Pricing Asian Options under a Hyper-Exponential Jump Diffusion Model

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June 2011

Abstract

We obtain a closed-form solution for the double-Laplace transform of Asian options under the hyper-exponential jump diffusion model (HEM). Similar results are only available previously in the special case of the Black-Scholes model (BSM). Even in the case of the BSM, our approach is simpler as we essentially use only Itô's formula and do not need more advanced results such as those of Bessel processes and Lamperti's representation. As a by-product we also show that a well-known recursion relating to Asian options has a unique solution in a probabilistic sense. The double-Laplace transform can be inverted numerically via a two-sided Euler inversion algorithm. Numerical results indicate that our pricing method is fast, stable, and accurate, and performs well even in the case of low volatilities.

Subject classifications: Finance: asset pricing. Probability: stochastic model applications.

Area of review: Financial engineering.

1   Introduction

Asian options (or average options), whose payoffs depend on the average of the underlying asset price over a pre-specified time period, are among the most popular path-dependent options traded in both exchanges and over-the-counter markets. A main difficulty in pricing Asian options is that the distribution of the average price may not be available analytically.

There is a large body of literature on Asian options under the Black-Scholes model (BSM). For example, approaches based on partial differential equations were given in Ingersoll [26], Rogers and Shi [38], Lewis [31], Dubois and Lelièvre [18], Zhang [51, 52]; Monte Carlo simulation techniques were discussed in Broadie and Glasserman [7], Glasserman [24] and Lapeyre and

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Temam [29]; analytical approximations were derived in Turnbull and Wakeman [46], Milevsky and Posner [35] and Ju [27]; lower and upper bounds were given in Curran [16], Henderson et al. [25], and Thompson [45]. Previous results that are related to ours are: (i) Linetsky [32] derived an elegant series expansion for Asian options via a one-dimensional affine diffusion. (ii) Vecer [47] obtained a one-dimensional partial differential equation (PDE) for Asian options which can be solved numerically in stable ways. (iii) Based on Bessel processes and Lamperti's representation, in a celebrated paper Geman and Yor [23] provided an analytical solution of a single-Laplace transform of the Asian option price with respect to the maturity; see also Yor [50], Matsumoto and Yor [33, 34], Carr and Schröder [13] and Schröder [40]. Significant progress has been made for the inversion of the single-Laplace transform in Shaw [41, 43]. Dewynne and Shaw [17] gave a simple derivation of the single-Laplace transform, and provided a matched asymptotic expansion, which performs well for extremely low volatilities. (iv) Dufresne [19, 20] obtained many interesting results including a Laguerre series expansion for both Asian and reciprocal Asian options. (v) Double-Laplace and Fourier-Laplace transforms were proposed in Fu et al. [21] and Fusai [22], respectively. For the differences between their methods and ours, see Section 2 and the online supplement (Section 3).

All the papers discussed above are within the Black-Scholes framework. There are only few papers for alternative models with jumps. For example, Albrecher [2], Albrecher and Predota [4] and Albrecher et al. [3] derived bounds and approximations for Asian options under certain exponential Lévy models; Carmona et al. [11] derived some theoretical representations for Asian options under some special Lévy processes; Vecer and Xu [49] gave some representations for Asian options under semi-martingale models via partial integro-differential equations; Bayraktar and Xing [6] proposed a numerical approach to Asian options for jump diffusions by constructing a sequence of converging functions.

In this paper we study the pricing of Asian options under the hyper-exponential jump diffusion model (HEM) where the jump sizes have a hyper-exponential distribution, i.e., a mixture of a finite number of exponential distributions. For background on the HEM, see

Levendorskiĭ [30] and Cai and Kou [9]. The contribution of the current paper is three-fold: (1) Even in the special case of the BSM, our approach is simpler as we essentially use only Itô's formula and do not need more advanced results such as those of Bessel processes and Lamperti's representation. See Section 3. (2) Our approach is more general as it applies to the HEM (see Section 4). As a by-product we also show that under the HEM a well-known recursion relating to Asian options has a unique solution in a probabilistic sense, and the integral of the underlying asset price process at the exponential time has the same distribution as a combination of a sequence of independent gamma and beta random variables; see Section 4.1. (3) The double-Laplace transform can be inverted numerically via a latest two-sided Euler inversion algorithm along with a scaling factor proposed in Petrella [37].

We analyze the algorithm's accuracy, stability, and low-volatility performance by conducting a detailed comparison with other existing methods. For example our pricing method is highly accurate compared with the benchmarks from the three existing pricing methods under the BSM: (i) Linetsky's method, (ii) Vecer's method, and (iii) Geman and Yor's single-Laplace method via Shaw's elegant Mathematica implementation. Moreover, our method performs well even for low volatilities, e.g., 0.05. See Section 5.

The rest of the paper is organized as follows. Section 2 contains a general formulation of the double-Laplace transform of Asian option prices. Section 3 concentrates on pricing Asian options under the BSM. In Section 4, we extend the results in Section 3 to the more general HEM. Section 5 is devoted to the implementation of our pricing algorithm via the latest two-sided, two-dimensional Euler inversion algorithm with a scaling factor. Some proofs and some numerical issues are presented in the appendices and the online supplement.

2   A Double-Laplace Transform

For simplicity, we shall focus on Asian call options, as Asian put options can be treated similarly. The payoff of a continuous Asian call option with a mature time t and a fixed strike K is $\left(\frac{S_0 A_t}{t} - K\right)^+$, where $A_t := \int_0^t e^{X(s)} ds$, $S(t)$ is the underlying asset price process with $S(0) \equiv$

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S_0, and $X(t) := \log(S(t)/S(0))$ is the return process. Standard finance theory says that the Asian option price at time zero can be expressed as $P(t,k) := e^{-rt}E\left(\frac{S_0 A_t}{t} - K\right)^+$, where E is expectation under a pricing probability measure $\mathbb{P}$. Under the BSM the measure $\mathbb{P}$ is the unique risk neutral measure, whereas under more general models $\mathbb{P}$ may be obtained in other ways, such as using utility functions or mean variance hedging arguments. For more details, see, e.g., Shreve [44].

A key component of our double-Laplace inversion method is a scaling factor $X > S_0$. More precisely, with $k := \ln(\frac{X}{Kt})$ we can rewrite the option price $P(t,k) = \frac{e^{-rt}X}{t}E\left(\frac{S_0}{X}A_t - e^{-k}\right)^+$. Note that k can be either positive or negative, so the Laplace transform w.r.t. k will be two-sided. The scaling factor X introduced by Petrella [37] is primarily to control the associated discretization errors and to let the inversion occur at a reasonable point k; see also Cai et al. [10], where they introduced a shift parameter for the two-sided Euler inversion algorithm. Thanks to the scaling factor, the resulting inversion algorithm appears to be accurate, fast, and stable even in the case of low volatility, e.g. $\sigma = 0.05$; see Section 5.

The following result presents an analytical representation for the double-Laplace transform of $f(t,k) := XE(\frac{S_0}{X}A_t - e^{-k})^+$ w.r.t. t and k. Note that Theorem 2.1 holds not only under the BSM but also under other stochastic models. The result reduces the problem of pricing Asian options to the study of real moments of the exponentially-stopped average $E[A_{T_\mu}^{\nu+1}]$.

Theorem 2.1. *Let $\mathcal{L}(\mu,\nu)$ be the double-Laplace transform of $f(t,k)$ w.r.t. t and k, respectively. More precisely, $\mathcal{L}(\mu,\nu) = \int_0^\infty \int_{-\infty}^\infty e^{-\mu t}e^{-\nu k}XE(\frac{S_0}{X}A_t - e^{-k})^+ dk dt$. Then we have that*

$$\mathcal{L}(\mu,\nu) = \frac{XE[A_{T_\mu}^{\nu+1}]}{\mu\nu(\nu+1)}\left(\frac{S_0}{X}\right)^{\nu+1}, \quad \mu > 0, \quad \nu > 0, \tag{1}$$

where $A_{T_\mu} = \int_0^{T_\mu} e^{X(s)}ds$ and T_μ is an exponential random variable with rate μ independent of $\{X(t) : t \geq 0\}$. Here $\mu > 0$ and $\nu > 0$ should satisfy $E[A_{T_\mu}^{\nu+1}] < +\infty$.

Proof: Applying Fubini's theorem yields

$$
\begin{aligned}
\mathcal{L}(\mu, \nu) &= X \int_0^\infty e^{-\mu t} E\left[\int_{-\ln(S_0 A_t/X)}^\infty e^{-\nu k} \left(\frac{S_0}{X} A_t - e^{-k} \right) dk \right] dt \\
&= X \int_0^\infty e^{-\mu t} E\left[\frac{S_0}{X} A_t \int_{-\ln(S_0 A_t/X)}^\infty e^{-\nu k} dk - \int_{-\ln(S_0 A_t/X)}^\infty e^{-(\nu+1)k} dk \right] dt \\
&= X \int_0^\infty e^{-\mu t} E\left[\frac{(S_0 A_t/X)^{\nu+1}}{\nu} - \frac{(S_0 A_t/X)^{\nu+1}}{\nu+1} \right] dt \\
&= X \int_0^\infty e^{-\mu t} \frac{E[A_t^{\nu+1}]}{\nu(\nu+1)} \left(\frac{S_0}{X} \right)^{\nu+1} dt = \frac{X}{\mu\nu(\nu+1)} \left(\frac{S_0}{X} \right)^{\nu+1} \cdot E[A_{T_\mu}^{\nu+1}],
\end{aligned}
$$

from which the proof is completed. $\square$

The idea of taking Laplace transform w.r.t. the log-strike $\ln(K)$ dates back to the work by Carr and Madan [12]. Here we use the scaled log-strike $\ln(X/(Kt))$ instead, as suggested in Petrella [37]. A different double-Laplace transform was given in Fu et al. [21] under the BSM, where the transform is taken w.r.t. t and K.

3 Pricing Asian Options under the BSM

In this section we study Asian option pricing under the BSM via the double-Laplace transforms. More precisely, we investigate the distribution of A_{T_μ} so that we can compute $E[A_{T_\mu}^{\nu+1}]$ explicitly and hence obtain analytical solutions for the double-Laplace transforms, thanks to Theorem 2.1.

3.1 Distribution of A_{T_μ} under the BSM

The classical BSM postulates that under the risk-neutral measure $\mathbb{P}$, the return process $\{X(t) = \log(S(t)/S(0)) : t \geq 0\}$ is given by $X(t) = \left(r - \frac{\sigma^2}{2} \right) t + \sigma W(t)$, $X(0) = 0$, where r is the risk-free rate, σ the volatility, and $\{W(t) : t \geq 0\}$ the standard Brownian motion. The infinitesimal generator of $\{S(t)\}$ is

$$
Lf(s) = \frac{\sigma^2}{2} s^2 f''(s) + rs f'(s) \tag{2}
$$

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for any twice continuously differentiable function $f(\cdot)$, and the Lévy exponent of $\{X(t)\}$ is

$$G(x) := \frac{E\left[e^{xX(t)}\right]}{t} = \frac{\sigma^2}{2}x^2 + \left(r - \frac{\sigma^2}{2}\right)x. \tag{3}$$

Let α_1 and α_2 be the two roots of the equation $G(x) = \mu(> 0)$ under the BSM. Then,

$$\alpha_1 = \frac{-\overline{v} + \sqrt{\overline{v}^2 + 2\overline{\mu}}}{2} > 0, \quad \alpha_2 = \frac{-\overline{v} - \sqrt{\overline{v}^2 + 2\overline{\mu}}}{2} < 0, \tag{4}$$

where $\overline{\mu} = \frac{4\mu}{\sigma^2}$ and $\overline{v} = \frac{2r}{\sigma^2} - 1$.

Consider the following nonhomogeneous ordinary differential equation (ODE)

$$Ly(s) = (s + \mu)y(s) - \mu, \quad \text{for } s \geq 0. \tag{5}$$

Note that the equation (5) has two singularities, a regular singularity at 0 and an irregular singularity at $+\infty$. Due to the singularity, the above equation has infinitely many solutions. However, if we impose an additional condition that the solution must be bounded, then the solution is unique.

