t)]^j}{j!} \{\zeta_1^{\alpha-1} + 1\}^j \\ &= \exp\{\lambda \zeta_1^{\alpha-1}(T-t)\}, \end{aligned}$$

Equation (A1) yields

$$B(t, T) = \exp\left[-(T-t)\left\{\theta - (\alpha-1)\left(\mu_1 - \frac{1}{2}\sigma_1^2\right) - \frac{1}{2}\sigma_1^2(\alpha-1)^2 - \lambda \zeta_1^{\alpha-1}\right\}\right],$$

from which (9) follows.

(2) Note that (A1) implies

$$e^{-r(T-t)} = \mathbf{E}(U_c(\delta(T), T)/U_c(\delta(t), t) | \mathcal{F}_t), \quad (A2)$$

which shows that  $Z(t)$  is a martingale under  $\mathbf{P}$ . Furthermore, (7) and (9) lead to

$$\begin{aligned} Z(t) &= (\delta(0))^{\alpha-1} e^{(r-\theta)t} \exp\left\{(\alpha-1)\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)t \right. \\ &\quad \left. + \sigma_1(\alpha-1)W_1(t)\right\} \prod_{i=1}^{N(t)} \tilde{V}_i^{\alpha-1} \\ &= (\delta(0))^{\alpha-1} \exp\left\{-\frac{1}{2}\sigma_1^2(\alpha-1)^2 - \lambda \zeta_1^{\alpha-1}\right\} t \\ &\quad + \sigma_1(\alpha-1)W_1(t) \prod_{i=1}^{N(t)} \tilde{V}_i^{\alpha-1}, \end{aligned}$$

from which (10) follows. Now by (8) and (A2),

$$\begin{aligned} \psi_S(t) &= \frac{\mathbf{E}(U_c(\delta(T), T)\psi_S(T) | \mathcal{F}_t)}{U_c(\delta(t), t)} = e^{-rT} \mathbf{E}\left\{\frac{Z(T)}{Z(t)} \psi_S(T) | \mathcal{F}_t\right\} \\ &= e^{-rT} \mathbf{E}^*(\psi_S(T) | \mathcal{F}_t). \quad \square \end{aligned}$$

PROOF OF THEOREM 1. The Girsanov theorem for jump-diffusion processes (see Björk et al. 1997) tells us that under  $\mathbf{P}^*$ ,  $W_1^*(t) := W_1(t) - \sigma_1(\alpha-1)t$  is a new Brownian motion, and under  $\mathbf{P}^*$  the jump rate of  $N(t)$  is  $\lambda^* = \lambda \mathbf{E}(\tilde{V}_i^{\alpha-1}) = \lambda(\zeta_1^{\alpha-1} + 1)$ , and  $\tilde{V}_i$  has a new

density  $f_{\tilde{V}}^*(x) = (1/(\zeta_1^{\alpha-1} + 1))x^{\alpha-1}f_{\tilde{V}}(x)$ . Therefore, the dynamics of  $S(t)$  is given by

$$\begin{aligned} \frac{dS(t)}{S(t-)} &= \mu dt + \sigma\{\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)\} + \Delta \left[ \sum_{i=1}^{N(t)} (\tilde{V}_i^\beta - 1) \right] \\ &= \{\mu + \sigma_1\sigma\rho(\alpha-1)\} dt + \sigma\{\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)\} \\ &\quad + \Delta \left[ \sum_{i=1}^{N(t)} (\tilde{V}_i^\beta - 1) \right]. \end{aligned}$$

