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CREDIT SPREADS, OPTIMAL CAPITAL STRUCTURE, AND IMPLIED VOLATILITY WITH ENDOGENOUS DEFAULT AND JUMP RISK

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We propose a two-sided jump model for credit risk by extending the Leland–Toft endogenous default model based on the geometric Brownian motion. The model shows that jump risk and endogenous default can have significant impacts on credit spreads, optimal capital structure, and implied volatility of equity options: (1) Jumps and endogenous default can produce a variety of non-zero credit spreads, including upward, humped, and downward shapes; interesting enough, the model can even produce, consistent with empirical findings, upward credit spreads for speculative grade bonds. (2) The jump risk leads to much lower optimal debt/equity ratio; in fact, with jump risk, highly risky firms tend to have very little debt. (3) The two-sided jumps lead to a variety of shapes for the implied volatility of equity options, even for long maturity options; although in general credit spreads and implied volatility tend to move in the same direction under exogenous default models, this may not be true in presence of endogenous default and jumps. Pricing formulae of credit default swaps and equity default swaps are also given. In terms of mathematical contribution, we give a proof of a version of the "smooth fitting" principle under the jump model, justifying a conjecture first suggested by Leland and Toft under the Brownian model.

KEY WORDS: credit risk, yield spreads, capital structure, implied volatility, jump diffusion, smooth pasting.

1. INTRODUCTION

Of great interest in both corporate finance and asset pricing is credit risk due to the possibility of default. In corporate finance the optimal capital structure for a firm may be selected by considering the trade-off between tax credits from coupon payments to debt

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holders and potential financial costs related to default. Credit risk also leads to yield spreads between defaultable and risk-free bonds. Furthermore, credit risk may affect firms' equity values, which in turn will contribute to implied volatility smiles in equity options.

There are basically two approaches to model credit risk, the structural approach and the reduced-form approach. Reduced-form models aim at providing a simple framework to fit a variety of credit spreads by abstracting from the firm-value process and postulating default as a single jump time.[1] Starting from Black and Scholes (1973) and Merton (1974), the structural approach aims at providing an intuitive understanding of credit risk by specifying a firm value process and modeling equity and defaultable bonds as contingent claims on the firm value. An important class of the structural models is the class of first-passage models, specifying default as the first time the firm value falls below a barrier level. Depending on whether the barrier is a decision variable or not, the default can be classified as endogenous or exogenous. For first-passage time models with exogenous default see, e.g., Black and Cox (1976), Longstaff and Schwartz (1995), and Collin-Dufresne and Goldstein (2001); and for endogenous default models see, e.g., Leland (1994), Leland and Toft (1996), Goldstein, Ju, and Leland (2001), and Ju and Ou-Yang (2005). There are also links between the two approaches by incorporating jumps or different information sets into structural models; see Duffie and Lando (2001) and Jarrow and Protter (2004). For further background on credit risk, see Bielecki and Rutkowski (2002), Das (1995), Duffie and Singleton (2003), Kijima (2002), Lando (2004), and Schönbucher (2003).

1.1. Related Empirical Facts

We try to build a model to incorporate some stylized facts related to credit spreads, optimal capital structure, and implied volatility. First, for credit spreads, related stylized facts are: (1) Credit spreads do not converge to zero even for very short maturity bonds, and there are many models addressing this issue.[2] (2) Credit spreads can have upward, humped, and downward shapes; as the firm's financial situation deteriorates, credit spread curves tend to change from upward, to humped, and maybe even to downward shapes, in presence of severe financial distresses; see, for example, Jones et al. (1984), Sarig and Warga (1989), and He et al. (2000). (3) For speculative grade bonds, in addition to humped and downward shapes, credit spreads can even be upwards; see Helwege and Turner (1999) and He et al. (2000). (4) Credit spreads tend to be negatively correlated with the risk-free rate (Longstaff and Schwartz 1995; Duffee 1998). (5) A comprehensive empirical study in Eom et al. (2004) suggests that empirical credit spread curves are much flatter than what are predicted from the Leland-Toft model. All of the above-mentioned empirical facts related to credit spreads will be incorporated in our model.

Second, we consider empirical facts related to capital structure, which is one of the most important areas in corporate finance, going back to the celebrated MM

[1] For reduced-form models, see, for example, Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), Duffie and Singleton (1999), Collin-Dufresne, Goldstein, and Hugonnier (2003), Madan and Unal (1998).

[2] For instance, Duffie and Lando (2001) and Huang and Huang (2003) suggest that incomplete accounting information or liquidity may lead to non-zero credit spreads. Leland (2004) supports the explanation of jumps by pointing out that including jumps "may solve the underestimation of both default probabilities and yield spreads."

theorem in Modigliani and Miller (1958). Many factors may affect the capital structure, such as taxes, bankruptcy costs, agency costs and conflicts between various investors, asymmetric information, corporate takeover and corporate control, and interactions between investment and production decisions.[3] It is too ambitious to try to build one model to address all these issues. Instead, in this paper we shall focus on a neoclassical view investigating the optimal capital structure as a trade-off between taxes and bankruptcy costs. Regarding this trade-off, empirical evidences seem to suggest that high volatility (Bradley et al. 1984; Friend and Lang 1988; Kim and Sorensen 1986) and low recovery values upon default (Bradley et al. 1984; Long and Malitz 1985; Kim and Sorensen 1986; Titman and Wessels 1988) generally lead to low debt/equity ratios. However, it is quite interesting to observe that high tech firms, such as Internet and biotechnology companies (which tend to have high volatility, large jump risk, and low recovery values upon default), have almost no debt despite the fact that tax credits would be given for coupon payments to debt holders. Large (perhaps even unrealistic) diffusion volatility parameters are needed for pure diffusion-type endogenous models to generate very low debt/equity ratio for these high tech firms. In this paper we show that jumps can lead to significantly lower debt/equity ratios, particularly for high tech companies.

Third, we move on to study the connection between implied volatility in equity options and credit spreads for the corresponding defaultable bonds. Discussion of this connection arguably went back at least to the leverage effect suggested in Black (1976). Recent empirical studies in Toft and Prucyk (1997), Hull et al. (2004), Cremers et al. (2005a) seem to indicate a significant connection between implied volatility and credit spreads. What we point out in this paper is that from a theoretical viewpoint, although in general there may be a positive connection between implied volatility in equity options and credit spreads with exogenous default and jumps, the situation may be reversed for short maturity bonds with endogenous default and jumps. Therefore, we should be careful about the difference between exogenous and endogenous defaults and control the variations of firm specific characteristics, when analyzing the connection between implied volatility and credit spreads.

1.2. Contribution of the Current Paper

Furthering the jump models in Hilberink and Rogers (2002), we extend the Leland–Toft endogenous default model (Leland 1994; Leland and Toft 1996) to a two-sided jump model, in which the jump sizes have a double exponential distribution (Kou 2002; Kou and Wang 2003). The model shows that jump risk and endogenous default can have significant impacts on credit spreads, optimal capital structure, and implied volatility of equity options. More precisely, we show: (1) Jumps and endogenous default can produce a variety of non-zero credit spreads, including upward, humped, and downward shapes; interesting enough, the model can even produce, consistent with empirical findings, upward credit spreads for speculative grade bonds. See Section 3.2 for details. (2) Compared with the diffusion model without jumps, the jump risk leads to much lower optimal debt/equity ratios; in fact, with jump risk, highly risky firms tend to have very little debt. This helps to explain why Internet and biotech firms have almost no debt. See Section 3.1 for details. (3) The two-sided jumps lead to a variety of shapes for the implied volatility of equity options, even for long maturity equity options; although in general

[3] For surveys, see Harris and Raviv (1991), Ravid (1988), Bradley et al. (1984), Masulis (1988), Brealey and Myers (2001), and a special issue of "Journal of Economic Perspectives," partly dedicated to the subject (Miller 1998; Stiglitz 1998; Ross 1988; Bhattacharya 1998; Modigliani 1998).

credit spreads and implied volatility tend to move in the same direction, this may not be true in presence of endogenous default and jumps. In fact, higher diffusion volatility may lead to a lower endogenous default barrier, resulting in smaller credit spreads for bonds with short maturities because the default is more likely to be caused by jumps for short maturity bonds. See Section 3.3 for details. (4) We also study pricing of credit and equity default swaps; see Section 3.4.

In terms of mathematical contribution, we give a proof of a version of the "smooth fitting" principle for the jump model, justifying a conjecture first suggested by Leland and Toft (1996) in the setting of the Brownian model. See Theorem 3.1 and Remark B.1 in Appendix B.

1.3. Related Literature

Kijima and Suzuki (2001) and Zhou (2001) introduced jump diffusions to exogenous default models in Merton (1974) and Black and Cox (1976), respectively. Here we consider endogenous default along with the optimal capital structure problem, under a different jump diffusion model. We also discuss the connection with implied volatility. Fouque et al. (2004) extended the exogenous default model of Black and Cox (1976) to include stochastic volatility.

A closely related paper is Hilberink and Rogers (2002), which extends the Leland–Toft model to Lévy processes with one-sided jumps and focuses on the study of capital structure.[4] Here we consider a jump diffusion model with two-sided jumps. In addition to capital structure, we discuss broader issues, such as various credit spread shapes and links between credit spreads and implied volatility; a mathematical justification of the smoothing-fitting principle is also given here. These issues are not discussed in Hilberink and Rogers (2002). Compared with one-sided jumps, two-sided jumps can generate more flexible implied volatility smiles, such as non-monotone implied volatility smiles.

Linetsky (2006) extended the firm model in Black and Scholes (1973) to include one jump to default, with the exogenous default intensity being a negative power function of the stock price. Carr and Linetsky (2005) generalized the CEV model to include one jump to default, with the exogenous default intensity being an affine function of the CEV variance. Both papers reach some similar conclusions about credit spreads and implied volatility as those in the current paper, but they do not consider the problem of optimal capital structure. Here we use endogenous default under a different model, and we also study optimal capital structure, various shapes of credit spreads, and the impact of endogenous default on implied volatility.

Structural models with jumps tend to have similar impacts on credit spreads as those from models with incomplete accounting information (Duffie and Lando 2001) or unobservable default barriers (Giesecke 2001, 2004). This is mainly because there are intrinsic connections between reduced-form models and structural models by changing information sets; see Section 3.2.1. Further references can be found in, for example, Collin-Dufresne, Goldstein, and Hugonnier (2003), Çetin et al. (2004), Jarrow and Protter (2004), and Guo et al. (2005). The main difference between the current paper and these papers is that we also discuss endogenous default, optimal capital structure, and related implied volatility in equity options.

[4] Various discussions and representations on the optimal capital structure (but not explicit calculation) for general Lévy processes are also given in Boyarchenko (2000); see also Le Courtois and Quittard-Pinon (2006).