Theorem 3.1. (Uniqueness of the solution of the ODE (5) via a stochastic representation) *A bounded solution to the ODE (5), if exists, must be unique. More precisely, suppose $a(s)$ solves the ODE (5) and $\sup_{s \in [0,\infty)} |a(s)| \leq C < \infty$ for some constant $C > 0$. Then we must have*

$$a(s) = E\left[\exp\left(-sA_{T_\mu}\right)\right] \quad \text{for any } s \geq 0. \tag{6}$$

Proof: In terms of $S(t)$, we can rewrite $E[\exp\left(-sA_{T_\mu}\right)]$ as

$$E[\exp\left(-sA_{T_\mu}\right)] = E_s\left[\int_0^\infty \mu \exp\left(-\int_0^t [\mu + S(u)]du\right) dt\right], \tag{7}$$

where the notation E_s means that the process $\{S(t)\}$ starts from s, i.e. $S(0) = s$. By Itô's formula, we have that

$$M_t := a(S(t)) \exp\left(-\int_0^t [\mu + S(u)]du\right) + \int_0^t \mu \exp\left(-\int_0^v [\mu + S(u)]du\right) dv$$

is a local martingale. Indeed, since

$$da(S(t)) = a'(S(t))\left[rS(t)dt + \sigma S(t)dW(t)\right] + \frac{1}{2}a''(S(t))\sigma^2 S^2(t)dt$$

$$= \left[(S(t) + \mu)a(S(t)) - \mu\right]dt + a'(S(t))\sigma S(t)dW(t),$$

where the last equality follows from the fact that $a(s)$ solves the ODE (5), and

$$d\exp\left(-\int_0^t [\mu + S(u)]du\right) = \exp\left(-\int_0^t [\mu + S(u)]du\right) \cdot \{-[\mu + S(t)]\}\,dt\;,$$

we obtain by some algebra that

$$dM_t = \exp\left(-\int_0^t [\mu + S(u)]du\right) \cdot a'(S(t))\sigma S(t)dW(t),$$

which implies that $\{M_t\}$ is a local martingale. Actually, $\{M_t\}$ is a true martingale as M_t is uniformly bounded, $\sup_{t\geq 0}|M_t| \leq \sup_{t\geq 0}\left\{C + \int_0^t \mu e^{-\mu v}dv\right\} = C + 1 < \infty$, because $\mu > 0$ and $S(u) \geq 0$. Thus, $a(s) = a(S(0)) = E_s[M_0] = E_s[M_t]$. Letting $t \to +\infty$, the first term in M_t goes to zero almost surely because $a(\cdot)$ is bounded, and therefore

$$M_t \to \int_0^\infty \mu \exp\left(-\int_0^v \{\mu + S(u)\}du\right)dv,$$

almost surely. Accordingly, by the dominated convergence theorem,

$$a(s) = E_s[\lim_{t\to\infty} M_t] = E_s\left[\int_0^\infty \mu \exp\left(-\int_0^v \{\mu + S(u)\}du\right)dv\right] = E[\exp\left(-sA_{T_\mu}\right)],$$

where the last equality holds due to (7). $\square$

Theorem 3.1 implies that if we can find a particular bounded solution to the ODE (5), it must have the stochastic representation in (6). To find such a one, consider a difference equation (or a recursion) for a function $H(\nu)$ defined on $(-1, \alpha_1)$

$$h(\nu)H(\nu) = \nu H(\nu - 1) \quad \text{for any } \nu \in (0, \alpha_1), \quad \text{and} \quad H(0) = 1, \tag{8}$$

$$h(\nu) \equiv \mu - G(\nu) = -\frac{\sigma^2}{2}\nu^2 - \left(r - \frac{\sigma^2}{2}\right)\nu + \mu = -\frac{\sigma^2}{2}(\nu - \alpha_1)(\nu - \alpha_2). \tag{9}$$

In general, if the above difference equation (8) has one solution $H_1(\nu)$, then there exist an infinite number of solutions to (8). In fact, any function in the following class

$$\left\{ H(\nu) = \theta(\nu)H_1(\nu) : \text{ for some periodic function } \theta(\nu) \text{ s.t. } \theta(\nu) = \theta(\nu - 1) \text{ for any } \nu \in (0, \alpha_1) \right\}$$

also solves (8). This also partly explains why very few people investigated Asian option pricing based on this recursion. However, we shall show next that the difference equation (8) has a unique solution if we restrict our attention to random variables.

Theorem 3.2. (A particular bounded solution to the ODE (5)) *If there exists a non-negative random variable X such that $H(\nu) := E[X^\nu]$ satisfies the difference equation (8), then the Laplace transform of X, i.e. $E[e^{-sX}]$, solves the nonhomogeneous ODE (5).*

Proof: Denote the Laplace transform of X by $y(s) = E[e^{-sX}]$, for $s \geq 0$. Note that for any $a \in (0, \min(\alpha_1, 1))$, we have

$$\int_0^{+\infty} s^{-a} e^{-sX} ds = \Gamma(1-a)X^{a-1} \quad \text{and} \quad \int_0^{+\infty} s^{-a-1}\left(e^{-sX} - 1\right) ds = -\frac{\Gamma(1-a)}{a}X^a,$$

where the second equality holds due to integration by parts. Taking expectations on both sides of the two equations above and applying Fubini's theorem yields

$$E[X^{a-1}] = \frac{1}{\Gamma(1-a)}\int_0^\infty s^{-a}y(s)ds \quad \text{and} \quad E[X^a] = -\frac{a}{\Gamma(1-a)}\int_0^\infty s^{-a-1}\left(y(s) - 1\right) ds.$$

Thus, by the difference equation (8), we have

$$-\frac{ah(a)}{\Gamma(1-a)}\int_0^\infty s^{-a-1}\left(y(s) - 1\right) ds = \frac{a}{\Gamma(1-a)}\int_0^\infty s^{-a}y(s)ds,$$

i.e.

$$0 = \int_0^\infty s^{-a-1}\left[sy(s) + h(a)(y(s) - 1)\right] ds,$$

where $h(a)$ is given by (9). Setting $s = e^{-x}$ and $z(x) = y(s) - 1$, we have

$$0 = \int_{-\infty}^\infty e^{ax}\left\{e^{-x}(z(x) + 1) + h(a)z(x)\right\} dx, \quad \text{for any } a \in (0, \min(\alpha_1, 1)).$$

For simplicity of notations, rewrite $h(a)$ as $h(a) = h_0 a^2 + h_1 a + h_2$, with $h_0 = -\frac{\sigma^2}{2}$, $h_1 = -r + \frac{\sigma^2}{2}$, and $h_2 = \mu$. Note that integration by parts yields

$$\int_{-\infty}^{\infty} e^{ax} a z(x) dx = -\int_{-\infty}^{\infty} e^{ax} z'(x) dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{ax} a^2 z(x) dx = \int_{-\infty}^{\infty} e^{ax} z''(x) dx$$

because $[z(x)e^{ax}]\Big|_{x=-\infty}^{+\infty} = 0$ and $[z'(x)e^{ax}]\Big|_{x=-\infty}^{\infty} = 0$. Then for any $a \in (0, \min(\alpha_1, 1))$,

$$
\begin{aligned}
0 &= \int_{-\infty}^{\infty} e^{ax} \left\{ e^{-x}(z(x) + 1) + \left(h_0 a^2 + h_1 a + h_2\right) z(x) \right\} dx \\
&= \int_{-\infty}^{\infty} e^{ax} \left\{ e^{-x}(z(x) + 1) + h_0 z''(x) - h_1 z'(x) + h_2 z(x) \right\} dx.
\end{aligned}
$$

By the uniqueness of the moment generating function, we have an ODE

$$h_0 z''(x) - h_1 z'(x) + h_2 z(x) + e^{-x}(z(x) + 1) = 0.$$

Now transferring the ODE for $z(x)$ back to that for $y(s)$, with $s = e^{-x}$ we have $z(x) = y(s) - 1$, $z'(x) = -sy'(s)$, and $z''(x) = sy'(s) + s^2 y''(s)$. Then the ODE becomes

$$h_0 s^2 y''(s) + (h_1 + h_0)sy'(s) + (h_2 + s)y(s) = h_2.$$

Substituting h_0, h_1, and h_2 into the above equation, we have the nonhomogeneous ODE (5). $\square$

Theorem 3.3. *Under the BSM, we have*

$$A_{T_\mu} =_d \frac{2}{\sigma^2} \frac{Z(1, -\alpha_2)}{Z(\alpha_1)} \tag{10}$$

and therefore

$$E[A_{T_\mu}^\nu] = \left(\frac{2}{\sigma^2}\right)^\nu \frac{\Gamma(\nu + 1)\Gamma(\alpha_1 - \nu)\Gamma(1 - \alpha_2)}{\Gamma(\alpha_1)\Gamma(-\alpha_2 + \nu + 1)}, \quad \text{for any } \nu \in (-1, \alpha_1). \tag{11}$$

Here $Z(a, b)$ denotes a beta random variable with parameters a and b, $Z(a)$ a gamma random variable with scale parameter 1 and shape parameter a, and $\Gamma(\cdot)$ the gamma function. Moreover, $Z(1, -\alpha_2)$ and $Z(\alpha_1)$ are independent with α_1 and α_2 given by (4).

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Proof: Consider a random variable χ such that $\chi =_d \frac{2}{\sigma^2} Z(1, -\alpha_2)/Z(\alpha_1)$. Then

$$E[\chi^\nu] = \left(\frac{2}{\sigma^2}\right)^\nu \frac{\Gamma(\nu+1)\Gamma(\alpha_1-\nu)\Gamma(1-\alpha_2)}{\Gamma(\alpha_1)\Gamma(-\alpha_2+\nu+1)}, \quad \text{for any } \nu \in (-1, \alpha_1),$$

and furthermore, it can be easily verified that $E[\chi^\nu]$ solves the difference equation (8). By Theorem 3.2, we conclude that $a^*(s) := E[e^{-s\chi}]$ for any $s \geq 0$ is a particular bounded solution to the ODE (5). As a result, it follows from Theorem 3.1 that $A_{T_\mu} =_d \chi =_d \frac{2}{\sigma^2} Z(1, -\alpha_2)/Z(\alpha_1)$, which gives the distribution of A_{T_μ}. $\square$

Remarks: 1. There are various ways to show that $E[A_{T_\mu}^\nu]$ satisfies recursions similar to (8) for general processes. For example, Dufresne [20] used time reversal and Itô's formula to derive the recursion (8) for the BSM, and Carmona et al. [11] obtained similar recursions for general Lévy-type processes. In Appendix C we shall give a new proof for a recursion similar to (8) under the HEM, although that proof is not needed to study Asian options.

2. The result (10) coincides with that in Yor [50] and Matsumoto and Yor [33]. However, compared with the existing proofs involving Bessel processes and Lamperti's representation, our approach is simpler and more elementary. Furthermore, we illustrate in Section 4 that our approach is more general, because it can be extended to the case of the HEM.