Because

$$\begin{aligned} E^*(\tilde{V}_i^\beta) &= \int_0^\infty x^\beta \frac{1}{\zeta_1^{\alpha-1} + 1} x^{\alpha-1} f_{\tilde{V}}(x) dx \\ &= \frac{1}{\zeta_1^{\alpha-1} + 1} E(\tilde{V}^{\alpha+\beta-1}) = \frac{\zeta_1^{\alpha+\beta-1} + 1}{\zeta_1^{\alpha-1} + 1}, \end{aligned}$$

we have  $\lambda^*\{E^*(\tilde{V}_i^\beta) - 1\} = \lambda(\zeta_1^{\alpha+\beta-1} - \zeta_1^{\alpha-1})$ . Therefore,

$$\begin{aligned} \frac{dS(t)}{S(t-)} &= \{\mu + \sigma_1\sigma\rho(\alpha-1) + \lambda(\zeta_{\alpha+\beta-1} - \zeta_{\alpha-1})\} dt \\ &\quad - \lambda^*\{E^*(\tilde{V}_i^\beta) - 1\} dt + \sigma\{\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)\} \\ &\quad + \Delta \left[ \sum_{i=1}^{N(t)} (\tilde{V}_i^\beta - 1) \right]. \end{aligned}$$

Hence, to satisfy the rational equilibrium requirement  $S(t) = e^{-r(T-t)}E^*(S(T) | \mathcal{F}_t)$ , we must have  $\mu + \sigma_1\sigma\rho(\alpha-1) + \lambda(\zeta_{\alpha+\beta-1} - \zeta_{\alpha-1}) = r$ , from which (14) follows. If (14) is satisfied, under the measure  $P^*$ , the dynamics of  $S(t)$  is given by

$$\begin{aligned} \frac{dS(t)}{S(t-)} &= r dt - \lambda^*\{E^*(\tilde{V}_i^\beta) - 1\} dt + \sigma\{\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)\} \\ &\quad + \Delta \left[ \sum_{i=1}^{N(t)} (\tilde{V}_i^\beta - 1) \right], \end{aligned}$$

from which (15) follows.  $\square$

## Appendix B: Derivation of the $\Upsilon$ Function

### B.1. Decomposition of the Sum of Double Exponential Random Variables

The memoryless property of exponential random variables yields  $(\xi^+ - \xi^- | \xi^+ > \xi^-) \stackrel{d}{=} \xi^+$  and  $(\xi^+ - \xi^- | \xi^+ < \xi^-) \stackrel{d}{=} -\xi^-$ , thus leading to the conclusion that

$$\xi^+ - \xi^- \stackrel{d}{=} \begin{cases} \xi^+, & \text{with probability } \eta_2/(\eta_1 + \eta_2) \\ -\xi^-, & \text{with probability } \eta_1/(\eta_1 + \eta_2) \end{cases}, \quad (B1)$$

because the probabilities of the events  $\xi^+ > \xi^-$  and  $\xi^+ < \xi^-$  are  $\eta_2/(\eta_1 + \eta_2)$  and  $\eta_1/(\eta_1 + \eta_2)$ , respectively. The following proposition extends (B1).

**PROPOSITION B.1.** For every  $n \geq 1$ , we have the following decomposition

$$\sum_{i=1}^n Y_i \stackrel{d}{=} \begin{cases} \sum_{i=1}^k \xi_i^+, & \text{with probability } P_{n,k}, k = 1, 2, \dots, n \\ -\sum_{i=1}^k \xi_i^-, & \text{with probability } Q_{n,k}, k = 1, 2, \dots, n \end{cases}, \quad (B2)$$

where  $P_{n,k}$  and  $Q_{n,k}$  are given by

$$\begin{aligned} P_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \cdot \left(\frac{\eta_1}{\eta_1 + \eta_2}\right)^{i-k} \left(\frac{\eta_2}{\eta_1 + \eta_2}\right)^{n-i} p^i q^{n-i}, \\ &\quad 1 \leq k \leq n-1, \\ Q_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \cdot \left(\frac{\eta_1}{\eta_1 + \eta_2}\right)^{n-i} \left(\frac{\eta_2}{\eta_1 + \eta_2}\right)^{i-k} p^{n-i} q^i, \\ &\quad 1 \leq k \leq n-1, P_{n,n} = p^n, \quad Q_{n,n} = q^n, \end{aligned}$$

and  $\binom{0}{0}$  is defined to be one. Here  $\xi_i^+$  and  $\xi_i^-$  are i.i.d. exponential random variables with rates  $\eta_1$  and  $\eta_2$ , respectively.