There are several other papers using the double exponential jump diffusion model to study credit risk. Mainly focusing on empirical aspects of exogenous default, Huang and Huang (2003) and Cremers et al. (2005b) used the double exponential jump diffusion model to extend the exogenous default model in Black and Cox (1976). Here we look at modeling aspects of endogenous default (by extending Leland and Toft 1996). Dao and Jeanblanc (2005) and Le Courtois and Quittard-Pinnon (2007) also studied endogenous default with the double exponential jump diffusion model but emphasize behavior finance aspects and default probabilities, respectively, while we investigate various shapes of credit spreads, analytical solution of optimal endogenous default boundary, and the relation between credit spreads and implied volatility. As a result, all these studies complement each other without many overlaps.

In summary, what differentiates this paper from the related literature is that we put optimal capital structure, credit spreads, and implied volatility into a unified framework with both endogenous default and jumps. The unified framework is potentially useful when analyzing a basket of securities (bonds, stocks, and options) on the same firm and trying to infer one price from the prices of the other securities (which is sometimes referred as "capital structural arbitrage" on Wall Street).

2. BASIC SETTING OF THE MODEL

2.1. Asset Model

To generalize the Leland and Toft (1996) model to include jumps, we will use a double exponential jump-diffusion model (Kou 2002; Kou and Wang 2003) for the firm asset. Essentially, it replaces the jump size distribution in Merton's (1976) normal jump-diffusion model by a double exponential distribution. Besides giving heavier tails, a main advantage of the double exponential distribution is that it leads to analytical solutions for debt and equity values, while they are difficult under the normal jump-diffusion model.

Since we view equity and debts as contingent claims on the asset, it is enough to specify dynamics under a risk-neutral probability measure P, which can be determined by using the rational expectations argument (Lucas 1978) with a HARA type of utility function for the representative agent, so that the equilibrium price of an asset is given by the expectation, under this risk-neutral measure P, of the discounted asset payoff. For a detailed justification of the rational expectations equilibrium argument,[5] see Kou (2002) and Naik and Lee (1990).

More precisely, we shall assume that under such a risk-neutral measure P, the asset value of the firm[6] V_t follows a double exponential jump-diffusion process

$$\frac{dV_t}{V_{t-}} = (r-\delta)\,dt + \sigma\,dW_t + d\left(\sum_{i=1}^{N_t}(Z_i - 1)\right) - \lambda\xi\,dt, \tag{2.1}$$

whose solution is given by

$$v_t = V_0 \exp\left\{\left(r - \delta - \frac{1}{2}\sigma^2 - \lambda\xi\right)t + \sigma W_t\right\}\prod_{i=1}^{N_t} Z_i,$$

[5] Instead of using the rational expectations equilibrium argument, alternative approaches of finding pricing measures for credit derivatives are given in Giesecke and Goldberg (2003) and Bielecki, Jeanblanc, and Rutkowski (2004, 2005).

[6] One can think of V_t as the value of an otherwise identical unleveraged firm.

with r being the constant risk-free interest rate, δ the total payment rate to the firm's investors (including to both bond and equity holders), and the mean percentage jump size ξ given by

$$\xi = E[Z-1] = E[e^Y - 1] = \frac{p_u \eta_u}{\eta_u - 1} + \frac{p_d \eta_d}{\eta_d + 1} - 1.$$

Here W_t is a standard Brownian motion under P, $\{N_t, t \geq 0\}$ is a Poisson process with rate λ, the Z_i's are i.i.d. random variables, and $Y_i := \ln(Z_i)$ has a double-exponential density:

$$f_Y(y) = p_u \eta_u e^{-\eta_u y} \mathbf{1}_{\{y \geq 0\}} + p_d \eta_d e^{\eta_d y} \mathbf{1}_{\{y<0\}}, \ \eta_u > 1, \eta_d > 0, \ p_u + p_d = 1.$$

Note that under the above risk-neutral measure, V_t is a martingale after proper discounting: $V_t = E[e^{-(r-\delta)(T-t)} V_T \mid \mathcal{F}_t]$, where $\mathcal{F}_t$ is the information available up to time t.

2.2. Debt Issuing and Coupon Payments

The setting of debt issuing and coupon payments follows Leland (1994) and Leland and Toft (1996). In the time interval $(t, t+dt)$, the firm issues new debt with par value pdt and maturity profile φ, where $\varphi(v) = me^{-mv}$, i.e., the maturity of a specific bond is chosen randomly according to an exponentially distributed random variable with mean $1/m$. Therefore, at any time t, the par value of debt maturing in the time period $(s, s + ds)$ is given by

$$\left(\int_{-\infty}^{t} p\varphi(s-u)\, du\right) ds = \left(\int_{-\infty}^{t} pme^{-m(s-u)}\, du\right) ds = pe^{-m(s-t)}\, ds, \ s \geq t. \tag{2.2}$$

The random maturity assumption is a standard assumption frequently used in the literature, starting from Leland (1994). Obviously the main motivation is that this assumption leads to analytical tractability. As Leland (1995) pointed out, a real-world equivalent of the random maturity is a sinking fund provision, which is quite common in corporate debt issues. Basically the provision means that part of the principal value of debt will be retired on a regular basis. In particular, if we assume that the principal will be retired at a rate m (the word "rate" means the constant in an ordinary differential equation $dx/dt = -mx$), then the par value of debt maturing in the time period $(s, s + ds)$ is given by

$$pe^{-m(s-t)} ds, \ s \geq t,$$

which is exactly what we get in (2.2) by using the random exponential maturity assumption.

Equation (2.2) has two implications. First, letting $s = t$ in (2.2), we have that at time t the par value of the debt maturing at the next moment $(t, t + dt)$ is given by $p\, dt$, which is the same as the par value of the newly issued debt, achieving a static balance. Second, using (2.2), at time t the par value of all pending debt (with maturity from t to ∞) is a constant, equal to

$$\int_t^{+\infty} pe^{-m(s-t)}\, ds = p/m \equiv P.$$

We also assume that before maturity bond holders will receive coupons at rate ρ until default.

At each moment the firm has two cash outflows and two cash inflows. The two cash outflows are after-tax coupon payment $(1-\kappa)\rho P dt$ and principal due $p\,dt$, where κ is the tax rate; the two cash inflows are $b_t dt$ from selling new debts (where b_t is the price of the total newly issued bonds) and the total payment from the asset $\delta V_t\,dt$. If the total cash inflow $(\delta V_t + b_t)\,dt$ is greater than the cash outflow $((1-\kappa)\rho P + p)\,dt$, we assume that the difference of the two goes to the equity holders as dividends; otherwise, additional equity will be issued to fulfill the due liability. Note that the difference $((1-\kappa)\rho P + p)\,dt - (\delta V_t + b_t)\,dt$ is an infinitesimal quantity. Thus, such a financing strategy is feasible as long as the total equity value remains positive before bankruptcy. Therefore, we need to impose the limited liability constraint on the dynamic of the firm's equity; we will discuss this constraint later in Section 3.1.

2.3. Default Payments

As in first-passage models, we assume that the default occurs at time $\tau = \inf\{t \geq 0 : V_t \leq V_B\}$. Upon default, the firm loses $(1-\alpha)$ of V_τ due to reorganization of the firm, and the debt holders as a whole get the rest of the value, αV_τ, after reorganization. Note that $V_\tau \neq V_B$ due to jumps.

To find the total debt value, total equity value, and optimal leverage ratio, it is not necessary to spell out the recovery value for individual bonds with different maturities. However, to model credit spreads we need to specify how the remaining asset of the firm αV_τ is distributed among bonds with different maturities. Three standard assumptions are recovery at a fraction of par value, of market values, and of corresponding treasury bonds; see Lando (2004), Duffie and Singleton (2003), and Jarrow and Turnbull (1995). Here we shall use the assumption of recovery at a fractional treasury (Jarrow and Turnbull 1995). We use this assumption because it makes analytical calculation easier. More precisely, upon default, the payoff of a bond with unit face value, coupon rate ρ, and maturity $T > \tau$ is given by

$$(2.3)\qquad c\left\{e^{-r(T-\tau)} + \int_0^{T-\tau} \rho e^{-rs}\,ds\right\} = c\left\{\left(1-\frac{\rho}{r}\right)e^{-r(T-\tau)} + \frac{\rho}{r}\right\},\quad 0 \leq c \leq 1.$$

To determine c, we shall match the total payment to bondholder with the total remaining asset αV_τ. By the memoryless property of exponential distribution, the conditional distribution of a bond's maturity does not change, given its maturity is beyond τ. Thus, we have by (2.2)

$$\int_\tau^{+\infty} c\left\{\left(1-\frac{\rho}{r}\right)e^{-r(T-\tau)} + \frac{\rho}{r}\right\}\cdot p e^{-m(T-\tau)}\,dT = \alpha V_\tau,$$

yielding

$$c = \frac{\alpha V_\tau}{\left(1-\frac{\rho}{r}\right)p\frac{1}{m+r} + \frac{\rho}{r}p\frac{1}{m}} = \frac{\alpha V_\tau (m+r)}{\left(1-\frac{\rho}{r}\right)Pm + \frac{\rho}{r}P(m+r)},$$

via $P = p/m$. Therefore,

$$c = \frac{m+r}{m+\rho}\frac{\alpha V_\tau}{P}.$$

To make sure that $0 \le c \le 1$, we shall impose that

$$\frac{m+r}{m+\rho}\frac{\alpha V_B}{P} \le 1. \tag{2.4}$$

We will see that this assumption is satisfied by the optimal default barrier; see (3.4).

We should emphasize that this recovery assumption is irrelevant if one is only interested in understanding debt values as a whole, but is relevant when we consider credit spreads which in turn depend on cash flows to bonds with different maturities. In particular, our results about optimal default boundary, optimal leverage level, and implied volatility will not be affected by this recovery assumption.

2.4. Debt, Equity, and Market Value of the Firm

At time 0 the price of a bond with face value 1 and maturity T when the firm asset $V_0 = V$ is given by

$$\begin{aligned}
B(V; V_B, T) &= E\left[e^{-rT}\mathbf{1}_{\{\tau>T\}} + e^{-r\tau}\cdot c\left\{\left(1-\frac{\rho}{r}\right)e^{-r(T-\tau)} + \frac{\rho}{r}\right\}\mathbf{1}_{\{\tau\le T\}}\right] \\
&\quad + E\left[\int_0^{\tau\wedge T}\rho e^{-rs}\,ds\right] \\
&= e^{-rT}P[\tau \ge T] + \frac{\alpha}{P}\frac{m+r}{m+\rho}\left(1-\frac{\rho}{r}\right)E\left[V_\tau e^{-rT}\mathbf{1}_{\{\tau\le T\}}\right] \\
&\quad + \frac{\alpha}{P}\frac{m+r}{m+\rho}\frac{\rho}{r}E\left[V_\tau e^{-r\tau}\mathbf{1}_{\{\tau\le T\}}\right] + \frac{\rho}{r}(1 - E[e^{-r(\tau\wedge T)}]) \\
&= \left(1-\frac{\rho}{r}\right)e^{-rT}P[\tau \ge T] + \frac{\alpha}{P}\frac{m+r}{m+\rho}\left(1-\frac{\rho}{r}\right)E\left[V_\tau e^{-rT}\mathbf{1}_{\{\tau\le T\}}\right] \\
&\quad + \frac{\alpha}{P}\frac{m+r}{m+\rho}\frac{\rho}{r}E\left[V_\tau e^{-r\tau}\mathbf{1}_{\{\tau\le T\}}\right] + \frac{\rho}{r}\left(1 - E\left[e^{-r\tau}\mathbf{1}_{\{\tau\le T\}}\right]\right).
\end{aligned}$$

In our endogenous model, both P (which is related to the optimal debt/equity ratio) and the endogenous default barrier V_B will be decision variables.