3.2 Pricing Formulae and Hedging Parameters under the BSM

Theorem 3.4. *Under the BSM, for every μ and ν such that $\mu > 0$ and $\nu \in (0, \alpha_1 - 1)$, the double-Laplace transform of $XE(\frac{S_0}{X} A_t - e^{-k})^+$ w.r.t. t and k is given by:*

$$\mathcal{L}(\mu, \nu) = \frac{X}{\mu\nu(\nu+1)} \left(\frac{2S_0}{X\sigma^2}\right)^{\nu+1} \frac{\Gamma(\nu+2)\Gamma(\alpha_1-\nu-1)\Gamma(1-\alpha_2)}{\Gamma(\alpha_1)\Gamma(\nu-\alpha_2+2)}. \tag{12}$$

Therefore, the Asian option price is equal to:

$$P(t, k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1}\left(\mathcal{L}(\mu, \nu)\right)\Big|_{k=\ln(X/Kt)},$$

where $\mathcal{L}^{-1}$, a function of t and k, denotes the Laplace inversion of $\mathcal{L}$. Furthermore, for any maturity t and strike K, two common greeks delta $\Delta(P(t, k))$ and gamma $\Gamma(P(t, k))$ can be

calculated as follows

$$\Delta(P(t,k)) = \frac{\partial}{\partial S_0} P(t,k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1} \left(\frac{XS_0^{\nu}}{\mu\nu} \left(\frac{2}{X\sigma^2} \right)^{\nu+1} \frac{\Gamma(\nu+2)\Gamma(\alpha_1 - \nu - 1)\Gamma(1-\alpha_2)}{\Gamma(\alpha_1)\Gamma(\nu - \alpha_2 + 2)} \right) \bigg|_{k=\ln(X/Kt)}$$

$$\Gamma(P(t,k)) = \frac{\partial^2}{\partial S_0^2} P(t,k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1} \left(\frac{XS_0^{\nu-1}}{\mu} \left(\frac{2}{X\sigma^2} \right)^{\nu+1} \frac{\Gamma(\nu+2)\Gamma(\alpha_1 - \nu - 1)\Gamma(1-\alpha_2)}{\Gamma(\alpha_1)\Gamma(\nu - \alpha_2 + 2)} \right) \bigg|_{k=\ln(X/Kt)}.$$

Proof. Combining (11) with (1) yields (12). The two greeks can be obtained by interchanging derivatives and integrals based on Theorem A. 12 on pp. 203-204 in Schiff [39]. □

Theorem 3.4 requires $(0, \alpha_1 - 1)$ to be nonempty, i.e. $\alpha_1 > 1$, which means some Laplace parameter $\mu > 0$ may be disqualified. Nonetheless, a broad range of μ meets the requirement. For example, it is sufficient to have $\mu > r$, because $G(1) = r$, α_1 solves the equation $G(x) = \mu$ and $G(x) - \mu$ is a continuous function. Furthermore, this restriction $\mu > r$ does not present any difficulty in term of numerical Laplace inversion.

4 Pricing Asian Options under the HEM

In the HEM, the asset return process $\{X(t) : t \geq 0\}$ under a risk-neutral measure $\mathbb{P}$ is given by

$$X(t) = \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta \right) t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \qquad X(0) = 0,$$

where r is the risk-free rate, σ the volatility, $\zeta := E(e^{Y_1}) - 1 = \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - 1} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + 1} - 1$, $\{W(t) : t \geq 0\}$ the standard Brownian motion, $\{N(t) : t \geq 0\}$ a Poisson process with rate λ, and $\{Y_i : i \in \mathbb{N}\}$ i.i.d. hyper-exponentially distributed random variables with the probability density function (pdf)

$$f_Y(y) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} I_{\{y \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{\theta_j y} I_{\{y < 0\}}, \tag{13}$$

with $p_i > 0$, $\eta_i > 1$, for $i = 1, \cdots, m$, $q_j > 0$, $\eta_j > 0$, for $j = 1, \cdots, n$, and $\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1$.

Due to the jumps, the risk-neutral measure is not unique. Here we assume the risk-neutral measure $\mathbb{P}$ is chosen within a rational expectations equilibrium setting such that the equilibrium price of an option is given by the expectation under $\mathbb{P}$ of the discounted option payoff. For

details, refer to Kou [28]. It is worth mentioning that when $m = n = 0$ the HEM is reduced to the BSM, and when $m = n = 1$ the HEM is reduced to the double exponential jump diffusion model. The Lévy exponent of $\{X_t\}$ is given by

$$G(x) := \frac{E\left[e^{xX(t)}\right]}{t} = \frac{1}{2}\sigma^2 x^2 + \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right)x + \lambda\left(\sum_{i=1}^{m}\frac{p_i\eta_i}{\eta_i - x} + \sum_{j=1}^{n}\frac{q_j\theta_j}{\theta_j + x} - 1\right) \quad (14)$$

for any $x \in (-\theta_1, \eta_1)$. It can be shown (see Cai [8]) that for any $\mu > 0$, the equation $G(x) = \mu$ has exactly $(m + n + 2)$ real roots $\beta_{1,\mu}, \cdots, \beta_{m+1,\mu}, \gamma_{1,\mu}, \cdots, \gamma_{n+1,\mu}$ satisfying

$$-\infty < \gamma_{n+1,\mu} < -\theta_n < \gamma_{n,\mu} < \cdots < -\theta_1 < \gamma_{1,\mu} < 0 < \beta_{1,\mu} < \eta_1 < \cdots < \beta_{m,\mu} < \eta_m < \beta_{m+1,\mu} < \infty. \quad (15)$$

Additionally, the infinitesimal generator of $\{S(t) = S(0)e^{X(t)} : t \geq 0\}$ is given by

$$Lf(s) = \frac{\sigma^2}{2}s^2 f''(s) + (r - \lambda\zeta)s f'(s) + \lambda\int_{-\infty}^{+\infty}[f(se^u) - f(s)]f_Y(u)du, \quad (16)$$

for any twice continuously differentiable function $f(\cdot)$.

4.1 Distribution of A_{T_μ} under the HEM

Consider the following nonhomogeneous ordinary integro-differential equation (OIDE)

$$Ly(s) = (s + \mu)y(s) - \mu, \quad (17)$$

where L is given by (16). Similarly as in the BSM, we have the following theorem.

Theorem 4.1. (Uniqueness of the solution of the OIDE (17) via a stochastic representation) *There is at most one bounded solution to the OIDE (17). More precisely, suppose $a(s)$ solves (17) and $\sup_{s\in[0,\infty)}|a(s)| \leq C < \infty$ for some constant $C > 0$. Then we must have*

$$a(s) = E\left[\exp\left(-sA_{T_\mu}\right)\right] \quad \text{for any } s \geq 0. \quad (18)$$

Proof: See Appendix A. $\square$

Since the proofs for Theorem 3.1 and 4.1 involve only Itô's formula (see Section 1.2 and 1.3 of Øksendal and Sulem [36] or Applebaum [5] for more general Itô formulae), the result

holds for more general underlying process $S(t)$ such as exponential Lévy processes and Lévy diffusions. Next, we look for a particular bounded solution to the OIDE (17), which has the stochastic representation in (18). Consider a difference equation (or a recursion) for a function $H(\nu)$ defined on $(-1, \beta_1)$

$$h(\nu)H(\nu) = \nu H(\nu - 1) \quad \text{for any } \nu \in (0, \beta_1), \quad \text{and} \quad H(0) = 1, \tag{19}$$

where

$$
\begin{aligned}
h(\nu) \equiv & \mu - G(\nu) = \mu - \frac{1}{2}\sigma^2\nu^2 - (r - \frac{1}{2}\sigma^2 - \lambda\zeta)\nu - \lambda\left(\sum_{i=1}^{m}\frac{p_i\eta_i}{\eta_i - \nu} + \sum_{j=1}^{n}\frac{q_j\theta_j}{\theta_j + \nu} - 1\right) \\
= & \left(\frac{\sigma^2}{2}\right)\frac{\prod_{i=1}^{m+1}(\beta_i - \nu)\prod_{j=1}^{n+1}(-\gamma_j + \nu)}{\prod_{i=1}^{m}(\eta_i - \nu)\prod_{j=1}^{n}(\theta_j + \nu)}.
\end{aligned}
\tag{20}
$$

Here $\beta_1, \cdots, \beta_{m+1}, \gamma_1, \cdots, \gamma_{n+1}$ are actually $\beta_{1,\mu}, \cdots, \beta_{m+1,\mu}, \gamma_{1,\mu}, \cdots, \gamma_{n+1,\mu}$, which are $(m + n + 2)$ real roots of the equation $G(x) = \mu$ satisfying the condition (15).

Theorem 4.2. (A particular bounded solution to the OIDE (17)) *If there is a nonnegative random variable X such that $H(\nu) := E[X^\nu]$ satisfies the difference equation (19), then the Laplace transform of X, i.e. $E[e^{-sX}]$, solves the nonhomogeneous OIDE (17).*

Proof: See Appendix B. $\square$

Theorem 4.3. *Under the HEM, we have*

$$A_{T_\mu} =_d \frac{2}{\sigma^2}\frac{Z(1, -\gamma_1)\prod_{j=1}^{n}Z(\theta_j + 1, -\gamma_{j+1} - \theta_j)}{Z(\beta_{m+1})\prod_{i=1}^{m}Z(\beta_i, \eta_i - \beta_i)}, \tag{21}$$

where all the gamma and beta random variables on the RHS are independent, and therefore for any $\nu \in (-1, \beta_1)$,

$$
\begin{aligned}
&E[A_{T_\mu}^\nu] \\
&= \left(\frac{2}{\sigma^2}\right)^\nu\frac{\Gamma(1 + \nu)\Gamma(1 - \gamma_1)}{\Gamma(1 - \gamma_1 + \nu)} \cdot \prod_{j=1}^{n}\left[\frac{\Gamma(\theta_j + 1 + \nu)\Gamma(1 - \gamma_{j+1})}{\Gamma(1 - \gamma_{j+1} + \nu)\Gamma(\theta_j + 1)}\right] \cdot \prod_{i=1}^{m}\left[\frac{\Gamma(\beta_i - \nu)\Gamma(\eta_i)}{\Gamma(\eta_i - \nu)\Gamma(\beta_i)}\right] \cdot \frac{\Gamma(\beta_{m+1} - \nu)}{\Gamma(\beta_{m+1})}.
\end{aligned}
\tag{22}
$$

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Proof: Consider a random variable χ that is defined by the right side of (21). Then simple algebra shows that $E[\chi^\nu]$ is given by the right side of (22). Moreover, we can verify that $E[\chi^\nu]$ solves the difference equation (19). By Theorem 4.2, $a^*(s) := E e^{-s\chi}$, for any $s \geq 0$, is a particular bounded solution to the OIDE (17), and the proof is terminated via Theorem 4.1. $\square$

4.2 Pricing Formulae for Asian Options and Hedging Parameters under the HEM

Theorem 4.4. *Under the HEM, for every μ and ν such that $\mu > 0$ and $\nu \in (0, \beta_1 - 1)$, the double-Laplace transform of $X E(\frac{S_0}{X} A_t - e^{-k})^+$ w.r.t. t and k is given by:*

$$\mathcal{L}(\mu, \nu) = \frac{X}{\mu\nu(\nu+1)} \left(\frac{2S_0}{X\sigma^2}\right)^{\nu+1} \frac{\Gamma(2+\nu)\Gamma(1-\gamma_1)}{\Gamma(2-\gamma_1+\nu)}$$
$$\cdot \prod_{j=1}^n \left[\frac{\Gamma(\theta_j+2+\nu)\Gamma(1-\gamma_{j+1})}{\Gamma(2-\gamma_{j+1}+\nu)\Gamma(\theta_j+1)}\right] \cdot \prod_{i=1}^m \left[\frac{\Gamma(\beta_i-\nu-1)\Gamma(\eta_i)}{\Gamma(\eta_i-\nu-1)\Gamma(\beta_i)}\right] \cdot \frac{\Gamma(\beta_{m+1}-\nu-1)}{\Gamma(\beta_{m+1})}. \quad (23)$$

Therefore, the Asian option price is equal to

$$P(t,k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1}\left(\mathcal{L}(\mu,\nu)\right)|_{k=\ln(X/Kt)}.$$

And two common greeks delta $\Delta(P(t,k))$ and gamma $\Gamma(P(t,k))$ can be calculated as follows

$$\Delta(P(t,k)) = \frac{\partial}{\partial S_0} P(t,k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1}\left(\frac{X S_0^\nu}{\mu\nu}\left(\frac{2}{X\sigma^2}\right)^{\nu+1} \cdot \frac{\Gamma(2+\nu)\Gamma(1-\gamma_1)}{\Gamma(2-\gamma_1+\nu)}\right.$$
$$\left.\cdot \prod_{j=1}^n \left[\frac{\Gamma(\theta_j+2+\nu)\Gamma(1-\gamma_{j+1})}{\Gamma(2-\gamma_{j+1}+\nu)\Gamma(\theta_j+1)}\right] \cdot \prod_{i=1}^m \left[\frac{\Gamma(\beta_i-\nu-1)\Gamma(\eta_i)}{\Gamma(\eta_i-\nu-1)\Gamma(\beta_i)}\right] \cdot \frac{\Gamma(\beta_{m+1}-\nu-1)}{\Gamma(\beta_{m+1})}\right)\Big|_{k=\ln(X/Kt)}$$

$$\Gamma(P(t,k)) = \frac{\partial^2}{\partial S_0^2} P(t,k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1}\left(\frac{X S_0^{\nu-1}}{\mu}\left(\frac{2}{X\sigma^2}\right)^{\nu+1} \cdot \frac{\Gamma(2+\nu)\Gamma(1-\gamma_1)}{\Gamma(2-\gamma_1+\nu)}\right.$$
$$\left.\cdot \prod_{j=1}^n \left[\frac{\Gamma(\theta_j+2+\nu)\Gamma(1-\gamma_{j+1})}{\Gamma(2-\gamma_{j+1}+\nu)\Gamma(\theta_j+1)}\right] \cdot \prod_{i=1}^m \left[\frac{\Gamma(\beta_i-\nu-1)\Gamma(\eta_i)}{\Gamma(\eta_i-\nu-1)\Gamma(\beta_i)}\right] \cdot \frac{\Gamma(\beta_{m+1}-\nu-1)}{\Gamma(\beta_{m+1})}\right)\Big|_{k=\ln(X/Kt)}$$

Proof. The proof is similar to that of Theorem 3.4. $\square$

5 Pricing Asian Options via a Two-Sided Euler Inversion Algorithm with a Scaling Factor

In this section, we intend to price Asian options under both the BSM and the HEM by inverting $\mathcal{L}(\mu,\nu)$ in (12) and (23) numerically. The algorithm used here is proposed in Petrella [37],

which, as a generalization of the one-sided Euler inversion algorithm ([1] and [14]), introduces a two-sided Euler inversion with a scaling factor.