As a key step in deriving closed-form solutions for call and put options, this proposition indicates that the sum of i.i.d. double exponential random variables can be written, in distribution, as a (randomly) mixed gamma random variable.<sup>10</sup> To prove Proposition B.1, the following lemma is needed.

**LEMMA B.1.**

$$\sum_{i=1}^n \xi_i^+ - \sum_{j=1}^m \xi_j^- \stackrel{d}{=} \left\{ \begin{array}{l} \sum_{i=1}^k \xi_i, \quad \text{with prob.} \\ \left(\frac{\eta_1}{\eta_1 + \eta_2}\right)^{n-k} \left(\frac{\eta_2}{\eta_1 + \eta_2}\right)^m \cdot \binom{n-k+m-1}{m-1}, \\ k = 1, \dots, n \\ \\ -\sum_{i=1}^l \xi_i, \quad \text{with prob.} \\ \left(\frac{\eta_1}{\eta_1 + \eta_2}\right)^n \left(\frac{\eta_2}{\eta_1 + \eta_2}\right)^{m-l} \cdot \binom{n-l+m-1}{n-1}, \\ l = 1, \dots, m \end{array} \right\}. \quad (B3)$$

**PROOF.** Introduce the random variables  $A(n, m) = \sum_{i=1}^n \xi_i - \sum_{j=1}^m \xi_j$ . Then

$$\begin{aligned} A(n, m) &\stackrel{d}{=} \begin{cases} A(n-1, m-1) + \xi^+, & \eta_2/(\eta_1 + \eta_2) \\ A(n-1, m-1) - \xi^-, & \eta_1/(\eta_1 + \eta_2) \end{cases} \\ &\stackrel{d}{=} \begin{cases} A(n, m-1), & \eta_2/(\eta_1 + \eta_2) \\ A(n-1, m), & \eta_1/(\eta_1 + \eta_2) \end{cases} \end{aligned}$$

via (B1). Now imagine a plane with the horizontal axis representing the number of  $\{\xi_i^+\}$  and the vertical axis representing the number of  $\{\xi_j^-\}$ . Suppose we have a random walk on the integer lattice points of the plane. Starting from any point  $(n, m)$ ,  $n, m \geq 1$ , the random walk goes either one step to the left with probability  $\eta_1/(\eta_1 + \eta_2)$  or one step down with probability  $\eta_2/(\eta_1 + \eta_2)$ , and

<sup>10</sup> A result similar to the decomposition (B2) was first discovered by Shanthikumar (1985), although (B2) gives a more explicit calculation of  $P_{n,k}$  and  $Q_{n,k}$ . Furthermore, the proofs are totally different: A combinatorial approach is used here, while the proof in Shanthikumar (1985) is based on the Laplace transform.

the random walk stops once it reaches either the horizontal or vertical axis. For any path from  $(n, m)$  to  $(k, 0)$ ,  $1 \leq k \leq n$ , it must reach  $(k, 1)$  first before it makes a final move to  $(k, 0)$ . Furthermore, all the paths going from  $(n, m)$  to  $(k, 1)$  must have exactly  $n - k$  lefts and  $m - 1$  downs, whence the total number of such paths is  $\binom{n-k+m-1}{m-1}$ . Similarly, the total number of paths from  $(n, m)$  to  $(0, l)$ ,  $1 \leq l \leq m$ , is  $\binom{n-l+m-1}{n-1}$ . Thus,

$$A(n, m) \stackrel{d}{=} \left\{ \begin{array}{l} \sum_{i=1}^k \xi_i, \quad \text{with prob.} \\ \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-k} \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^m \cdot \binom{n-k+m-1}{m-1}, \\ k = 1, \dots, n \\ - \sum_{i=1}^l \xi_i, \quad \text{with prob.} \\ \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^n \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{m-l} \cdot \binom{n-l+m-1}{n-1}, \\ l = 1, \dots, m \end{array} \right\},$$