According to the Modigliani–Miller theorem (Brealey and Myers 2001), the total market value of the firm is the firm asset value plus the tax benefit and minus the bankruptcy cost. More precisely, the market value of the firm at time 0 is

$$v(V; V_B) = V + E\left[\int_0^\tau \kappa\rho P e^{-rt}\,dt\right] - (1-\alpha)E\left[V_\tau e^{-r\tau}\right].$$

The total debt $D(V; V_B)$ is equal to

$$D(V; V_B) = P\int_0^{+\infty} me^{-mT}B(V; V_B, T)\,dT,$$

and the total equity value is

$$S(V; V_B) = v(V; V_B) - D(V; V_B).$$

REMARK 2.1. In this static model (thanks to the exponential maturity profile), the total firm value, total debt, and equity values are all Markovian and are independent of the time horizon.

2.5. Preliminary Results for Debt and Equity Values

To compute the total debt and equity values, one needs to compute the distribution of the default time τ and the joint distribution of V_τ and τ. The analytical solutions for these distributions depend on the roots of the following equation (which is essentially a four-degree polynomial):

$$G(x) = r + \beta,\ \beta > 0,\ G(x) := -\left(r - \delta - \tfrac{1}{2}\sigma^2 - \lambda\xi\right)x + \tfrac{1}{2}\sigma^2 x^2 + \lambda\left(\frac{p_d\eta_d}{\eta_d - x} + \frac{p_u\eta_u}{\eta_u + x} - 1\right).$$

Lemma 2.1 in Kou and Wang (2003) implies that the above equation has exactly four real roots. Denote the four roots to the equation above by $\gamma_{1,\beta}, \gamma_{2,\beta}, -\gamma_{3,\beta}, -\gamma_{4,\beta}$, with

$$0 < \gamma_{1,\beta} < \eta_d < \gamma_{2,\beta} < \infty,\quad 0 < \gamma_{3,\beta} < \eta_u < \gamma_{4,\beta} < \infty.$$

Analytical solutions for all these roots are given in the appendix of Kou, Petrella, and Wang (2005).

LEMMA 2.1. *The value of total debt at time* 0 *is*

$$\begin{aligned} D(V; V_B) &= P\int_0^{+\infty} me^{-mT}B(V; V_B, T)\,dT \\ &= \frac{P(\rho+m)}{r+m}\left\{1 - d_{1,m}\left(\frac{V_B}{V}\right)^{\gamma_{1,m}} - d_{2,m}\left(\frac{V_B}{V}\right)^{\gamma_{2,m}}\right\} \\ &\quad + \alpha V_B\left\{c_{1,m}\left(\frac{V_B}{V}\right)^{\gamma_{1,m}} + c_{2,m}\left(\frac{V_B}{V}\right)^{\gamma_{2,m}}\right\}. \end{aligned}$$

The Laplace transform of a bond price, $B(V; V_B, T)$, is given by

$$\begin{aligned} \int_0^{+\infty} e^{-\beta T}B(V; V_B, T)\,dT &= \frac{\rho+\beta}{\beta(r+\beta)}\left\{1 - d_{1,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{1,\beta}} - d_{2,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{2,\beta}}\right\} \\ &\quad + \frac{\alpha(m+r)}{P(m+\rho)}\frac{\rho+\beta}{\beta(r+\beta)} \\ &\quad \times V\left\{c_{1,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{1,\beta}+1} + c_{2,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{2,\beta}+1}\right\}, \end{aligned}$$

where

$$c_{1,\beta} = \frac{\eta_d - \gamma_{1,\beta}}{\gamma_{2,\beta} - \gamma_{1,\beta}}\frac{\gamma_{2,\beta}+1}{\eta_d+1},\quad c_{2,\beta} = \frac{\gamma_{2,\beta} - \eta_d}{\gamma_{2,\beta} - \gamma_{1,\beta}}\frac{\gamma_{1,\beta}+1}{\eta_d+1},\ d_{1,\beta} = \frac{\eta_d - \gamma_{1,\beta}}{\gamma_{2,\beta} - \gamma_{1,\beta}}\frac{\gamma_{2,\beta}}{\eta_d},$$

$$d_{2,\beta} = \frac{\gamma_{2,\beta} - \eta_d}{\gamma_{2,\beta} - \gamma_{1,\beta}}\frac{\gamma_{1,\beta}}{\eta_d}.$$

The total market value of the firm is given by

$$v(V; V_B) = V + \frac{P\kappa\rho}{r}\left\{1 - d_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} - d_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}}\right\}$$

$$- (1-\alpha)V_B\left\{c_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} + c_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}}\right\},$$

and the total equity of the firm is given by $S(V;V_B) = v(V;V_B) - D(V;V_B)$.

A short proof of this lemma, basically following from the calculation of the first passage times in Kou and Wang (2003), will be given in Appendix A. Various versions of the lemma and some similar results are also given in Huang and Huang (2003), Metayer (2003), Dao and Jeanblanc (2005), and Le Courtois and Quittard-Pinon (2007). The formulae above have interesting interpretations. For example, in the formula for the total debt value, note that $\frac{P(\rho+m)}{r+m}$ is the present value of the debt with face value P and maturity profile $\phi(t) = me^{-mt}$ in absence of bankruptcy. The term $1 - d_{1,m}(V_B/V)^{\gamma_{1,m}} - d_{2,m}(V_B/V)^{\gamma_{2,m}}$ in the first sum is the present value of \$1 contingent on future bankruptcy; the second sum is what the bondholders can get from the bankruptcy procedure. Note that due to jumps, the remaining asset after bankruptcy is not αV_B. A similar interpretation holds for the equity value.

3. MAIN RESULTS

We shall study four issues, the optimal capital structure with endogenous default, credit spreads, implied volatility generated by the model, and pricing of credit and equity default swaps. Both theoretical and numerical results will be reported in this section. Table 3.1 summarizes the parameters to be used in the numerical investigation. In addition, we set the number of shares of stocks is 100, and we assume that 1 year has 252 trading days.

TABLE 3.1
Basic Parameters for Numerical Illustration

Basic parameters: $\sigma = 0.2,\ \kappa = 35\%,\ r = 8\%,\ \alpha = 0.5,\ \rho = 8.162\%,\ \delta = 6\%,\ V_0 = 100,\ 1/m = 5.$
Case A: Pure Brownian case, i.e., the jump rate $\lambda = 0$. Case B: Small number of large jumps, $1/\eta_u = 1/3,\ 1/\eta_d = 1/2,\ p_u = 0.5,\ \lambda = 0.2$ Case C: Moderate number of small jumps, $1/\eta_u = 1/8,\ 1/\eta_d = 1/6,\ p_u = 0.25,\ \lambda = 1.$

The risk-free rate $r = 8\%$ is close to the average historical treasury rate during 1973–1998, and the coupon rate $\rho = 8.162\%$ is the par coupon rate for risk-free bonds with semi-annual coupon payments when the continuously compounded interest rate is 8% according to Huang and Huang (2003), who also set $\delta = 6\%$. The diffusion volatility $\sigma = 0.2$, corporate tax rate 35%, and the recovery fraction $\alpha = 0.5$ are chosen according to Leland and Toft (1996).

3.1. Optimal Capital Structure and Endogenous Default

In this subsection, we consider the problem of optimal capital structure with optimal endogenous default, i.e., the optimal choices of debt level P and bankruptcy trigger level V_B. After presenting results for the optimal P and V_B, we shall point out jump risk can significantly reduce the optimal debt/equity ratio. In particular, the model implies that for firms with high jump risk and few tangible assets (hence low recovery rate), such as Internet and biotech companies, the optimal P is close to zero.

3.1.1. The Solution of a Two-Stage Optimization Problem. Deciding optimal P and choosing optimal V_B are two entangled problems that cannot be separated easily. For example, when a firm chooses P to maximize the total firm value at time 0, the decision on P obviously depends on the bankruptcy trigger level V_B. Similarly, after the debt being issued, the equity holders will choose an optimal V_B, which of course depends on the debt level P. Here we shall choose P and V_B according to a two-stage optimization procedure as in Leland (1994) and Leland and Toft (1996).

More precisely, for a fixed P, the equity holders find the optimal default barrier by maximizing the equity value, subject to the limited liability constraint. It is the equity value that is maximized because after the debt being issued, it is the equity holders who control the firm. Mathematically the equity holders find the best V_B by solving

$$\max_{V_B} S(V; V_B), \tag{3.1}$$

subject to the limited liability constraint

$$S(V'; V_B) \geq 0, \quad \text{for all } V' \geq V_B \geq 0. \tag{3.2}$$

The limited liability constraint is imposed so that the equity value is always nonnegative for any future firm asset value V', as long as the asset value V' is above the bankruptcy trigger level V_B. This is the first-stage optimization problem.

Clearly the optimal $V_B^* \equiv V_B^*(P)$ to be chosen by the equity holders will depend on P. At time 0, the firm will conduct the second-stage optimization to maximize the firm value, as the debt holders will act in anticipation of what the equity holders may do later. More precisely, the second-stage optimization is

$$\max_{P} v\big(V; V_B^*(P)\big). \tag{3.3}$$

The two-stage optimization problem partly arises due to the conflict of interests between debt and equity holders; in fact the two-stage optimization can be viewed as a Stackelberg game (see Gibbons 1992) between the debt and equity holders. It is obvious that choosing P and V_B simultaneously will lead to a better market value of the firm. Leland (1998) used the difference of the two to explain agency costs due to conflicts between the equity and debt holders. Obviously the two-stage optimization is only a rough approximation to complicated decision and negotiation problems between debt and equity holders.[7] Here is the result for the two-stage optimization.

[7] An alternative approach is to model the strategic behavior by various investors of the firm; see Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997).