The inversion formula in Petrella [37] to get $f(t,k)$ from its Laplace transform $\mathcal{L}(\mu,\nu)$ is

$$
\begin{aligned}
f(t,k) &= \frac{\exp(A_1/2 + A_2/2)}{4tk} \\
&\times \left\{ \mathcal{L}\left(\frac{A_1}{2t}, \frac{A_2}{2k}\right) + 2\sum_{s=0}^{\infty}(-1)^s Re\left[-\mathcal{L}\left(\frac{A_1}{2t}, \frac{A_2}{2k} - \frac{i\pi}{k} - \frac{is\pi}{k}\right)\right] \right. \\
&+ 2\sum_{s=0}^{\infty}(-1)^s Re\left[-\mathcal{L}\left(\frac{A_1}{2t} - \frac{i\pi}{t} - \frac{is\pi}{t}, \frac{A_2}{2k}\right)\right] \\
&+ 2\sum_{j=0}^{\infty}(-1)^j Re\left[\sum_{s=0}^{\infty}(-1)^s \mathcal{L}\left(\frac{A_1}{2t} - \frac{i\pi}{t} - \frac{ij\pi}{t}, \frac{A_2}{2k} - \frac{i\pi}{k} - \frac{is\pi}{k}\right)\right] \\
&+ \left. 2\sum_{j=0}^{\infty}(-1)^j Re\left[\sum_{s=0}^{\infty}(-1)^s \mathcal{L}\left(\frac{A_1}{2t} - \frac{i\pi}{t} - \frac{ij\pi}{t}, \frac{A_2}{2k} + \frac{i\pi}{k} + \frac{is\pi}{k}\right)\right] \right\} \\
&- e_d^+ - e_d^-,
\end{aligned}
\tag{24}
$$

where the two errors are given by

$$
e_d^+ = \sum_{j_2=1}^{\infty}\sum_{j_1=0}^{\infty} e^{-(j_1 A_1 + j_2 A_2)} f((2j_1+1)t, (2j_2+1)k) + \sum_{j_1=1}^{\infty} e^{-j_1 A_1} f((2j_1+1)t, k), \tag{25}
$$

$$
e_d^- = \sum_{j_2=-\infty}^{-1}\sum_{j_1=0}^{\infty} e^{-(j_1 A_1 + j_2 A_2)} f((2j_1+1)t, (2j_2+1)k), \tag{26}
$$

and A_1 and A_2 (to be specified later) are some inversion parameters used to control the errors.

The inversion appears to be accurate, stable, and easy to implement. For example, under the BSM, the prices produced via our algorithm highly agree with benchmarks generated by the other three important methods by Linetsky, Geman-Yor-Shaw, and Vecer, and it is stable even for low volatilities (e.g. $\sigma = 0.05$). The algorithm is easy to implement primarily because the closed-form Laplace transform involves only gamma functions. We also study the algorithm's stability, derive a discretization error bound, and conduct a comparison with the Fourier and Laplace inversion algorithm by Fusai [22]; see the online supplement.

In the inversion, we have to calculate alternating series of the form $\sum_{i=0}^{\infty}(-1)^i a_i$ in the expression (24). To accelerate the convergence rate, we adopt the idea of Euler transformation

(see [1] and [14]), and approximate $\sum_{i=0}^{\infty}(-1)^i a_i$ by $E(m,n) := \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} 2^{-m} S_{n+k}$, where $S_j := \sum_{i=0}^{j}(-1)^i a_i$. Since there are two transforms, Euler transformation will be used twice. We use (n_1, m_1) and (n_2, m_2) to express parameters involved in Euler transformations for Euler inversion w.r.t. t and k, respectively. More precisely, for the first infinite sum and the inner series of both the third and fourth double sum on the RHS of (24), we use Euler transformation with parameters (n_2, m_2); while for the second infinite sum and the outer series of both the third and fourth double sum on the RHS of (24), we use Euler transformation with parameters (n_1, m_1). As suggested in Abate and Whitt [1], we shall set $m_1 = n_1 + 15$ and $m_2 = n_2 + 15$.

In the inversion algorithm there are several parameters to be chosen: (1) (n_1, n_2). The larger n_1 and n_2 will lead to better accuracy at the cost of computation time. In our experiments, $n_1 = n_2 = 35$ seems to achieve excellent accuracy even for low volatilities (e.g. $\sigma = 0.05$); usually even $n_1 = 15$ and $n_2 = 35$ can yield good accuracy if the volatility is not low. (2) (A_1, A_2). (3) The scaling factor X. One appealing feature of the inversion algorithm is its insensitivity to the choices of A_1, A_2, and X even for low volatilities. In other words, for wide ranges of parameters A_1, A_2, and X, the numerical results are almost identical. This is illustrated in Figure 1 in the online supplement. For convenience, in most cases we simply select $A_1 = 28$, $A_2 = 40$, and, as suggested in Petrella [37],

$$X = Kt \cdot e^k = Kt \cdot \exp\left\{\min\left(\frac{A_2}{2/(\sigma\sqrt{t})}, \frac{A_2}{4}\right)\right\}. \tag{27}$$

for practical implementation, although one can freely choose other values.

5.1 Pricing Asian Options under the BSM

5.1.1 Comparison of Accuracy with Other Methods

To check the accuracy of our double-Laplace (DL) inversion algorithm, we consider seven test cases in Table 1, which are frequently used in the literature ([21], [32], [43], [17], [48]). A detailed comparison of accuracy between our double-Laplace inversion algorithm with those obtained from six other existing methods is given in Table 2. From Table 2, we can see that our DL prices highly agree with benchmarks of existing methods. Specifically, our DL prices agree

Case No.	S_0	K	r	σ	t
1	2.0	2	0.02	0.10	1
2	2.0	2	0.18	0.30	1
3	2.0	2	0.0125	0.25	2
4	1.9	2	0.05	0.50	1
5	2.0	2	0.05	0.50	1
6	2.1	2	0.05	0.50	1
7	2.0	2	0.05	0.50	2

Table 1: Seven test cases in the literature.

with both Geman-Yor-Shaw's and Linetsky's results to ten decimal points and with Vecer's results to six decimal points. Moreover, our DL prices also agree with Zhang's PDE results to six decimal points. These consistence indicates that our double-Laplace inversion algorithm is very accurate and can also be used as benchmarks for pricing Asian options.

Case	DL Prices	Linetsky	GY-Shaw	Vecer	Zhang	GYS-Mellin	MAE3
1	0.0559860415	0.0559860415	0.0559860415	0.055986	0.055986	0.0559856	0.055986
2	0.2183875466	0.2183875466	0.2183875466	0.218388	0.218388	0.218359	0.218663
3	0.1722687410	0.1722687410	0.1722687410	0.172269	0.172269	0.172263	0.172263
4	0.1931737903	0.1931737903	0.1931737903	0.193174	0.193174	0.193060	0.193188
5	0.2464156905	0.2464156905	0.2464156905	0.246416	0.246416	0.246522	0.246382
6	0.3062203648	0.3062203648	0.3062203648	0.306220	0.306220	0.306501	0.306139
7	0.3500952190	0.3500952190	0.3500952190	0.350095	0.350095	0.348924	0.349909

Table 2: Comparison of accuracy with other existing methods. The parameters associated with our double-Laplace (DL) inversion method are $n_1 = 35$, $n_2 = 55$, $A_1 = 28$, $A_2 = 40$, and X given by (27). Results of Linetsky's eigenfunction expansion method are taken from Table 3 in [32]. Vecer's PDE results are from Table A in of [48]. The " GYS-Mellin" numbers are taken from a Mellin transformed-based approximation in [43]. The other three columns, including "GY-Shaw," "Zhang," and "MAE3", are all taken from the table on p. 383 in [17], and correspond to the methods in [41], [51], and [17], respectively. Our numerical DL prices are calculated using Matlab 7.1 on a desktop with Quad CPU 2.66 GHz.

Although Shaw's GYS-Mellin results and Dewynne and Shaw's MAE3 results seem less accurate than other methods, they are still sufficiently accurate in practice and moreover, these two methods have their own advantages. First, Shaw's GYS-Mellin method turns out to be very fast. For example, in Case 1 in Table 2 and on a desktop with Quad CPU 2.66 GHz, it takes only 0.047 seconds to produce one result; whereas our pricing method requires 0.563 seconds to match the GYS-Mellin result to five decimal points. Second, these two methods work better than most other methods when the volatility is extremely low; for details, see Section 5.1.2.

Note that although in our theorems the dividend $\delta = 0$, they can be easily extended to the case of nonzero dividends. Indeed for the two families of the extended seven cases in Section 6.2

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$(\delta > r)$ and Section 6.3 $(\delta = r)$ in [17], our DL prices, GY-Shaw prices (or its variant CIBess prices) and Zhang's results agree with one another to six or seven decimal points, hence being more accurate than MAE3; see Table 3.

Extension of Seven Cases in Table 1 as $\delta > r$ (see Section 6.2 in [17]).										
Case	S_0	K	r	δ	σ	t	DL Prices	GY-Shaw	MAE3	Zhang
1*	2.0	2	0.02	0.04	0.1	1	0.0357853	0.0357853	0.0357854	0.0357853
2*	2.0	2	0.18	0.36	0.3	1	0.0522607	0.0522607	0.0522755	0.0522607
3*	2.0	2	0.0125	0.025	0.25	2	0.145308	0.145308	0.145310	0.145308
4*	1.9	2	0.05	0.1	0.5	1	0.147562	0.147562	0.147618	0.147562
5*	2.0	2	0.05	0.1	0.5	1	0.191747	0.191747	0.191760	0.191747
6*	2.1	2	0.05	0.1	0.5	1	0.242316	0.242316	0.242283	0.242316
7*	2.0	2	0.05	0.1	0.5	2	0.240495	0.240495	0.240564	0.240495
Extension of Seven Cases in Table 1 as $\delta = r$ (see Section 6.3 in [17]).										
Case	S_0	K	r	δ	σ	t	DL Prices	CIBess	MAE3	Zhang
1**	2.0	2	0.02	0.02	0.1	1	0.0451431	0.0451431	0.0451431	0.0451431
2**	2.0	2	0.18	0.18	0.3	1	0.115188	0.115188	0.115188	0.115188
3**	2.0	2	0.0125	0.0125	0.25	2	0.158380	0.158380	0.158378	0.158380
4**	1.9	2	0.05	0.05	0.5	1	0.169202	0.169202	0.169238	0.169202
5**	2.0	2	0.05	0.05	0.5	1	0.217815	0.217815	0.217805	0.217815
6**	2.1	2	0.05	0.05	0.5	1	0.272924	0.272924	0.272869	0.272924
7**	2.0	2	0.05	0.05	0.5	2	0.291315	0.291315	0.291264	0.291315

Table 3: Comparison of accuracy in extended cases when $\delta > r$ or $\delta = r$

5.1.2 Comparison of Behaviors for Low Volatilities

It is well known that many numerical methods for Asian option pricing do not perform well for low volatilities (see [15, 21, 42]). Here we would like to conduct cross-comparisons of behaviors for reasonably low (e.g., $\sigma = 0.05$) and extremely low volatilities (e.g., $\sigma \leq 0.01$) between our double-Laplace inversion method and the three methods discussed in [17], GY-Shaw (or its variants GYS-Full and CIBess), MAE3, and Zhang's method, for three cases $\delta < r$, $\delta > r$ and $\delta = r$, where δ denotes the dividend (in [17] the dividend is denoted by q).