and the lemma is proven.  $\square$

PROOF OF PROPOSITION B.1. By the same analogy used in Lemma B.1 to compute probability  $P_{n,k}$ ,  $1 \leq k \leq n$ , the probability weight assigned to  $\sum_{i=1}^k \xi_i^+$  when we decompose  $\sum_{i=1}^n Y_i$ , it is equivalent to consider the probability of the random walk ever reach  $(k, 0)$  starting from the point  $(i, n - i)$ ,  $0 \leq i \leq n$ , with probability of starting from  $(i, n - i)$  being  $\binom{n}{i} p^i q^{n-i}$ . Note that the point  $(k, 0)$  can only be reached from points  $(i, n - i)$  such that  $k \leq i \leq n - 1$ , because the random walk can only go left or down, and stops once it reaches the horizontal axis. Therefore, for  $1 \leq k \leq n - 1$ , (B3) leads to

$$\begin{aligned} P_{n,k} &= \sum_{i=k}^{n-1} \mathbf{P}(\text{going from } (i, n - i) \text{ to } (k, 0)) \cdot \mathbf{P}(\text{starting from } (i, n - i)) \\ &= \sum_{i=k}^{n-1} \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{i-k} \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{n-i} \binom{i+(n-i)-k-1}{(n-i)-1} \cdot \binom{n}{i} p^i q^{n-i} \\ &= \sum_{i=k}^{n-1} \binom{n-k-1}{n-i-1} \binom{n}{i} \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{i-k} \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{n-i} p^i q^{n-i} \\ &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{i-k} \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{n-i} p^i q^{n-i}. \end{aligned}$$

Of course,  $P_{n,n} = p^n$ . Similarly, we can compute  $Q_{n,k}$ :

$$\begin{aligned} Q_{n,k} &= \sum_{i=k}^{n-1} \mathbf{P}(\text{going from } (n - i, i) \text{ to } (0, k)) \cdot \mathbf{P}(\text{starting from } (n - i, i)) \\ &= \sum_{i=k}^{n-1} \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-i} \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{i-k} \binom{(n-i)+i-k-1}{(n-i)-1} \cdot \binom{n}{n-i} p^{n-i} q^i \\ &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-i} \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{i-k} p^{n-i} q^i, \end{aligned}$$

with  $Q_{n,n} = q^n$ . Incidentally, we have also shown that  $\sum_{k=1}^n (P_{n,k} + Q_{n,k}) = 1$ .  $\square$

**B.2. Results on Hh Functions**

First of all, note that  $\text{Hh}_n(x) \rightarrow 0$ , as  $x \rightarrow \infty$ , for  $n \geq -1$ ; and  $\text{Hh}_n(x) \rightarrow \infty$ , as  $x \rightarrow -\infty$ , for  $n \geq 1$ ; and  $\text{Hh}_0(x) = \sqrt{2\pi}\Phi(-x) \rightarrow \sqrt{2\pi}$ , as  $x \rightarrow -\infty$ . Also, for every  $n \geq -1$ , as  $x \rightarrow \infty$ ,

$$\lim_{x \rightarrow \infty} \text{Hh}_n(x) / \left\{ \frac{1}{x^{n+1}} e^{-x^2/2} \right\} = 1, \tag{B4}$$

and as  $x \rightarrow -\infty$ ,

$$\text{Hh}_n(x) = O(|x|^n). \tag{B5}$$

Here (B4) follows from Equations (19.14.3) and (19.8.1) in Abramowitz and Stegun (1972); and (B5) is clearly true for  $n = -1$ , while for  $n \geq 0$  note that as  $x \rightarrow -\infty$ ,

$$\begin{aligned} \text{Hh}_n(x) &= \frac{1}{n!} \int_x^\infty (t-x)^n e^{-t^2/2} dt \\ &\leq \frac{2^n}{n!} \int_{-\infty}^\infty |t|^n e^{-t^2/2} dt + \frac{2^n}{n!} \int_{-\infty}^\infty |x|^n e^{-t^2/2} dt = O(|x|^n). \end{aligned}$$

For option pricing it is important to evaluate the integral  $I_n(c; \alpha, \beta, \delta)$ ,

$$I_n(c; \alpha, \beta, \delta) := \int_c^\infty e^{\alpha x} \text{Hh}_n(\beta x - \delta) dx, \quad n \geq 0, \tag{B6}$$

for arbitrary constants  $\alpha, c$ , and  $\beta$ .