THEOREM 3.1. (*a*) *The first-stage optimization: Define*

$$\epsilon := \frac{\frac{\rho+m}{r+m}(d_{1,m}\gamma_{1,m}+d_{2,m}\gamma_{2,m}) - \frac{\kappa\rho}{r}(d_{1,0}\gamma_{1,0}+d_{2,0}\gamma_{2,0})}{(1-\alpha)(c_{1,0}\gamma_{1,0}+c_{2,0}\gamma_{2,0})+\alpha(c_{1,m}\gamma_{1,m}+c_{2,m}\gamma_{2,m})+1}.$$

Given the debt level P, if $\epsilon P < V$, *then the optimal barrier level* V_B^* *solving* (*3.1*) *with the constraint in* (*3.2*) *is given by* $V_B^* = \epsilon P$; *otherwise, the optimal choice of* V_B *is to make* $\tau_B =$ 0. *(b) The second-stage optimization: The bondholders will choose such P that* $\epsilon P < V$. *After plugging the optimal* $V_B^* = \epsilon P$ *into the second-stage optimization, we have that* $v(V;\epsilon P)$ *is a concave function of* $P \in (0, V/\epsilon)$, *which implies that we can find a unique optimal debt level P for the problem* (*3.3*).

The proof, which is one of the main results in the paper, will be given in Appendix B. A mathematical contribution of the paper is that the proof also gives a rigorous justification of a smooth-pasting principle; more precisely, V_B^* is the solution of $\frac{\partial S(V;V_B^*)}{\partial V}|_{V=V_B^*} = 0$. Even under the pure Brownian model, Leland and Toft (1996) did not prove the result, mainly because the proof needs the local convexity at V_B, i.e., $\frac{\partial^2 S(V;V_B)}{\partial V^2}|_{V=V_B} \geq 0$. Instead, Leland and Toft (1996, footnote 9) verified the above local convexity numerically and made a conjecture that the smooth-pasting principle should hold. Hilberink and Rogers (2002) gave a numerical verification of the local convexity and conjectured that the smooth-pasting principle should hold under a one-sided jump model. Here we are able to prove the smooth-pasting principle by first proving the local convexity. See Remark B.1 at the end of Appendix B for details.[8]

REMARK 3.1. It is easy to see that V_B^* satisfies (2.4) because

$$0 < \epsilon < \frac{\frac{\rho+m}{r+m}(d_{1,m}\gamma_{1,m}+d_{2,m}\gamma_{2,m})}{\alpha(c_{1,m}\gamma_{1,m}+c_{2,m}\gamma_{2,m}+1)} < \frac{1}{\alpha}\frac{\rho+m}{r+m}, \tag{3.4}$$

which follows from the definitions of $d_{i,m}$ and $c_{i,m}$ as

$$d_{1,m}\gamma_{1,m}+d_{2,m}\gamma_{2,m} = \frac{\gamma_{1,m}\gamma_{2,m}}{\eta_d},\quad c_{1,m}\gamma_{1,m}+c_{2,m}\gamma_{2,m}+1 = \frac{(\gamma_{1,m}+1)(\gamma_{2,m}+1)}{\eta_d+1},$$

$$\frac{\gamma_{1,m}}{\eta_d} < \frac{\gamma_{1,m}+1}{\eta_d+1},\quad \frac{\rho+m}{r+m} > \frac{\kappa\rho}{r},\quad \gamma_{1,m}\gamma_{2,m} > \gamma_{1,0}\gamma_{2,0}.$$

3.1.2. The Impact of Jumps on Optimal Capital Structure. Table 3.2 shows the effect of various parameters on the optimal leverage level. Consistent with our intuition, the table shows that the optimal leverage ratio is an increasing function of the fractional remaining asset α and the average maturity profile $1/m$, and it is a decreasing function of the jump rate λ and the diffusion volatility σ. More importantly, the table shows that jump risk leads to much lower leverage ratios. In particular, with infrequent large jumps (case B), the optimal leverage ratio is close to zero, even in the case of only one jump every year ($\lambda = 1$), recovery rate $\alpha = 25\%$, and maturity profile $1/m$ not more than

[8] After the current paper was tentatively accepted in 2006, Dr. A. Kyprianou also pointed out to us an interesting and independent paper by Kyprianou and Surya (2007), in which they show that the smooth-pasting principle may not hold for other jump-diffusion processes.

TABLE 3.2
Effects of Various Parameters on the Optimal Debt/Equity Ratio

		$m^{-1} = 0.5$		$m^{-1} = 1$		$m^{-1} = 2$		$m^{-1} = 5$	
		$\sigma = 0.2$ (%)	$\sigma = 0.4$ (%)	$\sigma = 0.2$ (%)	$\sigma = 0.4$ (%)	$\sigma = 0.2$ (%)	$\sigma = 0.4$ (%)	$\sigma = 0.2$ (%)	$\sigma = 0.4$ (%)
Case B									
$\alpha = 5\%$	$\lambda = 0$	7.15	1.12	11.21	2.38	17.58	5.07	30.69	12.94
	$\lambda = 0.5$	0.68	0.20	1.64	0.61	3.96	1.87	11.75	7.36
	$\lambda = 1$	0.12	0.04	0.42	0.20	1.46	0.84	6.58	4.82
	$\lambda = 2$	0.01	0.001	0.05	0.03	0.35	0.24	3.16	2.64
$\alpha = 25\%$	$\lambda = 0$	13.81	3.22	18.35	5.31	25.12	9.17	38.44	19.11
	$\lambda = 0.5$	2.50	0.88	4.37	1.89	8.11	4.31	18.71	12.69
	$\lambda = 1$	0.66	0.28	1.5	0.78	3.69	2.31	11.94	9.27
	$\lambda = 2$	0.07	0.04	0.29	0.18	1.19	0.87	6.91	6.01
$\alpha = 50\%$	$\lambda = 0$	25.48	9.26	30.34	12.67	37.33	18.33	50.54	31.24
	$\lambda = 0.5$	8.93	4.16	12.44	6.69	18.57	11.62	33.40	25.27
	$\lambda = 1$	3.74	1.99	6.13	3.77	11.01	7.84	25.33	21.33
	$\lambda = 2$	0.88	0.54	2.04	1.43	5.26	4.22	18.55	16.99
Case C									
$\alpha = 5\%$	$\lambda = 0$	7.15	1.12	11.21	2.38	17.58	5.07	30.69	12.94
	$\lambda = 0.5$	4.77	0.88	7.94	1.96	13.22	4.36	24.93	11.79
	$\lambda = 1$	3.39	0.70	5.94	1.62	10.44	3.79	21.08	10.80
	$\lambda = 2$	1.88	0.45	3.64	1.14	7.05	2.91	16.13	9.21
$\alpha = 25\%$	$\lambda = 0$	13.81	3.22	18.35	5.31	25.12	9.17	38.44	19.11
	$\lambda = 0.5$	10.13	2.67	13.99	4.55	20.01	8.15	32.53	17.78
	$\lambda = 1$	7.76	2.23	11.11	3.93	16.54	7.28	28.42	16.62
	$\lambda = 2$	4.90	1.58	7.52	2.97	12.07	5.91	22.94	14.71
$\alpha = 50\%$	$\lambda = 0$	25.48	9.26	30.34	12.67	37.33	18.33	50.54	31.24
	$\lambda = 0.5$	20.66	8.17	25.19	11.44	31.90	16.96	45.13	29.90
	$\lambda = 1$	17.21	7.24	21.46	10.36	27.93	15.75	41.15	28.70
	$\lambda = 2$	12.56	5.76	16.38	8.60	22.45	13.72	35.66	26.67

The basic parameters are given by Table 3.1. $\lambda = 0$ is the pure Brownian case (case A). Note that comparing to the pure Brownian case the jump risk reduces the optimal leverage ratio significantly, often even making the ratio close to zero, especially in the case of infrequent large jumps (case B).

2 years. Internet and biotech companies typically have low α (as they do not have many "tangible" assets) and short maturity profile (as they do not have long operating history to secure long-term debt even if they want to issue debts). Therefore, jump risk can lead to a much lower debt/equity ratio, even making it close to zero, especially in the case of infrequent large jumps.

3.2. Flexible Credit Spreads

In this subsection we try to incorporate four stylized facts related to credit spreads as outlined in Section 1.1, namely, (1) non-zero credit spreads for very short maturity bonds; (2) flexible credit spreads with possible upward, humped, and downward shapes;

(3) upward credit spreads even for speculative grade bonds; and (4) the problem of over- and underprediction of credit spreads for long- and short-maturity bonds, respectively, in the Leland–Toft model.

By analogy to the case of discrete coupons, in the case of continuous coupon rate, we shall define the yield to maturity, $\nu(T)$, of a defaultable bond with maturity T and coupon rate ρ as the one satisfies

$$B(V; V_B, T) = e^{-\nu(T)T} + \int_0^T \rho e^{-\nu(T)s} ds = e^{-\nu(T)T} + \frac{\rho}{\nu(T)}(1 - e^{-\nu(T)T}).$$

The credit spread is defined as $\nu(T) - r$. The bond price $B(V; V_B, T)$ can be computed by using Lemma 2.1 and numerical Laplace inversion algorithms, such as the Euler inversion algorithm in Abate and Whitt (1992).

3.2.1. Non-Zero Credit Spreads and a Connection with Reduced-Form Models. For very short maturity bonds, the analytical solution of credit spreads is available.

THEOREM 3.2. *We have*

$$\lim_{T\to 0} \nu(T) - r = \lambda p_d \left(\frac{V_B}{V}\right)^{\eta_d} \left[1 - \frac{\alpha V_B}{P} \frac{m+r}{m+\rho} \frac{\eta_d}{\eta_d + 1}\right].$$

In particular, by (2.4) *the above limit for the short-maturity credit spread is strictly positive as long as there is a downside jump risk* (*i.e.*, $\lambda p_d > 0$).

The proof will be given in Appendix C. As we mentioned in Section 1.2, there are well-recognized connections linking reduced-form models and structural models. This is perhaps one of the main reasons that adding jumps can produce flexible shapes of non-zero credit spreads, just as in reduced-form models. In our particular setting,

$$P[\tau \le t + \Delta t \mid \tau > t] = \lambda p_d \left(\frac{V_B}{V_t}\right)^{\eta_d} \Delta t + o(\Delta t).$$

Since the intensity h_t in a reduced-form model satisfies

$$P[\tau \le t + \Delta t \mid \tau > t] = h_t \Delta t + o(\Delta t),$$

we have, to the first-order approximation, that the model behaves like a reduced-form model with $h_t = \lambda p_d (V_B/V_t)^{\eta_d}$, despite that the model also has a predictable component (i.e., the diffusion component).

3.2.2. Upward, Humped, and Downward Credit Spreads. By adding jumps the model can produce flexible credit spreads, including upward, humped, and downwards shapes.[9] Normally, the credit spread shape is upward; as the firm's financial situation deteriorates, it becomes humped and even downward in face of immediate financial distress. Figure 3.1 illustrates that our model can reproduce this phenomenon.

[9] All three kinds of shapes may prevail not only for corporate bonds but also for investment grade sovereign bonds. For example, Schmid (2004, p. 277) shows that the credit spreads between Italian bonds (with S&P rating AA) and German bonds (with AAA, rating which may be effectively treated as "risk-free") can have both upward and humped shapes, while the spreads between Greek bonds (with A− rating) in general have downward shapes.