To investigate the behaviors of these algorithms in the case of low volatilities, we modify the test case 1 in Table 1 and the test cases 1* and 1** in Table 3 by letting the volatility σ be small; see Table 4. Note that the parameter settings are the same as in Section 6.1–6.3 in [17]. Table 4 demonstrates that when the volatility is reasonably small $(= 0.05)$, the numerical results obtained by the four methods coincide with one another to six or seven decimal points.

Nonetheless, when the volatility is extremely small (≤ 0.01), MAE3 and Zhang's results are still available and highly agree with each other, but our double-Laplace inversion method and GY-Shaw methods no longer work. Consequently, we conclude that our methods and GY-Shaw methods are reliable in the case of normal or reasonably low volatilities ($\sigma \geq 0.05$). However, we may use Dewynne and Shaw's asymptotic method or Zhang's PDE method for the extremely low volatilities ($\sigma < 0.05$).

Extension of Case 1 in Table 1 when $\delta < r$ and σ is extremely small (Section 6.1 in [17]).										
Case	S_0	K	r	δ	σ	t	DL Prices	GY-Shaw	MAE3	Zhang
1	2.0	2	0.02	0	0.1	1	0.0559860	0.0559860	0.0559860	0.0559860
1A	2.0	2	0.02	0	0.05	1	0.0339412	0.0339412	0.0339412	0.0339412
1B	2.0	2	0.02	0	0.01	1	NA	NA	0.0199278	0.0199278
1C	2.0	2	0.02	0	0.005	1	NA	NA	0.0197357	0.0197357
1D	2.0	2	0.02	0	0.001	1	NA	NA	0.0197353	0.0197353
Extension of Case 1* in Table 3 when $\delta > r$ and σ is extremely small (see Section 6.2 in [17]).										
Case	S_0	K	r	δ	σ	t	DL Prices	GYS-full	MAE3	Zhang
1 *	2.0	2	0.02	0.04	0.1	1	0.0357853	0.0357853	0.0357854	0.0357853
1A*	2.0	2	0.02	0.04	0.05	1	0.0140247	0.0140247	0.0140248	0.0140247
1B*	2.0	2	0.02	0.04	0.01	1	NA	NA	0.000190254	0.000190254
1C*	2.0	2	0.02	0.04	0.005	1	NA	NA	3.7991×10^{-7}	3.7993×10^{-7}
1D*	2.0	2	0.02	0.04	0.001	1	NA	NA	$o(10^{-70})$	$o(10^{-72})$
Extension of Case 1** in Table 3 when $\delta = r$ and σ is extremely small (see Section 6.3 in [17]).										
Case	S_0	K	r	δ	σ	t	DL Prices	CIBess	MAE3	Zhang
1**	2.0	2	0.02	0.02	0.1	1	0.0451431	0.0451431	0.0451431	0.0451431
1A**	2.0	2	0.02	0.02	0.05	1	0.0225755	0.0225755	0.0225755	0.0225755
1B**	2.0	2	0.02	0.02	0.01	1	NA	NA	0.00451536	0.00451536
1C**	2.0	2	0.02	0.02	0.005	1	NA	NA	0.00225768	0.00225768
1D**	2.0	2	0.02	0.02	0.001	1	NA	NA	0.000451537	0.000451537

Table 4: Comparison of accuracy for extremely low volatilities when $\delta < r$, $\delta > r$ or $\delta = r$

5.2 Pricing Asian Options under the HEM

In this section, we price Asian options numerically under the HEM by inverting the double-Laplace transform (23) via the two-sided Euler inversion algorithm. Without loss of generality, we concentrate on Kou's model, which along with the BSM are the most important special cases of the HEM. Table 5 and Table 6 provide the numerical results of Asian option prices using double-Laplace transform in the cases of $\lambda = 3$ and $\lambda = 5$, respectively. The double-Laplace inversion method seems to be still accurate under Kou's model. Furthermore, our algorithm is

efficient because it takes only around 6 seconds to produce one numerical result on a desktop with Quad CPU 2.66 GHz.

σ	K	DL Prices	MC Prices	Std Err	Abs Err	Rel Err
0.05	90	13.41924	13.42054	0.00048	-0.00130	0.0097%
0.05	95	8.98812	8.98730	0.00095	0.00082	0.0091%
0.05	100	4.95673	4.95681	0.00162	-0.00008	0.0016%
0.05	105	2.13611	2.13453	0.00208	0.00158	0.0740%
0.05	110	0.83091	0.82995	0.00193	0.00096	0.1200%
0.1	90	13.48451	13.47574	0.00071	0.00877	0.0651%
0.1	95	9.20478	9.20559	0.00135	-0.00081	0.0088%
0.1	100	5.53662	5.53619	0.00207	0.00043	0.0078%
0.1	105	2.88896	2.88890	0.00249	0.00006	0.0021%
0.1	110	1.33809	1.33781	0.00238	0.00028	0.0210%
0.2	90	14.03280	14.03489	0.00193	-0.00289	0.0206%
0.2	95	10.32293	10.32461	0.00276	-0.00168	0.0163%
0.2	100	7.21244	7.21556	0.00343	-0.00312	0.0432%
0.2	105	4.78516	4.78822	0.00380	-0.00306	0.0638%
0.2	110	3.02270	3.02558	0.00380	-0.00288	0.0952%
0.3	90	15.19639	15.19689	0.00350	-0.00050	0.0033%
0.3	95	11.92926	11.93168	0.00431	-0.00242	0.0203%
0.3	100	9.14769	9.15063	0.00495	-0.00294	0.0321%
0.3	105	6.86049	6.86412	0.00533	-0.00363	0.0529%
0.3	110	5.04029	5.04400	0.00545	-0.00331	0.0656%
0.4	90	16.68984	16.69294	0.00506	-0.00310	0.0186%
0.4	95	13.73384	13.73747	0.00586	-0.00363	0.0264%
0.4	100	11.17115	11.17579	0.00649	-0.00464	0.0415%
0.4	105	8.99114	8.99645	0.00692	-0.00531	0.0590%
0.4	110	7.16816	7.17317	0.00716	-0.00501	0.0698%
0.5	90	18.35379	18.35851	0.00658	-0.00472	0.0257%
0.5	95	15.62810	15.63389	0.00738	-0.00579	0.0370%
0.5	100	13.22860	13.23485	0.00804	-0.00625	0.0472%
0.5	105	11.13944	11.14602	0.00853	-0.00658	0.0590%
0.5	110	9.33799	9.34396	0.00887	-0.00597	0.0639%

Table 5: Numerical results of Asian option prices under Kou's model with $\lambda = 3$. Other parameters of the model are set as: $S_0 = 100$, $r = 0.09$, $t = 1.0$, $p_1 = 0.6$, $q_1 = 0.4$, and $\eta_1 = \theta_1 = 25$. Parameters of the algorithm are set as: $n_1 = 35$, $n_2 = 55$, $A_1 = 38.9$, $A_2 = 40$, and $X = Kt \exp\{\min(A_2/\theta, A_2/10)\}$, where $\theta = 2/\sqrt{(\sigma^2 + 2p_1\lambda/\eta_1^2 + 2q_1\lambda/\theta_1^2)t}$; DL prices are obtained by double-Laplace inversion; MC price are Monte Carlo simulation estimates obtained by simulating 1 million paths and setting the discretization step size to be 0.0001; Std Err is the standard error of the MC price; Abs Err and Rel Err are absolute and relative errors, respectively.

Appendix A. Proof of Theorem 4.1

Proof: For the HEM, the argument is similar to that for the BSM in Theorem 3.1 except that we need to show the process $\{M(t)\}$ is still a local martingale in the jump diffusion case. Indeed,

σ	K	LL Price	MC Price	Std Err	Error
0.05	90	13.47952	13.47729	0.00075	0.00223
0.05	95	9.16588	9.16468	0.00135	0.00120
0.05	100	5.38761	5.38519	0.00208	0.00242
0.05	105	2.72681	2.72275	0.00252	0.00406
0.05	110	1.28264	1.28034	0.00242	0.00230
0.1	90	13.55964	13.56384	0.00102	-0.00420
0.1	95	9.41962	9.42350	0.00173	-0.00388
0.1	100	5.91537	5.91707	0.00246	-0.00170
0.1	105	3.35071	3.35124	0.00287	-0.00053
0.1	110	1.74896	1.74934	0.00281	-0.00038
0.2	90	14.17380	14.17586	0.00217	-0.00206
0.2	95	10.53795	10.53973	0.00300	-0.00178
0.2	100	7.48805	7.48864	0.00367	-0.00059
0.2	105	5.09001	5.09000	0.00405	0.00001
0.2	110	3.32061	3.31967	0.00409	0.00096
0.3	90	15.33688	15.33728	0.00367	-0.00040
0.3	95	12.10723	12.10732	0.00448	-0.00009
0.3	100	9.35336	9.35297	0.00511	0.00039
0.3	105	7.08059	7.07908	0.00551	0.00151
0.3	110	5.26109	5.25875	0.00565	0.00234
0.4	90	16.81490	16.81475	0.00518	0.00015
0.4	95	13.87995	13.87950	0.00598	0.00045
0.4	100	11.33257	11.33142	0.00662	0.00115
0.4	105	9.16131	9.15913	0.00706	0.00218
0.4	110	7.34063	7.33721	0.00730	0.00342
0.5	90	18.46259	18.46136	0.00667	0.00123
0.5	95	15.75006	15.74865	0.00748	0.00141
0.5	100	13.36027	13.35842	0.00814	0.00185
0.5	105	11.27716	11.27410	0.00864	0.00306
0.5	110	9.47826	9.47384	0.00898	0.00443

Table 6: Numerical results of Asian option prices under Kou's model with $\lambda = 5$. Other parameters of the model are set as: $S_0 = 100$, $r = 0.09$, $t = 1.0$, $p_1 = 0.6$, $q_1 = 0.4$, and $\eta_1 = \theta_1 = 25$. Parameters of the algorithms are $n_1 = 35$, $n_2 = 55$, $A_1 = 35.9$, $A_2 = 40$, and $X = Kt \exp\{\min(A_2/\theta, A_2/10)\}$, where $\theta = 2/\sqrt{(\sigma^2 + 2p_1\lambda/\eta_1^2 + 2q_1\lambda/\theta_1^2)t}$. DL prices are obtained by the double Laplace inversion; MC prices are Monte Carlo simulation estimates by using 1 million paths and setting the discretization step size to be 0.0001.

by Ito's formula for jump diffusions, we have

$$
\begin{aligned}
da(S(t)) &= a'(S(t-))dS^c(t) + \frac{1}{2}a''(S(t-))d\langle S^c, S^c\rangle(t) + d\sum_{0<u\leq t}[a(S(u)) - a(S(u-))] \\
&= \left[(r - \lambda\zeta)S(t-)a'(S(t-)) + \frac{1}{2}\sigma^2 S^2(t-)a''(S(t-))\right]dt + \sigma S(t-)a'(S(t-))dW(t) \\
&\quad + d\sum_{0<u\leq t}[a(S(u)) - a(S(u-))].
\end{aligned}
$$

Since $a(s)$ solves the OIDE (17), we have

$$
\begin{aligned}
da(S(t)) &= [(S(t-)+\mu)a(S(t-))-\mu]dt + \sigma S(t-)a'(S(t-))dW(t) \\
&\quad +d\sum_{0<u\leq t}[a(S(u))-a(S(u-))] - \lambda\int_{-\infty}^{+\infty}[a(S(t-)e^y)-a(S(t-))]\,f_Y(y)dydt.
\end{aligned}
$$