PROPOSITION<sup>11</sup> B.2. (1) If  $\beta > 0$  and  $\alpha \neq 0$ , then for all  $n \geq -1$ ,

$$\begin{aligned} I_n(c; \alpha, \beta, \delta) &= -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^n \left( \frac{\beta}{\alpha} \right)^{n-i} \text{Hh}_i(\beta c - \delta) \\ &\quad + \left( \frac{\beta}{\alpha} \right)^{n+1} \frac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}} \Phi\left(-\beta c + \delta + \frac{\alpha}{\beta}\right). \end{aligned} \tag{B7}$$

(2) If  $\beta < 0$  and  $\alpha < 0$ , then for all  $n \geq -1$ ,

$$\begin{aligned} I_n(c; \alpha, \beta, \delta) &= -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^n \left( \frac{\beta}{\alpha} \right)^{n-i} \text{Hh}_i(\beta c - \delta) \\ &\quad - \left( \frac{\beta}{\alpha} \right)^{n+1} \frac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}} \Phi\left(\beta c - \delta - \frac{\alpha}{\beta}\right). \end{aligned} \tag{B8}$$

PROOF. Case 1.  $\beta > 0$  and  $\alpha \neq 0$ . Since, for any constant  $\alpha$  and  $n \geq 0$ ,  $e^{\alpha x} \text{Hh}_n(\beta x - \delta) \rightarrow 0$  as  $x \rightarrow \infty$  thanks to (B4), integration by parts leads to

$$\begin{aligned} I_n &= \frac{1}{\alpha} \int_c^\infty \text{Hh}_n(\beta x - \delta) d e^{\alpha x} \\ &= -\frac{1}{\alpha} \text{Hh}_n(\beta c - \delta) e^{\alpha c} + \frac{\beta}{\alpha} \int_c^\infty e^{\alpha x} \text{Hh}_{n-1}(\beta x - \delta) dx. \end{aligned}$$

<sup>11</sup> If  $\beta > 0$  and  $\alpha = 0$ , then for all  $n \geq 0$ ,  $I_n(c; \alpha, \beta, \delta) = \frac{1}{\beta} \text{Hh}_{n+1}(\beta c - \delta)$ . If  $\beta \leq 0$  and  $\alpha \geq 0$ , then for all  $n \geq 0$ ,  $I_n(c; \alpha, \beta, \delta) = \infty$ . If  $\beta = 0$  and  $\alpha < 0$ , then for all  $n \geq 0$ ,  $I_n(c; \alpha, \beta, \delta) = \int_c^\infty e^{\alpha x} \text{Hh}_n(-\delta) dx = \text{Hh}_n(-\delta) e^{\alpha c}$ .

In other words, we have a recursion, for  $n \geq 0, I_n = -(e^{\alpha c}/\alpha) \text{Hh}_n(\beta c - \delta) + (\beta/\alpha) I_{n-1}$  with

$$I_{-1} = \sqrt{2\pi} \int_c^\infty e^{\alpha x} \varphi(-\beta x + \delta) dx$$

$$= \frac{\sqrt{2\pi}}{\beta} \exp\left\{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}\right\} \Phi\left(-\beta c + \delta + \frac{\alpha}{\beta}\right).$$

Solving it yields, for  $n \geq -1$ ,

$$I_n = -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^n \left(\frac{\beta}{\alpha}\right)^i \text{Hh}_{n-i}(\beta c - \delta) + \left(\frac{\beta}{\alpha}\right)^{n+1} I_{-1}$$

$$= -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^n \left(\frac{\beta}{\alpha}\right)^{n-i} \text{Hh}_i(\beta c - \delta)$$

$$+ \left(\frac{\beta}{\alpha}\right)^{n+1} \frac{\sqrt{2\pi}}{\beta} \exp\left\{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}\right\} \Phi\left(-\beta c + \delta + \frac{\alpha}{\beta}\right),$$

where the sum over an empty set is defined to be zero.