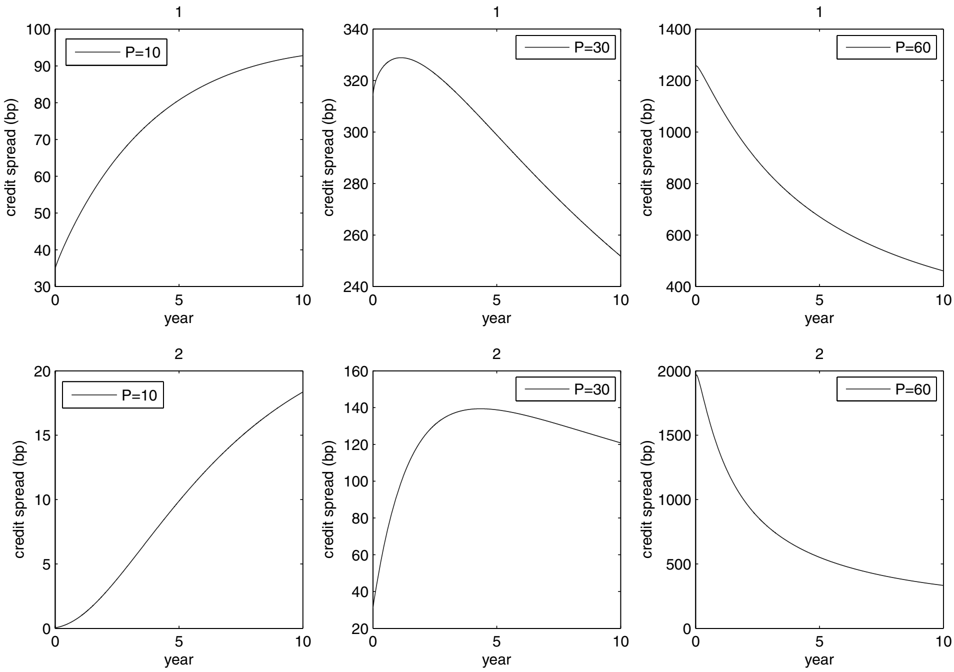

FIGURE 3.1. Various shapes of credit spreads for two-sided jumps. The parameters used are risk-free rate $r = 8\%$, coupon rate $\rho = 1\%$, payment rate $\delta = 1\%$, volatility $\sigma = 10\ \%$, corporate tax rate $\kappa = 35\%$, bankruptcy loss fraction $\alpha = 50\%$, and average maturity $m^{-1} = 1/3$ years for the first row and $m^{-1} = 0.25$ for the second row. In the first row, the jump parameters are specified as $1/\eta_u = 1/3,\ 1/\eta_d = 1/2,\ p_u = 0.75$ and $\lambda = 0.5$, while in the second row, the jump parameters are given in Case C with jump rates $\lambda = 2$.

The shape of credit spreads for speculative bonds is a somewhat controversial subject. Traditional empirical studies based on aggregated data for speculative bonds suggested downward and humped shapes (e.g. Sarig and Warga 1989; Fons 1994). Helwege and Turner (1999) argued that possible maturity bias (as healthier firms are able to issue longer maturity bonds) may affect the aggregated empirical studies; instead, they suggested that even speculative bonds may have upward shapes during normal times. However, He et al. (2000) later argued that empirically humped and downward shapes should be seen for speculative bonds, even after being adjusted for the maturity bias.

For theoretical models, it is desirable to have all three shapes for speculative bonds. We have seen that the model leads to humped and downward credit spreads for firms with large leverage levels. Figure 3.2 shows that we can generate an upward-shaped curve for the leverage level $P/V = 50\%$ and total volatility[10] $\sigma_{\text{total}} = 40\%$. In pure Brownian model with $\lambda = 0$ (the dash line), the credit spread curve is humped with a zero credit spread,

[10] The total variance is denoted by $\sigma^2_{\text{total}} = \sigma^2 + \sigma^2_{\text{jump}}$, where

$$\sigma^2_{\text{jump}} = \frac{1}{t}\text{Var}\left(\sum_{i=1}^{N(t)}(Z_i - 1)\right) = \lambda\left\{\left[\frac{p_u\eta_u}{\eta_u - 2} + \frac{p_d\eta_d}{\eta_d + 2}\right] - \left[\frac{p_u\eta_u}{\eta_u - 1} + \frac{p_d\eta_d}{\eta_d + 1}\right]^2\right\},\ \eta_u > 2.$$

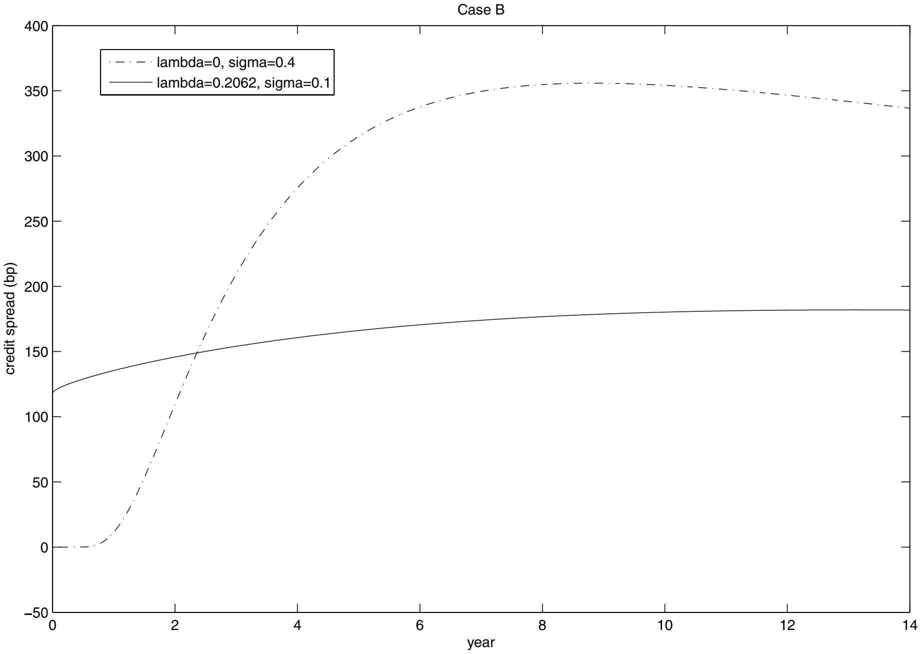

FIGURE 3.2. Upward credit spread curves for speculative bonds. The parameters used are the leverage ratio $P/V = 50\%$, total volatility $\sigma_{\text{total}} = 40\%$, average bonds maturity $m^{-1} = 5$ years, coupon rate $\rho = 2\%$, and all the other parameters are the same as Case B and basic parameters in Table 3.1.

as the maturity goes to zero. However, with jumps, the credit spread becomes upward. Collin-Dufresne and Goldstein (2001) generated a similar upward credit spread curve for speculative bonds (see Figure 3.3 in their paper) using the Brownian motion with a stochastic exogenous default barrier; however, being a diffusion model, their model leads to zero credit spread as the maturity goes to zero.

3.2.3. Effects of Various Parameters on Credit Spreads. Figure 3.3 shows that in our model credit spreads decrease as the interest rate increases. This is consistent with the negative correlation between credit spreads and risk-free rate found in empirical studies mentioned in Section 1.2. Furthermore, if the risk-free rate has mean reversion, then the credit spreads will be so too since they are negatively related in the model.

Figure 3.4 illustrates the effect of various parameters on credit spreads. In particular, credit spreads decrease in α, increase in λ, and decrease in average maturity $1/m$. All of these are consistent with our intuition. However, it is interesting to point out that in Case B for short-maturity bonds, the credit spread is actually a decreasing function of diffusion volatility σ. To explain this, note that the endogenous optimal bankrupt barriers are $V_B^* = 21.6947$ ($\sigma = 0.2$), 19.5422 ($\sigma = 0.3$), and 17.3502 ($\sigma = 0.4$), respectively. For short-maturity bonds, the defaults will be caused mainly by jumps rather than by the diffusion part. Therefore, the low diffusion volatility case of $\sigma = 0.2$ has the largest probability of defaulting in a short time because it is easier to cross the resulting higher barrier V_B^* by jumps. On the other hand, for long-maturity bonds, the diffusion part of the process

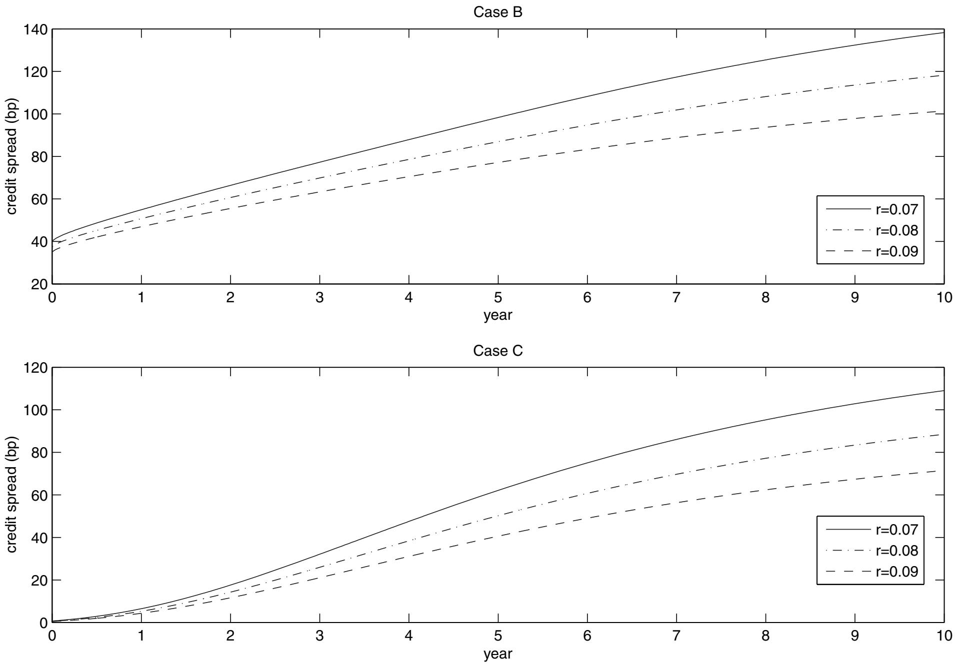

FIGURE 3.3. The effect of the risk-free rate on credit spreads. The parameters used are the leverage level $P/V = 30\%$, average bonds maturity $m^{-1} = 5$ years, and all the other parameters are the same as those in Table 3.1.

plays a more important role to determine credit spreads; therefore, higher volatility in diffusion part may lead to larger credit spreads.

In Figure 3.5, we fix the total volatility of the asset process to see the impacts of jump and diffusion parts to credit spreads. Eom et al. (2004) found that empirical credit spread curves are much flatter than what are predicted from the Leland–Toft model after their calibration.[11] Figure 3.5 seems to suggest that large infrequent jumps, as in Case B, lead to results that might be more consistent with the empirical credit spread curves, as in that case, jumps significantly reduce credit spreads for long-maturity bonds while lift up credit spreads for short-maturity bonds.