Note that

$$
\begin{aligned}
dM(t) &= \exp\left(-\int_0^t\{\mu+S(u)\}du\right)\cdot da(S(t)) + a(S(t-))\cdot d\exp\left(-\int_0^t\{\mu+S(u)\}du\right) \\
&\quad +d\left\langle a(S(t),\ \exp\left(-\int_0^t\{\mu+S(u)\}du\right)\right\rangle + \mu\exp\left(-\int_0^t\{\mu+S(u)\}du\right)dt.
\end{aligned}
$$

Plugging $d(a(S(t)))$ into $dM(t)$ yields

$$
\begin{aligned}
dM(t) &= \exp\left(-\int_0^t\{\mu+S(u)\}du\right)a'(S(t-))\sigma S(t-)dW(t) \\
&\quad +\exp\left(-\int_0^t\{\mu+S(u)\}du\right)d\sum_{0<u\leq t}[a(S(u))-a(S(u-))] \\
&\quad -\lambda\exp\left(-\int_0^t\{\mu+S(u)\}du\right)\int_{-\infty}^{+\infty}[a(S(t-)e^y)-a(S(t-))]\,f_Y(y)dydt,
\end{aligned}
$$

which is a local martingale. Then the same proof as in Theorem 3.1 applies. $\square$

Appendix B. Proof of Theorem 4.2

Proof: Similar algebra as in Theorem 3.2 yields that

$$
0 = \int_{-\infty}^{\infty}e^{ax}\left\{e^{-x}(z(x)+1)+h(a)z(x)\right\}dx, \quad \text{for any } a\in(0,\min(\beta_1,1)), \tag{28}
$$

where β_1 is the smallest positive root of $G(x)=\mu$ and $z(x)=y(s)-1$. Plugging $h(a)$ in (20) into (28), we have that for all $a\in(0,\min(\beta_1,1))$,

$$
\int_{-\infty}^{\infty}e^{ax}\left\{e^{-x}(z(x)+1)+\left[-\frac{1}{2}\sigma^2a^2-\left(r-\frac{\sigma^2}{2}-\lambda\zeta\right)a+\mu\right]z(x)\right\}dx
$$
$$
+\ \int_{-\infty}^{\infty}e^{ax}\left\{-\lambda\left(\sum_{i=1}^{m}\frac{p_i\eta_i}{\eta_i-a}+\sum_{j=1}^{n}\frac{q_j\theta_j}{\theta_j+a}-1\right)z(x)\right\}dx = 0.
$$

Using the same technique as in the proof of Theorem 3.2, we can show that the first integral on the left hand side of the above is equal to

$$
\int_{-\infty}^{\infty}e^{ax}\left\{e^{-x}(z(x)+1)-\frac{1}{2}\sigma^2z''(x)+\left(r-\frac{\sigma^2}{2}-\lambda\zeta\right)z'(x)+\mu z(x)\right\}dx.
$$

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In addition, we claim that the second integral is equal to

$$\int_{-\infty}^{\infty} e^{ax} \left\{ -\lambda \int_{-\infty}^{\infty} [z(x-u) - z(x)] f_Y(u) du \right\} dx.$$

Indeed, this is because

$$\int_{-\infty}^{\infty} e^{ax} \left\{ -\lambda \int_{-\infty}^{\infty} [z(x-u) - z(x)] f_Y(u) du \right\} dx$$

$$= -\lambda \int_{-\infty}^{\infty} f_Y(u) \left\{ \int_{-\infty}^{\infty} e^{ax} z(x-u) dx \right\} du + \int_{-\infty}^{\infty} e^{ax} \lambda z(x) dx$$

$$= -\lambda \int_{-\infty}^{\infty} f_Y(u) \left\{ \int_{-\infty}^{\infty} e^{a(u+\pi)} z(\pi) d\pi \right\} du + \int_{-\infty}^{\infty} e^{ax} \lambda z(x) dx$$

$$= \int_{-\infty}^{\infty} e^{a\pi} z(\pi) \left\{ -\lambda \int_{-\infty}^{\infty} e^{au} f_Y(u) du \right\} d\pi + \int_{-\infty}^{\infty} e^{ax} \lambda z(x) dx$$

$$= \int_{-\infty}^{\infty} e^{ax} \left\{ -\lambda \left(\sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - a} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + a} - 1 \right) z(x) \right\} dx,$$

where the second equality is via change of variable $x - u = \pi$, and the last equality holds because $0 < a < \beta_1 < \eta_1$.

Thus we have

$$\int_{-\infty}^{\infty} e^{ax} \left\{ e^{-x}(z(x) + 1) - \frac{1}{2}\sigma^2 z''(x) + \left(r - \frac{\sigma^2}{2} - \lambda\zeta \right) z'(x) + \mu z(x) \right.$$

$$\left. - \lambda \int_{-\infty}^{\infty} [z(x-u) - z(x)] f_Y(u) du \right\} dx = 0, \quad \text{for all } a \in (0, \min(\beta_1, 1)).$$

By the uniqueness of the moment generating function we have an OIDE as follows

$$e^{-x}(z(x)+1) - \frac{1}{2}\sigma^2 z''(x) + \left(r - \frac{\sigma^2}{2} - \lambda\zeta \right) z'(x) + \mu z(x) - \lambda \int_{-\infty}^{\infty} [z(x-u) - z(x)] f_Y(u) du = 0.$$

Now transferring the OIDE back to $y(s)$, with $s = e^{-x}$ we have

$$z(x) = y(s) - 1, \quad z'(x) = -sy'(s), \quad z''(x) = sy'(s) + s^2 y''(s),$$

$$z(x - u) = z(-\log s - u) = z(-\log(se^u)) = y(se^u) - 1$$

and the OIDE becomes

$$-\frac{\sigma^2}{2} s^2 y''(s) - (r - \lambda\zeta) sy'(s) + (s + \mu)y(s) - \lambda \int_{-\infty}^{\infty} [y(se^u) - y(s)] f_Y(u) du = \mu,$$

i.e., $Ly(s) = (s + \mu)y(s) - \mu$, from which the proof is completed. $\square$

Appendix C. On the Recursion (19) under the HEM

In this section, we shall give an alternative proof that under the HEM, $E[A_{T_\mu}^\nu]$ solves the recursion (19) using the Feynman-Kac formula. More precisely, under the HEM, $E[A_{T_\mu}^\nu]$ satisfies the recursion (19), i.e.,

$$h(\nu)E[A_{T_\mu}^\nu] = \nu E[A_{T_\mu}^{\nu-1}] \quad \text{for any } \nu \in (0, \beta_1), \tag{29}$$

where $h(\nu)$ is given by (20).

Proof: Under the HEM, define $Y(t) = \int_0^t S(u)du$, where $S(u)$ represents the underlying stock price at the time u. Consider a function $f^*(t, x, y)$ defined as follows:

$$f^*(t, x, y) := E\left(Y_T^\nu \mid S(t) = x, Y(t) = y\right), \qquad t \in [0, T], \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R},$$

where T is any fixed positive real number and $\nu \in (-1, \beta_1)$ is a constant. Note that $\{(S(t), Y(t)) : t \geq 0\}$ is a two-dimensional Markov process under the HEM so that $f^*(t, S(t), Y(t)) = E\left(Y_T^\nu \mid \mathcal{F}_t\right)$. Accordingly, the multivariate Feynman-Kac theorem implies that $f^*(t, x, y)$ solves the following PIDE:

$$f_t^*(t, x, y) + (r - \lambda\zeta)x f_x^*(t, x, y) + x f_y^*(t, x, y) + \frac{\sigma^2}{2}x^2 f_{xx}^*(t, x, y)$$
$$+ \lambda \int_{-\infty}^{+\infty} [f^*(t, xe^z, y) - f^*(t, x, y)] f_Y^*(z) dz = 0, \quad t \in [0, T], \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R}.$$

On the other hand, by the Markovian property, we can rewrite $f^*(t, x, y)$ as follows.

$$f^*(t, x, y) = E\left[\left(y + x \int_0^{T-t} e^{(r - \lambda\zeta - \frac{\sigma^2}{2})u + \sigma W(u) + \sum_{i=1}^{N(u)} Y_i} du\right)^\nu\right] = E\left[(y + xA_{T-t})^\nu\right].$$

Introduce a new function

$$\hat{f}^*(t, x, y) := f^*(T - t, x, y) = E\left[(y + xA_t)^\nu\right], \qquad t \in [0, T], \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R},$$

which then satisfies the following PIDE

$$-\hat{f}_t^*(t, x, y) + (r - \lambda\zeta)x \hat{f}_x^*(t, x, y) + x \hat{f}_y^*(t, x, y) + \frac{\sigma^2}{2}x^2 \hat{f}_{xx}^*(t, x, y)$$
$$+ \lambda \int_{-\infty}^{+\infty} [\hat{f}^*(t, xe^z, y) - \hat{f}^*(t, x, y)] f_Y^*(z) dz = 0, \quad t \in [0, T], \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R}, \tag{30}$$

with the initial value

$$\hat{f}^*(0, x, y) = y^\nu.$$

It is worth noting that the "T" in the PIDE (30) can be any positive real number.

Next, for any $\nu \in (0, \beta_1)$, define $h^*(x, y) = E\left[(y + xA_{T_\mu})^\nu\right] = \int_0^\infty \mu e^{-\mu t} \hat{f}^*(t, x, y)dt$, $x \in \mathbb{R}^+$, $y \in \mathbb{R}$. Taking the Laplace transform w.r.t. t on both sides of the PIDE (30) and then multiplying both sides by μ, we can show that $h^*(x, y)$ solves a PIDE as follows.

$$\mu y^\nu - \mu h^*(x, y) + (r - \lambda\zeta)xh_x^*(x, y) + xh_y^*(x, y) + \frac{\sigma^2}{2}x^2h_{xx}^*(x, y)$$
$$+ \lambda \int_{-\infty}^{+\infty} [h^*(xe^z, y) - h^*(x, y)]f_Y^*(z)dz = 0, \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R}. \quad (31)$$

Interchanging the derivatives and integrals by using Theorem A. 12 on the pp. 203-204 in Schiff [39], we have

$$xh_x^*(x, y) = \nu E\left[(y + xA_{T_\mu})^\nu\right] - \nu y E\left[(y + xA_{T_\mu})^{\nu-1}\right],$$
$$xh_y^*(x, y) = \nu x E\left[(y + xA_{T_\mu})^{\nu-1}\right],$$
$$x^2 h_{xx}^*(x, y) = \nu(\nu-1)E\left[(y + xA_{T_\mu})^\nu\right] + \nu(\nu-1)y^2 E\left[(y + xA_{T_\mu})^{\nu-2}\right]$$
$$-\nu(\nu-1)2y E\left[(y + xA_{T_\mu})^{\nu-1}\right].$$

Substituting above into (31) yields

$$\mu y^\nu + \left(-\mu + (r - \lambda\zeta)\nu + \frac{\sigma^2}{2}\nu(\nu-1)\right)E\left[(y + xA_{T_\mu})^\nu\right]$$
$$+ \nu\left(x - (r - \lambda\zeta)y - \sigma^2(\nu-1)y\right)E\left[(y + xA_{T_\mu})^{\nu-1}\right] + \frac{\sigma^2}{2}\nu(\nu-1)y^2 E\left[(y + xA_{T_\mu})^{\nu-2}\right]$$
$$+ \lambda \int_{-\infty}^{+\infty}\left\{E\left[(y + xe^zA_{T_\mu})^\nu\right] - E\left[(y + xA_{T_\mu})^\nu\right]\right\}f_Y(z)dz = 0.$$

In the special case $x = 1$ and $y = 0$, since

$$\int_{-\infty}^{+\infty}\left[E\left(e^{z\nu}A_{T_\mu}^\nu\right) - E\left(A_{T_\mu}^\nu\right)\right]f_Y(z)dz = E\left(A_{T_\mu}^\nu\right)\left(\sum_{i=1}^m \frac{p_i\eta_i}{\eta_i - \nu} + \sum_{j=1}^n \frac{q_j\theta_j}{\theta_j + \nu} - 1\right),$$

we obtain that for any $\nu \in (0, \beta_1)$,

$$\left[\frac{\sigma^2}{2}\nu^2 + (r - \lambda\zeta - \frac{\sigma^2}{2})\nu + \lambda\left(\sum_{i=1}^m \frac{p_i\eta_i}{\eta_i - \nu} + \sum_{j=1}^n \frac{q_j\theta_j}{\theta_j + \nu} - 1\right) - \mu\right]E\left(A_{T_\mu}^\nu\right) = -\nu E\left(A_{T_\mu}^{\nu-1}\right),$$

which is exactly (29) and (19). The proof is completed. $\square$

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Acknowledgements

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Online Supplement

Pricing Asian Options under a Hyper-Exponential Jump Diffusion Model

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More Discussion on the Numerical Algorithm

1 Stability of the Method

To study the stability of the method, we perform some numerical experiments under the BSM to show the absolute and relative errors of our double-Laplace inversion method against various choices of parameters A_1, A_2 and X. The "true" prices are obtained by using Monte Carlo simulation with a control variate being $\int_0^t e^{X(s)}ds$, because it is easy to compute $E\left[\int_0^t e^{X(s)}ds\right] = (e^{rt}-1)/r$, and $\int_0^t e^{X(s)}ds$ has a high degree of correlation with the payoff function. In addition, Richardson extrapolation is also employed to reduce the discretization bias generated when we discretize the sample path to approximate the integral. More precisely, let $M(h)$ be the Monte Carlo estimator without Richardson extrapolation when the discretization step size is set to be h. Then we use $(4M(h) - M(2h))/3$ rather than $M(h)$ as the final estimator to achieve the discretization bias reduction. For more details about the technique of control variates and Richardson extrapolation, see Glasserman [24].