Case 2.  $\beta < 0$  and  $\alpha < 0$ . In this case, we must also have, for  $n \geq 0$  and any constant  $\alpha < 0, e^{\alpha x} \text{Hh}_n(\beta x - \delta) \rightarrow 0$  as  $x \rightarrow \infty$ , thanks to (B5). Using integration by parts, we again have the same recursion, for  $n \geq 0, I_n = -(e^{\alpha c}/\alpha) \text{Hh}_n(\beta c - \delta) + (\beta/\alpha) I_{n-1}$ , but with a different initial condition

$$I_{-1} = \sqrt{2\pi} \int_c^\infty e^{\alpha x} \varphi(-\beta x + \delta) dx$$

$$= -\frac{\sqrt{2\pi}}{\beta} \exp\left\{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}\right\} \Phi\left(\beta c - \delta - \frac{\alpha}{\beta}\right).$$

Solving it yields (B8), for  $n \geq -1$ .  $\square$

### B.3. Sum of Double Exponential and the Normal Random Variables

PROPOSITION B.3. Suppose  $\{\xi_1, \xi_2, \dots\}$  is a sequence of i.i.d. exponential random variables with rate  $\eta > 0$ , and  $Z$  is a normal random variable with distribution  $N(0, \sigma^2)$ . Then for every  $n \geq 1$ , we have: (1) The density functions are given by

$$f_{Z+\sum_{i=1}^n \xi_i}(t) = (\sigma\eta)^n \frac{e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} e^{-t\eta} \text{Hh}_{n-1}\left(-\frac{t}{\sigma} + \sigma\eta\right), \quad (\text{B9})$$

$$f_{Z-\sum_{i=1}^n \xi_i}(t) = (\sigma\eta)^n \frac{e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} e^{t\eta} \text{Hh}_{n-1}\left(\frac{t}{\sigma} + \sigma\eta\right). \quad (\text{B10})$$

(2) The tail probabilities are given by

$$\mathbb{P}\left(Z + \sum_{i=1}^n \xi_i \geq x\right) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} I_{n-1}\left(x; -\eta, -\frac{1}{\sigma}, -\sigma\eta\right), \quad (\text{B11})$$

$$\mathbb{P}\left(Z - \sum_{i=1}^n \xi_i \geq x\right) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} I_{n-1}\left(x; \eta, \frac{1}{\sigma}, -\sigma\eta\right). \quad (\text{B12})$$

PROOF. Case 1. The densities of  $Z + \sum_{i=1}^n \xi_i$  and  $Z - \sum_{i=1}^n \xi_i$ . We have

$$f_{Z+\sum_{i=1}^n \xi_i}(t) = \int_{-\infty}^\infty f_{\sum_{i=1}^n \xi_i}(t-x) f_Z(x) dx$$

$$= e^{-t\eta} (\eta^n) \int_{-\infty}^t \frac{e^{x\eta} (t-x)^{n-1}}{(n-1)!} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)} dx$$

$$= e^{-t\eta} (\eta^n) e^{(\sigma\eta)^2/2} \int_{-\infty}^t \frac{(t-x)^{n-1}}{(n-1)!} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\sigma^2\eta)^2/(2\sigma^2)} dx.$$

Letting  $y = (x - \sigma^2\eta)/\sigma$  yields

$$f_{Z+\sum_{i=1}^n \xi_i}(t) = e^{-t\eta} e^{(\sigma\eta)^2/2} \sigma^{n-1} \eta^n$$

$$\times \int_{-\infty}^{t/\sigma - \sigma\eta} \frac{(t/\sigma - y - \sigma\eta)^{n-1}}{(n-1)!} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$= \frac{e^{(\sigma\eta)^2/2}}{\sqrt{2\pi}} (\sigma^{n-1} \eta^n) e^{-t\eta} \text{Hh}_{n-1}\left(-\frac{t}{\sigma} + \sigma\eta\right),$$

because  $(1/(n-1)!) \int_{-\infty}^a (a-y)^{n-1} e^{-y^2/2} dy = \text{Hh}_{n-1}(a)$ . The derivation of  $f_{Z-\sum_{i=1}^n \xi_i}(t)$  is similar.