3.3. Volatility Smile

In this subsection, we study the connection between credit spreads and implied volatility. The connection went back at least to Black (1976); for recent empirical studies of the connection see, e.g., Toft and Prucyk (1997), Hull, Nelken, and White (2004), Cremers et al. (2005a), and Carr and Linetsky (2005). The interesting points in this subsection are: (1) We should carefully distinguish exogenous and endogenous defaults when we study the possible connection between credit spreads and implied volatility; see Figure 3.6. (2) Default and jumps together can generate significant volatility smiles even for long-maturity equity options; see Figure 3.8.

[11] Eom et al. (2004) did not report what calibration parameters were used in that paper.

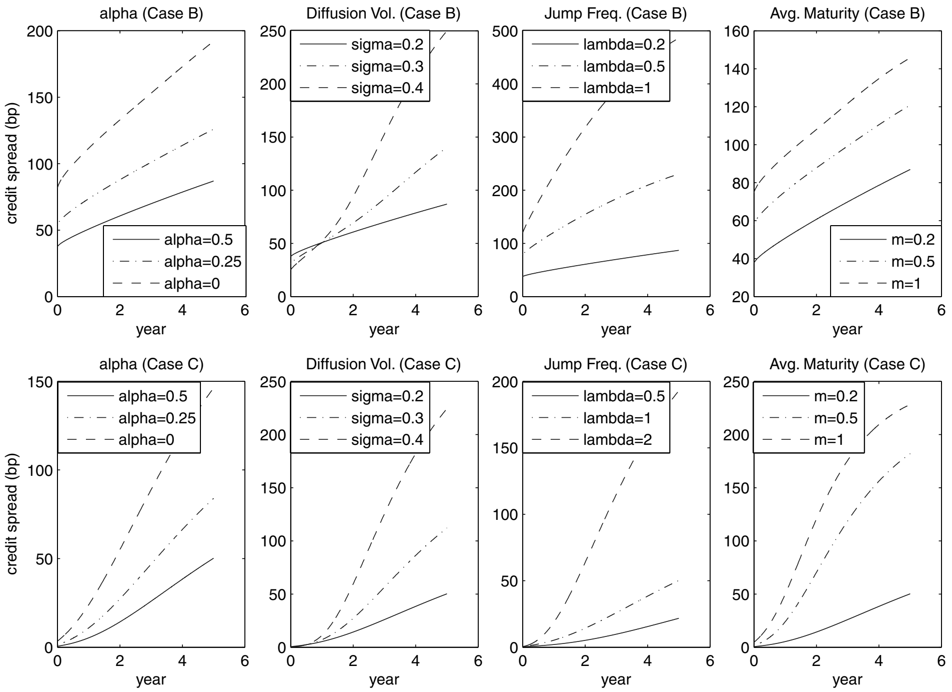

FIGURE 3.4. The effects of various parameters on credit spreads. The defaulting parameters used are the leverage ratio $P/V = 30\%$, and all the other parameters are the same as those in Table 3.1. Note the non-monotonicity in Case B in terms of σ, partly because the optimal V_B^* is decreasing in σ.

The value of a call option written on the whole equity (or undiluted stock shares) is

$$e^{-rT} E[\max(S(V_T) - K) \cdot \mathbf{1}_{\{\tau > T\}}], \tag{3.5}$$

where $S(V_T)$ is the equity value at the maturity of the option with the asset value of the firm being V_T, K is the strike price, and τ is the bankruptcy time. We assume that the option becomes worthless if the bankruptcy happens before the option expires.

In our investigation, we use Monte-Carlo simulation to estimate the prices of such call options. More precisely, 100,000 simulation runs of V_t with $V_0 = 100$ are generated, and we price European call options with 60 different strike prices, $K_i = S(V_0) - 2 \times (i - 30)$, $i = 1, \ldots, 60$, according to (3.5). These options cover the cases of deep-in-the-money, at-the-money, and deep-out-of-the money. In simulation of the dynamic of V_t in (2.1) and the corresponding default time τ, we choose the discrete time unit to be 1 trading day (i.e., 1/252 year). After getting the call prices, we calculate the corresponding implied volatility from the Black–Scholes formula (without jumps and default), assuming these simulated call prices to be the true market prices.[12]

[12] To get the implied volatility from the Black–Scholes formula, we need to compute the dividend rate of the stock. The average dividend rate d of the stock over [0, T] must satisfy $S_0 = e^{-(r-d)} E[S_T]$, where S_0 and S_T are the stock prices at time 0 and T, respectively. In the Monte-Carlo simulation, we use the average dividend $d = r + \log(S_0/E[S_T])/T$ in the Black–Scholes formula to compute the implied volatility.

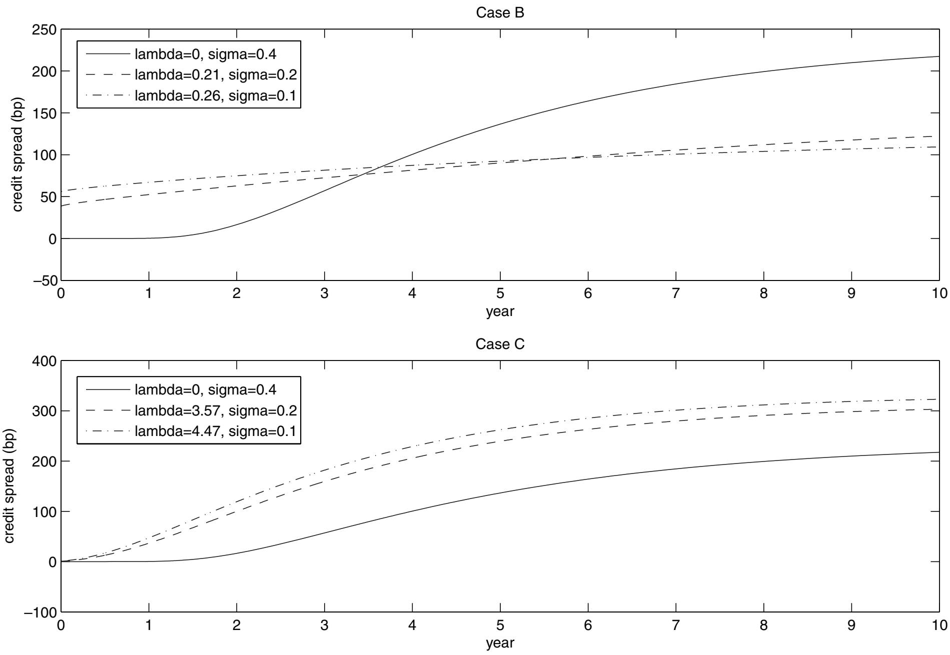

FIGURE 3.5. Credit spreads: Jump volatility vs. diffusion volatility. The parameters are $P/V = 30\%$, $\sigma_{\text{total}} = 0.40$, and all the other parameters are the same as those in Table 3.1. The plot for Case B seems to be more consistent with the empirical finding in Eom et al. (2004).

3.3.1. Connection between Implied Volatility and Credit Spreads. With exogenous default, the default barrier V_B is a fixed constant independent of firm parameters, such as σ; while with endogenous default, the default barrier V_B is chosen to be the optimal value V_B^* in Theorem 3.1. In particular, with exogenous default the barrier V_B is independent of σ, while with endogenous default the barrier V_B^* is a function of σ.

In terms of the connection between implied volatility and credit spreads, Figure 3.6 shows a clear difference between exogenous and endogenous defaults. In the case of exogenous default, we set the default boundary $V_B = P = 30$ for all σ's; in the case of endogenous defaults, different σ's lead to different choices of the optimal default barrier V_B^*; in fact, $V_B^* = 21.6947$ ($\sigma = 0.2$), $V_B^* = 19.5422$ ($\sigma = 0.3$), $V_B^* = 17.3502$ ($\sigma = 0.4$), respectively. Notice that in Figure 3.6, the lower σ is, the higher is V_B.

Figure 3.6 shows that the implied volatility and the credit spreads tend to be positively correlated with exogenous default. However, this is not true with endogenous default. In particular, for short-maturity bonds with exogenous default both credit spreads and implied volatility are increasing functions of the diffusion volatility, while this is not true with endogenous default. The reasons are: (1) The higher σ leads to lower optimal default barrier V_B^*. (2) For a short-maturity bond, the default is mainly caused by jumps rather than by the diffusion part. Therefore, combining them together, with higher volatility σ, it is more difficult for the process to cross the (resulting lower) default level V_B^* in short time period, leading to lower credit spreads for short-maturity bonds.

Of course, in practice all defaults must be endogenous, as the default decision will clearly depend on the firm-specific parameters such as σ. Thus, the practical message from

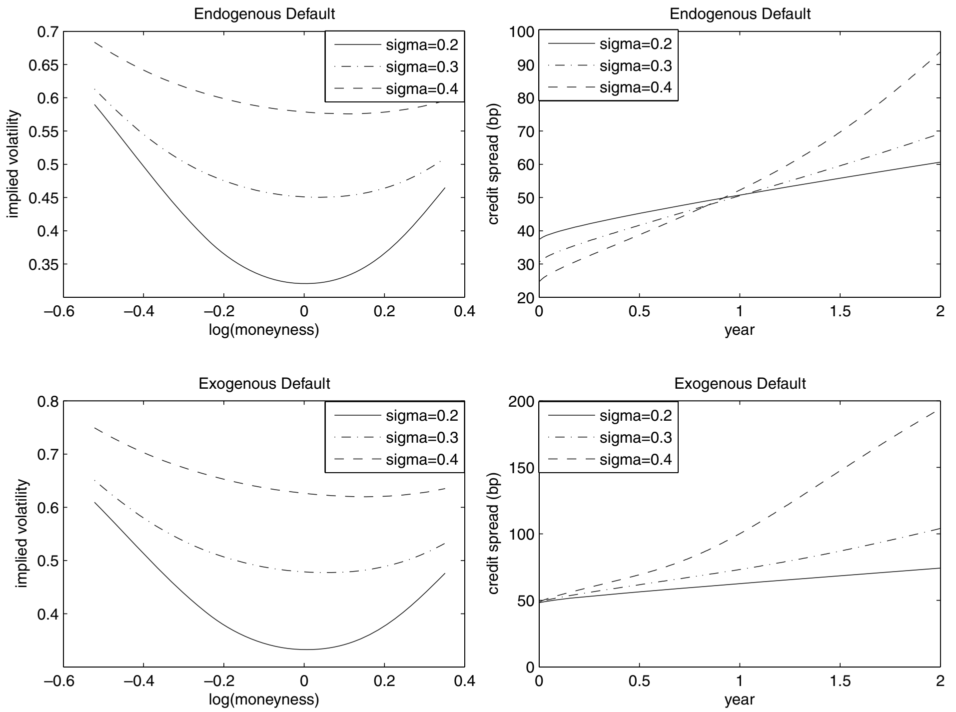

FIGURE 3.6. Exogenous default boundary vs. endogenous boundary. We use leverage ratio $P/V = 30\%$, the maturity of option $T = 0.25$ year, and the rest of parameters as in Case B in Table 3.1. Note the non-monotonicity of the credit spreads under endogenous default. For exogenous default we set $V_B = P = 30$, and for endogenous default we choose V_B optimally.

here is that when analyzing the relation between credit spreads and implied volatility it is important to control for variation in firm-specific characteristics, such as changing in σ.