Figure 1 shows how the absolute and relative errors change as A_1, A_2 and X vary in the case of low volatility $\sigma = 0.05$, illustrating that our algorithm is insensitive to the selection of parameters A_1, A_2 and X. For normal volatilities, our method becomes even more stable and associated plots can be obtained on request.

2 Discretization Error Bounds of Euler Inversion Algorithm under the BSM

The discretization error bound of the Euler inversion algorithm was first studied by Abate and Whitt [1], and was extended to a two-sided Laplace inversion case by Petrella [37]. In this
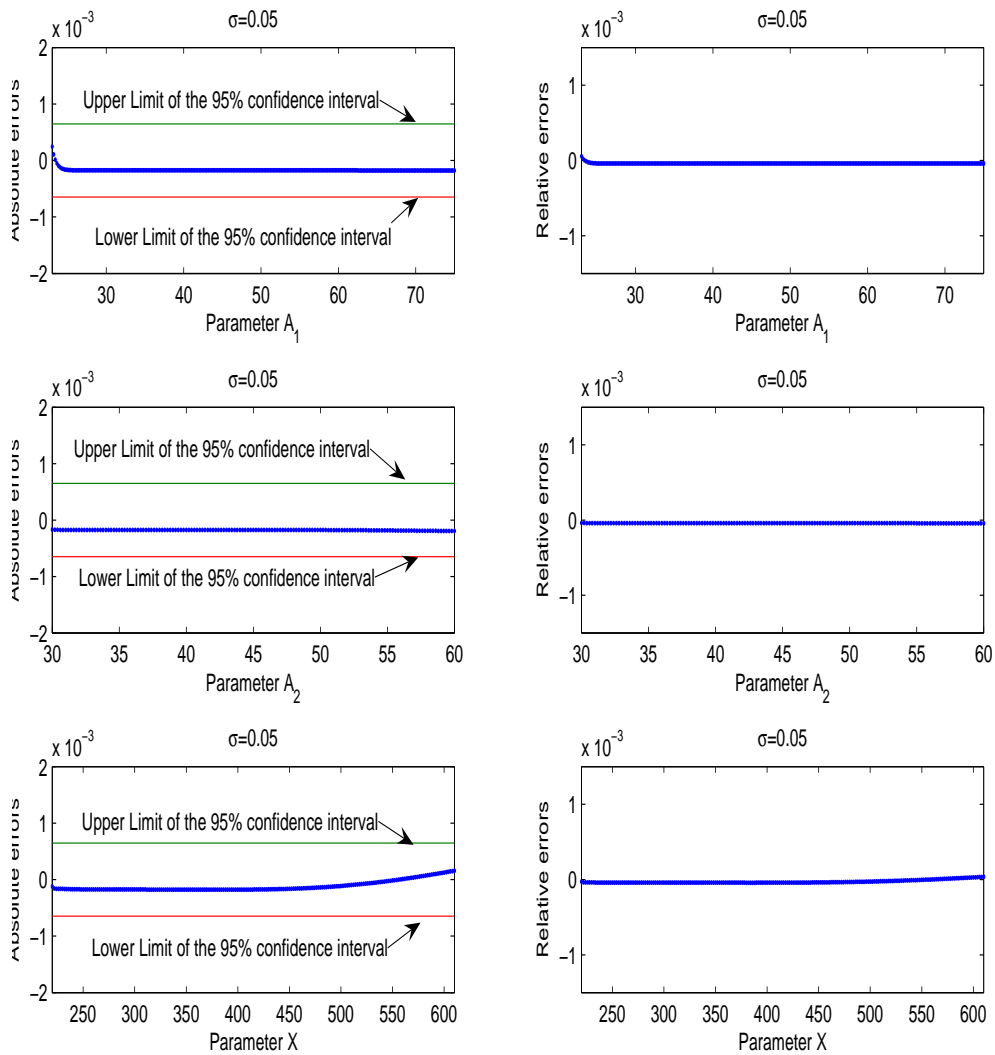
Figure 1: The stability and accuracy of the algorithm as A_1, A_2 and X vary in the case of low volatility $\sigma = 0.05$. The default choices for unvarying algorithm parameters are $A_1 = 28$, $A_2 = 40$ and X given by (27). The absolute errors and relative errors are reported on the left and right graphs, respectively. For broad regions of A_1, A_2 and X our algorithm appears to be stable and accurate, all within the 95% confidence intervals. In fact, the relative errors are all smaller than 0.02%.

subsection, by extending the results in Petrella [37], we provide discretization error bounds of the inversion algorithm for our specific case of Asian option pricing under the BSM. The discretization error bounds decay exponentially, therefore leading to a fast convergence.

Recall that what we want to invert is $\mathcal{L}(\mu, \nu) = \int_0^\infty \int_{-\infty}^\infty e^{-\mu t} e^{-\nu k} f(t, k) dk dt$, where $f(t, k) = XE(\frac{S_0}{X} A_t - e^{-k})^+$. Then we can prove the following theorem for the error bounds.

Theorem 2.1. *Suppose* $t \in (0, \frac{A_1}{2(\theta_1 + \theta_2)})$ *and* $k > \frac{A_2}{\theta_2}$, *for some constant* $\theta_2 > 0$, *where* $\theta_1 = 1 + \tilde{r} + \sigma^2/2 > 0$ *and* $\tilde{r} = \max(r - \frac{\sigma^2}{2}, 0)$. *Then the discretization error bounds* e_d^+ *and* e_d^- *satisfy*

$$e_d^+ \leq \frac{C^+(\theta_1)}{1 - e^{-(A_1 - 2\theta_1 t)}} \left\{ \frac{e^{-A_2}}{1 - e^{-A_2}} + e^{-(A_1 - 2\theta_1 t)} \right\}, \tag{32}$$

$$e_d^- \leq C^-(\theta_1, \theta_2) \frac{1}{1 - e^{-(A_1 - 2(\theta_1 + \theta_2)t)}} \frac{e^{-(\theta_2 k - A_2)}}{1 - e^{-(\theta_2 k - A_2)}}, \tag{33}$$

with $C^+(\theta_1) := 2S_0 e^{\theta_1 t}$ *and* $C^-(\theta_1, \theta_2) := 2S_0 e^{t\sigma^2 \theta_2^2/2 + [(1 + \tilde{r} + \sigma^2)t - 1]\theta_2 + \theta_1 t}$.

Before proving this theorem, we give an example to illustrate how to apply it in real situations. Consider the case where $r = 0.09$, $\sigma = 0.2$ and $t = 1$, and we use $A_1 = 50$, $A_2 = 40$ and $\theta_2 = 20$. Then $\tilde{r} = 0.07$, $\theta_1 = 1.09$, $t = 1 \in \left(0, \frac{A_1}{2\theta_1}\right) \equiv (0, 1.19)$, and $k = 4 \in \left(\frac{A_2}{\theta_2}, +\infty\right) \equiv (2, +\infty)$. Simple algebra yields that $C^+(\theta_1) \approx e^{6.39}$ and $C^-(\theta_1, \theta_2) \approx e^{16.59}$. Plugging them into (32) and (33), we can get discretization error bounds: $e_d^+ \leq 2.53 \times 10^{-15}$ and $e_d^- \leq 6.80 \times 10^{-11}$. Hence, the discretization error for Asian option price is theoretically no more than $6.80 \times 10^{-11} \times e^{-rt}/t \approx 6.22 \times 10^{-11}$.

Proof of Theorem 2.1: First, since the scaling factor $X > S_0$, we have that

$$f(t, k) = XE\left(\frac{S_0}{X} A_t - e^{-k}\right)^+ \leq XE\left(\frac{S_0}{X} A_t - \frac{S_0}{X} e^{-k}\right)^+ = S_0 E(A_t - e^{-k})^+,$$

where $k = \log(\frac{X}{Kt})$. On the other hand, we can bound A_t as follows

$$A_t = \int_0^t e^{(r - \frac{\sigma^2}{2})s + \sigma W(s)} ds \leq \int_0^t \exp\left(\tilde{r}t + \sigma \max_{\{0 \leq s \leq t\}} W(s)\right) ds = t \exp\left\{\tilde{r}t + \sigma \max_{\{0 \leq s \leq t\}} W(s)\right\},$$

where $\tilde{r} := \max(r - \frac{\sigma^2}{2}, 0)$. Since $\max_{\{0 \leq s \leq t\}} W(s) =_d |W(t)|$, it follows that

$$
\begin{aligned}
f(t,k) \;\leq\;& S_0 E\left[t \exp\{\tilde{r}t + \sigma|W(t)|\} - e^{-k}\right]^+ \\
=\;& S_0 E\left[\left(t \exp\{\tilde{r}t + \sigma W(t)\} - e^{-k}\right) I_{\{W(t) \geq 0,\, t\exp(\tilde{r}t + \sigma W(t)) > e^{-k}\}}\right] \\
& + S_0 E\left[\left(t \exp\{\tilde{r}t - \sigma W(t)\} - e^{-k}\right) I_{\{W(t) < 0,\, t\exp(\tilde{r}t - \sigma W(t)) > e^{-k}\}}\right] \\
\leq\;& S_0 E\left[\left(t \exp\{\tilde{r}t + \sigma W(t)\} - e^{-k}\right) I_{\{t\exp(\tilde{r}t + \sigma W(t)) > e^{-k}\}}\right] \\
& + S_0 E\left[\left(t \exp\{\tilde{r}t - \sigma W(t)\} - e^{-k}\right) I_{\{t\exp(\tilde{r}t - \sigma W(t)) > e^{-k}\}}\right] \\
=\;& 2 S_0 E\left[\left(t \exp\{\tilde{r}t + \sigma W(t)\} - e^{-k}\right) I_{\{t\exp(\tilde{r}t + \sigma W(t)) > e^{-k}\}}\right],
\end{aligned}
$$

via the symmetric property of standard Brownian motion.

Next, introduce a new measure $\bar{P}$ such that $\frac{d\bar{P}}{dP} = e^{Y_t - (\tilde{r} + \sigma^2/2)t}$, where $Y_t := \tilde{r}t + \sigma W(t)$. Then the change of measure leads to

$$
\begin{aligned}
f(t,k) \;\leq\;& 2 S_0 \bar{E}\left[t \exp\{\tilde{r}t + \sigma W_t\} I_{\{t\exp(\tilde{r}t + \sigma W(t)) > e^{-k}\}} \times e^{-Y_t + (\tilde{r} + \sigma^2/2)t}\right] \\
=\;& 2 S_0 t e^{(\tilde{r} + \sigma^2/2)t} \bar{P}\{t \exp(Y_t) > e^{-k}\} \\
\leq\;& 2 S_0 e^{(1 + \tilde{r} + \sigma^2/2)t} \bar{P}\{Y_t > -k - \log(t)\} \\
=\;& 2 S_0 e^{\theta_1 t} \bar{P}\{Y_t > -k - \log(t)\},
\end{aligned}
$$

where $\theta_1 := 1 + \tilde{r} + \sigma^2/2 > 0$ and the last inequality holds because $t < e^t$ for any $t > 0$.