Case 2.  $\mathbb{P}(Z + \sum_{i=1}^n \xi_i \geq x)$  and  $\mathbb{P}(Z - \sum_{i=1}^n \xi_i \geq x)$ . From (B9), it is clear that

$$\mathbb{P}\left(Z + \sum_{i=1}^n \xi_i \geq x\right) = \frac{(\sigma\eta)^n e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} \int_x^\infty e^{-t\eta} \text{Hh}_{n-1}\left(-\frac{t}{\sigma} + \sigma\eta\right) dt$$

$$= \frac{(\sigma\eta)^n e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} I_{n-1}\left(x; -\eta, -\frac{1}{\sigma}, -\sigma\eta\right),$$

by (B6). We can compute  $\mathbb{P}(Z - \sum_{i=1}^n \xi_i \geq x)$  similarly.  $\square$

THEOREM B.1. With  $\pi_n := \mathbb{P}(N(T) = n) = e^{-\lambda T} (\lambda T)^n / n!$  and  $I_n$  in Proposition B.2, we have

$$\mathbb{P}\{Z(T) \geq a\} = \frac{e^{(\sigma\eta_1)^2 T/2}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^\infty \pi_n \sum_{k=1}^n P_{n,k}(\sigma\sqrt{T}\eta_1)^k$$

$$\times I_{k-1}\left(a - \mu T; -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\sigma\eta_1\sqrt{T}\right)$$

$$+ \frac{e^{(\sigma\eta_2)^2 T/2}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^\infty \pi_n \sum_{k=1}^n Q_{n,k}(\sigma\sqrt{T}\eta_2)^k$$

$$\times I_{k-1}\left(a - \mu T; \eta_2, \frac{1}{\sigma\sqrt{T}}, -\sigma\eta_2\sqrt{T}\right)$$

$$+ \pi_0 \Phi\left(-\frac{a - \mu T}{\sigma\sqrt{T}}\right).$$

PROOF. By the decomposition (B2),

$$\mathbb{P}\{Z(T) \geq a\} = \sum_{n=0}^\infty \pi_n \mathbb{P}\left(\mu T + \sigma\sqrt{T}Z + \sum_{j=1}^n Y_j \geq a\right)$$

$$= \pi_0 \mathbb{P}(\mu T + \sigma\sqrt{T}Z \geq a)$$

$$+ \sum_{n=1}^\infty \pi_n \sum_{k=1}^n P_{n,k} \mathbb{P}\left(\mu T + \sigma\sqrt{T}Z + \sum_{j=1}^k \xi_j^+ \geq a\right)$$

$$+ \sum_{n=1}^\infty \pi_n \sum_{k=1}^n Q_{n,k} \mathbb{P}\left(\mu T + \sigma\sqrt{T}Z - \sum_{j=1}^k \xi_j^- \geq a\right).$$

The result now follows via (B11) and (B12) for  $\eta_1 > 1$  and  $\eta_2 > 0$ .  $\square$

### References

Abramowitz, M., I. A. Stegun. 1972. *Handbook of Mathematical Function*, 10th Printing. U.S. National Bureau of Standards, Washington, D.C.