3.3.2. Effects of Various Parameters on Implied Volatility. Figure 3.7 illustrates the effects of various parameters on the implied volatility, which seems to be increasing in σ and λ, and decreasing in $1/m$ and α; all of these make sense intuitively. Figure 3.8 aims at comparing the impacts of jump volatility and diffusion volatility by fixing the total volatility. It is interesting to observe that even for very long-maturity options, such as $T =$ 8 years, the implied volatility smile is still significant, due to the combination of default and jump risks. This should be compared with Lévy process models without default risk, in which case the implied volatility smile tends to disappear for long-maturity options, as the jump impacts are gradually washed out; see Cont and Tankov (2003). However, the combination of jump and default seems to prolong the effect of implied volatility significantly.

3.4. One-Sided Jumps vs. Two-Sided Jumps

We build a model with both upside and downside jumps. There are some differences between one-sided jumps and two-sided jumps.

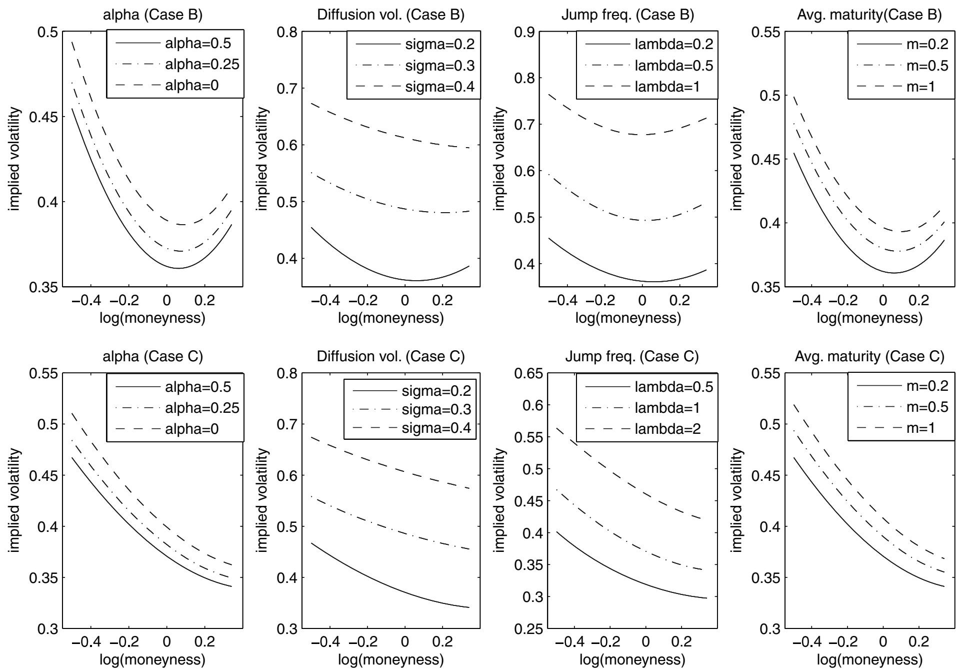

FIGURE 3.7. Effects of various parameters on implied volatility. We use $P/V = 30\%$, call option maturity $T = 1$, and all other parameters as in Table 3.1.

(1) As we have shown in Theorem 3.2, the non-zero credit spreads are due to downside jumps. Therefore, a jump-diffusion model with only upside jumps may not produce non-zero credit spreads.
(2) One-sided downside jumps can also produce some flexible credit spreads, as in the two-sided jump case; see Figure 3.9. However, models with two-sided jumps can easily produce these pictures, as they have some extra parameters.
(3) The main difference between two-sided jumps and one-sided (downside) jumps lies in implied volatility; see Figure 3.10. More precisely, similar to the case of equity options, Figure 3.10 shows that two-sided jumps can generate more flexible implied volatility curves compared with one-sided jumps, which tend to generate monotone curves.

Although people have known for a long time that, numerically, two-sided jumps can produce flexible volatility smiles for call and put options on stocks, a complete theoretical justification of this is still needed. For example, only recently Benaim and Friz (2006a, 2006b) gave a rigorous proof of the asymptotic shape of the implied volatility curves, as the strike price K goes to infinity or to zero under various models, including jump models. On the other hand, Renault and Touzi (1996) and Sircar and Papanicolaou (1999) proved that under a stochastic volatility model, if K is within the neighborhood of the initial stock price S_0, then the implied volatility curves should be convex. However, to our best knowledge, it is still an open problem to give a complete theoretical investigation of the whole implied volatility curves, not just for the cases of $K \approx S_0$, $K \to \infty$, or $K \to 0$.

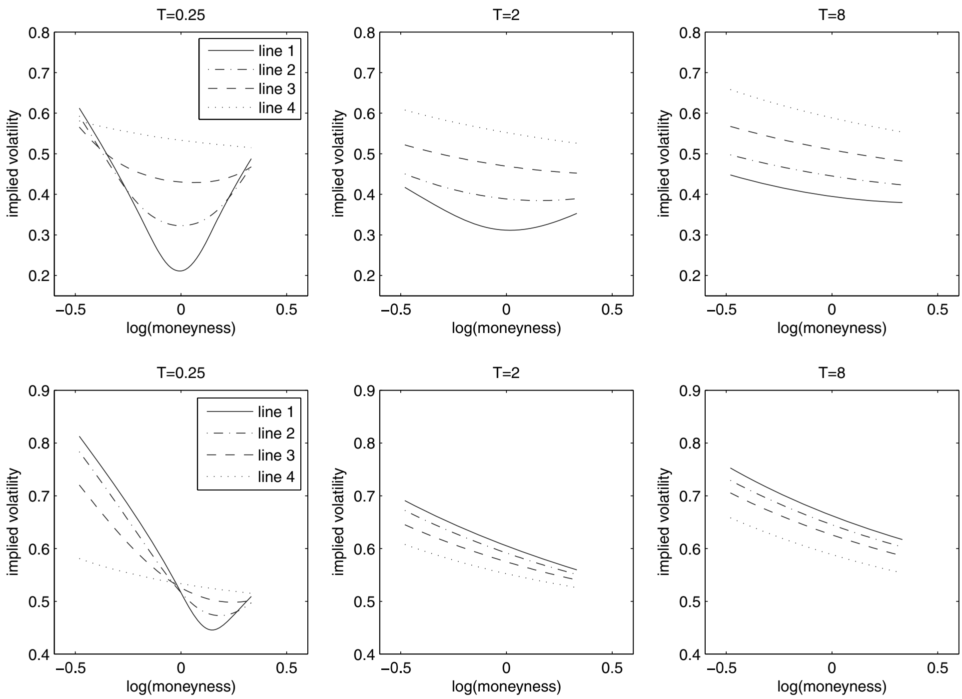

FIGURE 3.8. Implied volatility with fixed total volatility $\sigma_{\text{total}} = 40\%$ and $P = 30$. The first row is for Case B, with line 1 for $\sigma = 0.1,\ \lambda = 0.26$, line 2 for $\sigma = 0.2,\ \lambda = 0.20$, line 3 for $\sigma = 0.3,\ \lambda = 0.12$, line 4 for $\sigma = 0.4,\ \lambda = 0$. The second row is for Case C, with line 1 for $\sigma = 0.1,\ \lambda = 4.47$, line 2 for $\sigma = 0.2,\ \lambda = 3.57$, line 3 for $\sigma = 0.3,\ \lambda = 2.08$, line 4 for $\sigma = 0.4,\ \lambda = 0$. All the other parameters follow Table 3.1. Note that the implied volatility is still significant even for $T = 8$ years.

3.5. Pricing of Credit and Equity Default Swaps

In this subsection we study credit and equity default swaps. Further background on these derivatives can be found in Lando (2004), Medova and Smith (2004), and Schönbucher (2003).

Consider a credit default swap (CDS) with maturity t on a T-year coupon bond ($t < T$) issued by the firm with the asset value in (2.1). The buyer of the CDS makes continuous payments at rate s, called the CDS spread, to the seller until either the end of the life of the CDS (time t) or the default, whichever comes first; in return the seller will cover the possible loss of the buyer due to the default. A central issue for CDS is the calculation of the CDS spread s.

PROPOSITION 3.1. *The CDS spread s is given by*

$$s = r\,\frac{\left(1-\frac{\rho}{r}\right)e^{-r(T-t)}A_5(t) + \frac{\rho}{r}A_3(t) - \frac{m+r}{m+\rho}\left(\frac{\alpha}{P}\right)\left\{\left(1-\frac{\rho}{r}\right)e^{-r(T-t)}A_2(t) + \frac{\rho}{r}A_4(t)\right\}}{1 - A_3(t) - A_1(t)},$$

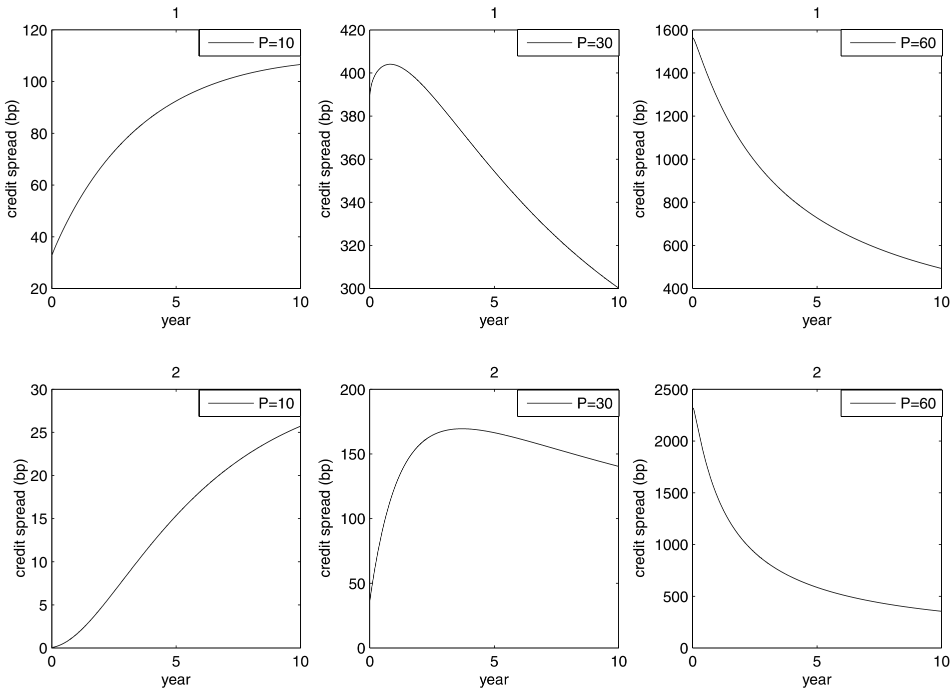

FIGURE 3.9. Various shapes of credit spreads with one-sided (downsided) jumps. The parameters used are risk-free rate $r = 8\%$, coupon rate $\rho = 1\%$, payment rate $\delta = 1\%$, volatility $\sigma = 10\ \%$, corporate tax rate $\kappa = 35\%$, bankruptcy loss fraction $\alpha = 50\%$, and average maturity $m^{-1} = 1/3$ years for the first row and $m^{-1} = 0.25$ for the second row. In the first row, the jump parameters are specified as $1/\eta_d = 1/2$, $p_d = 100\%$, and $\lambda = 0.5$, while in the second row the jump parameters are given as $1/\eta_d = 1/6$, $p_d = 100\%$, and $\lambda = 2$.