Therefore, when $j_1 \geq 0$ and $j_2 \geq 0$, we have

$$
\begin{aligned}
f((2j_1 + 1)t, (2j_2 + 1)k) \;\leq\;& 2 S_0 e^{\theta_1 (2j_1 + 1)t} \bar{P}\{Y_t > -(2j_2 + 1)k - \log((2j_1 + 1)t)\} \\
\leq\;& 2 S_0 e^{\theta_1 t} e^{2\theta_1 j_1 t} = C^+(\theta_1) e^{2\theta_1 j_1 t},
\end{aligned}
$$

where $C^+(\theta_1) := 2 S_0 e^{\theta_1 t}$. On the other hand when $j_1 \geq 0$ and $j_2 \leq -1$, we have that for any $\theta_2 > 0$,

$$
\begin{aligned}
f((2j_1 + 1)t, (2j_2 + 1)k) \;\leq\;& 2 S_0 e^{\theta_1 (2j_1 + 1)t} \bar{P}\{Y_t > -(2j_2 + 1)k - \log((2j_1 + 1)t)\} \\
\leq\;& 2 S_0 e^{\theta_1 (2j_1 + 1)t} \bar{P}\{Y_t > -j_2 k - \log((2j_1 + 1)t)\} \\
\leq\;& 2 S_0 e^{\theta_1 (2j_1 + 1)t} \bar{E}\left(e^{\theta_2 Y_t}\right) e^{\theta_2 j_2 k + \theta_2 \log((2j_1 + 1)t)},
\end{aligned}
$$

where the second inequality holds as $j_2 \leq -1$ and the third inequality comes from Markov's inequality. Since $x + 1 \leq e^x$ for any $x > -1$, we obtain that $e^{\theta_2 \log((2j_1+1)t)} \leq e^{\theta_2[(2j_1+1)t-1]}$ and

$$
\begin{aligned}
f((2j_1+1)t, (2j_2+1)k) &\leq 2S_0 e^{\theta_1(2j_1+1)t} \bar{E}\left(e^{\theta_2 Y_t}\right) e^{\theta_2 j_2 k + \theta_2[(2j_1+1)t-1]} \\
&= 2S_0 e^{(\theta_1+\theta_2)t-\theta_2} \bar{E}\left(e^{\theta_2 Y_t}\right) \cdot e^{2(\theta_1+\theta_2)j_1 t} e^{\theta_2 j_2 k} \\
&= C^-(\theta_1, \theta_2) e^{\theta_2 j_2 k + 2(\theta_1+\theta_2)j_1 t},
\end{aligned}
$$

where

$$
C^-(\theta_1, \theta_2) := 2S_0 e^{(\theta_1+\theta_2)t-\theta_2} \cdot \bar{E}\left(e^{\theta_2 Y_t}\right) = 2S_0 e^{(\theta_1+\theta_2)t-\theta_2} \cdot E\left(e^{(\theta_2+1)Y_t - (\tilde{r}+\sigma^2/2)t}\right).
$$

Recall that $Y_t = \tilde{r}t + \sigma W(t)$. Simple algebra yields

$$
C^-(\theta_1, \theta_2) = 2S_0 e^{t\sigma^2 \theta_2^2/2 + [(1+\tilde{r}+\sigma^2)t-1]\theta_2 + \theta_1 t}.
$$

If we have $t \in \left(0, \frac{A_1}{2\theta_1}\right)$, according to the definition of e_d^+ and the bound of function $f((2j_1+1)t, (2j_2+1)k)$ obtained above, we can get

$$
\begin{aligned}
e_d^+ &\leq \sum_{j_2=1}^{\infty} \sum_{j_1=0}^{\infty} e^{-(j_1 A_1 + j_2 A_2)} C^+(\theta_1) e^{2\theta_1 j_1 t} + \sum_{j_1=1}^{\infty} e^{-j_1 A_1} C^+(\theta_1) e^{2\theta_1 j_1 t} \\
&= C^+(\theta_1) \sum_{j_2=1}^{\infty} \sum_{j_1=0}^{\infty} e^{-(A_1-2\theta_1 t)j_1 - j_2 A_2} + C^+(\theta_1) \sum_{j_1=1}^{\infty} e^{-(A_1-2\theta_1 t)j_1} \\
&= C^+(\theta_1) \frac{e^{-A_2}}{1-e^{-A_2}} \frac{1}{1-e^{-(A_1-2\theta_1 t)}} + C^+(\theta_1) \frac{e^{-(A_1-2\theta_1 t)}}{1-e^{-(A_1-2\theta_1 t)}} \\
&= \frac{C^+(\theta_1)}{1-e^{-(A_1-2\theta_1 t)}} \left\{ \frac{e^{-A_2}}{1-e^{-A_2}} + e^{-(A_1-2\theta_1 t)} \right\},
\end{aligned}
$$

which is exactly (32).

For e_d^- we have for any $t \in \left(0, \frac{A_1}{2(\theta_1+\theta_2)}\right)$ and $k > \frac{A_2}{\theta_2}$,

$$
\begin{aligned}
e_d^- &\leq \sum_{j_2=-\infty}^{-1} \sum_{j_1=0}^{\infty} e^{-(j_1 A_1 + j_2 A_2)} C^-(\theta_1, \theta_2) e^{\theta_2 j_2 k + 2(\theta_1+\theta_2)j_1 t} \\
&= C^-(\theta_1, \theta_2) \sum_{j_1=0}^{\infty} e^{-(A_1-2(\theta_1+\theta_2)t)j_1} \sum_{j_2=-\infty}^{-1} e^{j_2(\theta_2 k - A_2)} \\
&= C^-(\theta_1, \theta_2) \frac{1}{1-e^{-(A_1-2(\theta_1+\theta_2)t)}} \frac{e^{-(\theta_2 k - A_2)}}{1-e^{-(\theta_2 k - A_2)}},
\end{aligned}
$$

from which (33) is proved. □

	Double-Laplace Inversion Method			
	2 decimal	3 decimal	4 decimal	5 decimal
σ=0.05	$(n_1,n_2)=(35,35)^*$	(35,35)	(35,35)	(35,35)
(CPU time)	$(3.5 \text{ secs})^{**}$	(3.5 secs)	(3.5 secs)	(3.5 secs)
σ=0.1	$(n_1,n_2)=(15;15)$	(15;35)	(15;35)	(15;35)
(CPU time)	(1.2 secs)	(2.0 secs)	(2.0 secs)	(2.0 secs)
	Fourier-Laplace Inversion Method			
	2 decimal	3 decimal	4 decimal	5 decimal
σ=0.05	$(n_l,n_f)=(35,115)$	(35,135)	(35,195)	(35,195)
(CPU time)	(17.5 secs)	(20.1 secs)	(28.6 secs)	(28.6 secs)
σ=0.1	$(n_l,n_f)=(15,55)$	(15,75)	(15,95)	(15,95)
(CPU time)	(4.6 secs)	(6.4 secs)	(7.7 secs)	(7.7 secs)

Table 7: Comparison of the efficiency between our double-Laplace inversion and Fusai's Fourier-Laplace inversion method. In this table, $(n_1,n_2)=(35,35)^*$ means that (n_1,n_2) should be set roughly at least $(35,35)$ to achieve 2-decimal accuracy. $(3.5 \text{ secs})^{**}$ below $(n_1,n_2)=(35,35)^*$ means that the corresponding CPU time is 3.5 seconds. The CPU times associated with Fusai's method are obtained using the code implemented by Matlab 7.1. All computations in Table 7 are conducted on an IBM laptop with a Pentium M 1.86GHz processor. We can see that to achieve the same accuracy, our method is more efficient than Fusai's.
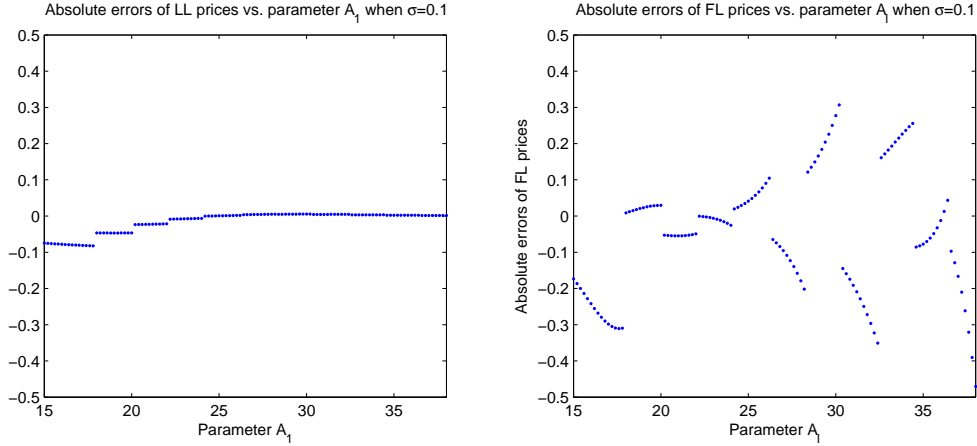

Figure 2: Comparison of the stability and accuracy between the double-Laplace inversion and the Fourier-Laplace inversion method in the case of low volatility $\sigma = 0.1$ under Kou's model, where other parameters are $K = 100$, $S_0 = 100$, $t = 1$, $\lambda = 3$, $r = 0.09$, $p_1 = 0.6$, $q_1 = 0.4$, and $\eta_1 = \theta_1 = 25$;. The absolute errors of LL prices (obtained by the double-Laplace inversion method) and FL prices (obtained by the Fourier-Laplace inversion method) are reported on the left and right graphs, respectively. Other parameters for the left graph are $n_1 = 35$, $n_2 = 55$, $A_2 = 40$ and $X = 5460$; while other parameters for the right graph are $n_l = 35$, $n_f = 135$ and $A_f = 40$. We can see that LL prices are quite accurate and stable when A_1 varies between [22.2,38], but FL prices are so desultory that we cannot decide which A_l we should choose.

3 Comparison with the Fourier and Laplace Inversion Algorithm

Under the BSM, Fusai [22] gave a closed-form for the Fourier-Laplace transform of Asian option price w.r.t. $k = \ln(\sigma^2 K t/(4S_0))$ and $h = \sigma^2 t/4$, respectively. Despite some similarities, there are some key differences between our method and Fusai's method. (1) Our method performs better for low volatility, e.g., $\sigma = 0.05$ or 0.1. Specifically, for Fusai's method, a large number of terms are needed to do the Euler inversion to achieve a desired accuracy. In comparison, our algorithm in general requires far fewer terms in computation, especially for low volatility (See Table 7). This is mainly because we use the latest inversion method with a scaling factor in Petrella [37]. (2) Our method performs better in jump diffusion models. Specifically, the Fourier-Laplace inversion method seems unstable in the case of low volatilities under Kou's model because it is sensitive to parameters. To illustrate the sensitivity, we fix $A_f = 40$ and let A_l change from 15 to 38. The right panel of Figure 2 illustrates how the difference between the numerical result and the true value changes as A_l varies in the case of $\sigma = 0.1$ and $K = 100$. In comparison with the double-Laplace inversion on the left panel of Figure 2, FL prices seem unstable. Figure 2 seems to indicate that, with jumps, our double-Laplace inversion method works in a more stable manner than the Fourier-Laplace method, especially in the case of low volatility. (3) Under the BSM, we can derive a theoretical discretization error bound for the double-Laplace inversion. See Section 2 in the online supplement. (4) In terms of the main theoretical difference, note that the recursion used in Fusai's paper, namely (8), has no unique but infinitely many solutions. We spend considerable efforts to overcome this difficulty; see Section 3.