- Andersen, L., J. Andreasen. 2000. Volatility skews and extensions of the libor market model. *Appl. Math. Finance* 7 1–32.
- Andersen, T., L. Benzoni, J. Lund. 1999. Estimating jump-diffusions for equity returns. Working paper, Northwestern University, Evanston, IL.
- Barberis, N., A. Shleifer, R. Vishny. 1998. A model of investor sentiment. *J. Financial Econom.* 49 307–343.
- Barndorff-Nielsen, O. E., N. Shephard. 2001. Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics (with discussion). *J. Roy. Statist. Soc., Ser. B.* 63 167–241.
- Björk, T., Y. Kabanov, W. Runggaldier. 1997. Bond market structure in the presence of marked point processes. *Math. Finance* 7 211–239.
- Blattberg, R. C., N. J. Gonedes. 1974. A comparison of the stable and student distributions as statistical models for stock prices. *J. Bus.* 47 244–280.
- Boyle, P., M. Broadie, P. Glasserman. 1997. Simulation methods for security pricing. *J. Econom. Dynam. Control* 21 1267–1321.
- Clark, P. K. 1973. A subordinated stochastic process model with finite variance for speculative prices. *Econometrica* 41 135–155.
- Cox, J. C., S. A. Ross. 1976. The valuation of options for alternative stochastic processes. *J. Financial Econom.* 3 145–166.
- , J. E. Ingersoll, S. A. Ross. 1985. A theory of the term structure of interest rates. *Econometrica* 53 385–407.
- Das, S. R., S. Foresi. 1996. Exact solutions for bond and option prices with systematic jump risk. *Rev. Derivatives Res.* 1 7–24.
- Davydov, D., V. Linetsky. 2001. The valuation and hedging of path-dependent options under the CEV process. *Management Sci.* 47 949–965.
- Derman, E., I. Kani. 1994. Riding on a smile. *RISK* (Feb) 7 32–39.
- Duffie, D., J. Pan, K. Singleton. 2000. Transform analysis and option pricing for affine jump-diffusions. *Econometrica* 68 1343–1376.
- Dupire, B. 1994. Pricing with a smile. *RISK* (Feb) 7 18–20.
- Engle, R. 1995. *ARCH, Selected Readings*. Oxford University Press, Oxford, U.K.
- Fama, E. 1998. Market efficiency, long-term returns, and behavioral finance. *J. Financial Econom.* 49 283–306.
- Fouque, J.-P., G. Papanicolaou, K. R. Sircar. 2000. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press, Cambridge, U.K.
- Geman, H., D. B. Madan, M. Yor. 2001. Time changes for Lévy processes. *Math. Finance* 11 79–96.
- Glasserman, P., S. G. Kou. 1999. The term structure of simple interest rates with jump risk. Working paper, Columbia University, New York.
- Grünewald, B., S. Trautmann. 1996. Option hedging in the presence of jump risk. Preprint, Johannes Gutenberg-Universität Mainz, Germany.
- Heston, S. 1993. A closed-form solution of options with stochastic volatility with applications to bond and currency options. *Rev. Financial Stud.* 6 327–343.
- Heyde, C. C. 2000. A risky asset model with strong dependence through fractal activity time. *J. Appl. Probab.* 36 1234–1239.
- Hull, J., A. White. 1987. The pricing of options on assets with stochastic volatilities. *J. Finance* 42 281–300.
- Johnson, N., S. Kotz, N. Balakrishnan. 1995. *Continuous Univariate Distribution*, Vol. 2, 2nd ed. Wiley, New York.
- Kou, S. G., H. Wang. 2000. First passage times for a jump diffusion process. Working paper, Columbia University, New York.
- , —. 2001. Option pricing under a double exponential jump diffusion model. Working paper, Columbia University, New York.
- Lucas, R. E. 1978. Asset prices in an exchange economy. *Econometrica* 46 1429–1445.
- Madan, D. B., E. Seneta. 1990. The variance gamma (V.G.) model for share market returns. *J. Bus.* 63 511–524.
- , P. Carr, E. C. Chang. 1998. The variance gamma process and option pricing. *Eur. Finance Rev.* 2 79–105.
- Mandelbrot, B. 1963. The variation of certain speculative prices. *J. Bus.* 36 394–419.
- Merton, R. C. 1976. Option pricing when underlying stock returns are discontinuous. *J. Financial Econom.* 3 125–144.
- Naik, V., M. Lee. 1990. General equilibrium pricing of options on the market portfolio with discontinuous returns. *Rev. Financial Stud.* 3 493–521.
- Ramezani, C. A., Y. Zeng. 1999. Maximum likelihood estimation of asymmetric jump-diffusion process: Application to security prices. Working paper, Department of Statistics, University of Wisconsin, Madison, WI.
- Rogers, L. C. G. 1997. Arbitrage from fractional Brownian motion. *Math. Finance* 7 95–105.
- Samorodnitsky, G., M. S. Taqqu. 1994. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman and Hall, New York.
- Shanthikumar, J. G. 1985. Bilateral phase-type distributions. *Naval Res. Logist. Quart.* 32 119–136.
- Stokey, N. L., R. E. Lucas. 1989. *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, MA.

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