where the Laplace transforms of the terms in the above equation are

$$\mathcal{L}[A_1(t)](\beta) := \mathcal{L}[e^{-rt} P[\tau > t]](\beta) = \frac{1}{r+\beta}\left\{1 - \left[d_{1,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{1,\beta}} + d_{2,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{2,\beta}}\right]\right\},$$

$$\mathcal{L}[A_2(t)](\beta) := \mathcal{L}\big[e^{-rt} E\big[V_\tau \mathbf{1}_{\{\tau\le t\}}\big]\big](\beta) = \frac{V_B}{r+\beta}\left\{c_{1,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{1,\beta}} + c_{2,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{2,\beta}}\right\},$$

$$\mathcal{L}[A_3(t)](\beta) := \mathcal{L}\big[E\big[e^{-r\tau}\mathbf{1}_{\{\tau\le t\}}\big]\big](\beta) = \frac{1}{\beta}\left\{d_{1,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{1,\beta}} + d_{2,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{2,\beta}}\right\},$$

$$\mathcal{L}[A_4(t)]\{\beta\} := \mathcal{L}\big[E\big[V_\tau e^{-r\tau}\mathbf{1}_{\{\tau\le t\}}\big]\big]\{\beta\} = \frac{V_B}{\beta}\left\{c_{1,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{1,\beta}} + c_{2,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{2,\beta}}\right\},$$

$$\mathcal{L}[A_5(t)](\beta) := \mathcal{L}[e^{-rt} P[\tau \le t]](\beta) = \frac{1}{r+\beta}\left\{d_{1,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{1,\beta}} + d_{2,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{2,\beta}}\right\},$$

with $d_{1,\beta}, d_{2,\beta}, c_{1,\beta}, c_{1,\beta}, \gamma_{1,\beta}, \gamma_{2,\beta}$ *defined as in Lemma 2.1.*

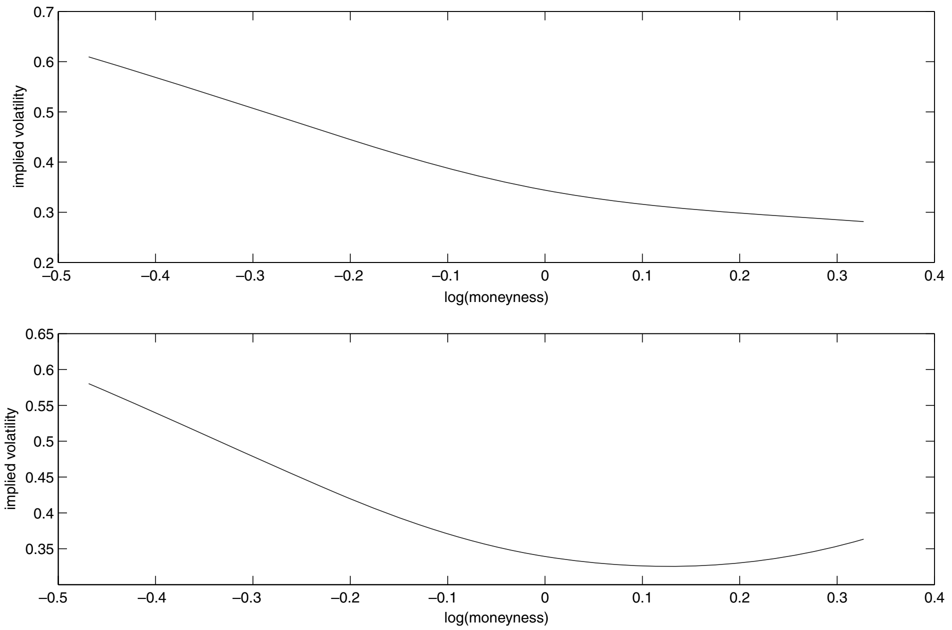

FIGURE 3.10. One-sided jumps (upper panel) vs. two-sided jumps (lower panel). The parameters used are the same basic parameters in Table 3.1 and the leverage ratio $P/V = 30\%$, the option maturity $T = 0.25$. The jump parameters follow Case C, except in the upper panel (the one-sided jump case), where we set $p_u = 0$. In all the plots, the "moneyness" is defined to be the ratio of strike price to stock price.

Proof. The value of continuous spread payments from the buyer to the seller is

$$E\left[\int_0^t se^{-ru} \cdot \mathbf{1}_{\{\tau>u\}}\, du\right] = sE\left[\int_0^{\tau\wedge t} e^{-ru} du\right] = \frac{s}{r} E[1 - e^{-r(\tau\wedge t)}]$$
$$= \frac{s}{r} E\big[1 - e^{-r\tau}\mathbf{1}_{\{\tau\le t\}}\big] - \frac{s}{r} e^{-rt} P[\tau > t].$$

Upon default the loss for the bond holder is $(1-c)\{(1-\frac{\rho}{r})e^{-r(T-\tau)} + \frac{\rho}{r}\}$ via (2.3), and the seller of the CDS will cover the loss. The value of the default payment from the seller to the buyer is

$$E\left[e^{-r\tau}(1-c)\left\{\left(1-\frac{\rho}{r}\right)e^{-r(T-\tau)} + \frac{\rho}{r}\right\}\mathbf{1}_{\{\tau\le t\}}\right]$$
$$= \left(1-\frac{\rho}{r}\right)e^{-rT}P(\tau\le t) + \frac{\rho}{r}E\left[e^{-r\tau}\mathbf{1}_{\{\tau\le t\}}\right]$$
$$- \frac{m+r}{m+\rho}\left(\frac{\alpha}{P}\right)\left\{\left(1-\frac{\rho}{r}\right)e^{-rT}E[V_\tau\mathbf{1}_{\{\tau\le t\}}] + \frac{\rho}{r}E\big[V_\tau e^{-r\tau}\mathbf{1}_{\{\tau\le t\}}\big]\right\}.$$

The CDS spread s is the one that makes two values equal, yielding the pricing formula of s. The Laplace transforms of the terms involved can be found in Appendix A. □

Next, we study the pricing of equity default swap (EDS). The buyer of an EDS makes continuous payments until either a payment event occurs or the EDS expires, while the seller makes a single payment if the payment event occurs before the expiry of the EDS.

Unlike a CDS, which has only one payment event (i.e., default), there are two possible payment events in an EDS: a default on the bonds or a fall in the equity value below some predefined level. For simplicity, if we focus on the case that the credit event is the default, then the first payment event is a special case of the second with the predefined level simply being zero, because the value of the equity becomes zero upon default in our model.

More precisely, suppose that the notional value of an EDS is N, the expiry of the EDS is t. The buyer of the EDS has to make a continuous payment at rate $\tilde{s}$, which is called the EDS spread, while the seller of the EDS has to make a payment wN to the buyer when the payment event is being triggered. We assume that the payment event is the first time the equity value (or equivalently the undiluted stock value) falls below a constant S^*, i.e., at the time $\inf\{t \geq 0 : S(V_t; V_B) \leq S^*\}$.

PROPOSITION 3.2. *Let V^* be the unique solution to the equation $S(V; V_B) = S^*$, which is a strictly increasing function in V for $V > V_B$, and let $\varsigma = \inf\{t \geq 0 : V_t \leq V^*\}$. The EDS spread $\tilde{s}$ is given by*

$$\tilde{s} = rwN \cdot \frac{E\big[e^{-r\varsigma} \cdot \mathbf{1}_{\{\varsigma \leq t\}}\big]}{1 - E\big[e^{-r\varsigma}\mathbf{1}_{\{\varsigma \leq t\}}\big] - e^{-rt}P[\varsigma > t]},$$

where the Laplace transforms of the terms involved are

$$\begin{aligned} \mathcal{L}[e^{-rt}P[\varsigma > t]](\beta) &= \frac{1}{r+\beta}\left\{1 - \left[d_{1,\beta}\left(\frac{V^*}{V}\right)^{\gamma_{1,\beta}} + d_{2,\beta}\left(\frac{V^*}{V}\right)^{\gamma_{2,\beta}}\right]\right\}, \\ \mathcal{L}\big[E\big[e^{-r\varsigma}\mathbf{1}_{\{\varsigma \leq t\}}\big]\big](\beta) &= \frac{1}{\beta}\left\{d_{1,\beta}\left(\frac{V^*}{V}\right)^{\gamma_{1,\beta}} + d_{2,\beta}\left(\frac{V^*}{V}\right)^{\gamma_{2,\beta}}\right\}. \end{aligned}$$

Proof. Because $S(V; V_B)$ is strictly increasing in V for $V > V_B$ (see Fact (iii) in Appendix B) and $S(V; V_B)$ is Markovian and static in V_t, we can represent the payment event in S in terms of asset value V: The payment of the EDS is triggered at $\varsigma = \inf\{t \geq 0 : V_t \leq V^*\}$. The value of continuous spread payments from the buyer to the seller is

$$E\left[\int_0^t \tilde{s}e^{-ru} \cdot \mathbf{1}_{\{\varsigma > u\}}du\right] = \frac{\tilde{s}}{r}E\big[1 - e^{-r\varsigma}\mathbf{1}_{\{\varsigma \leq t\}}\big] - \frac{\tilde{s}}{r}e^{-rt}P[\varsigma > t].$$

The value of the payment from the seller to the buyer is

$$E\big[e^{-r\varsigma} \cdot wN \cdot \mathbf{1}_{\{\varsigma \leq t\}}\big].$$

Setting the two values being equal leads to the solution of $\tilde{s}$. The Laplace transforms of the terms involved can be found in Appendix A. □

4. CONCLUSION

We have demonstrated significant impacts of jump risk and endogenous default on credit spreads, on optimal capital structure, and on implied volatility of equity options. Jump and endogenous default can produce a variety of non-zero credit spreads, upward, downward, and humped shapes, consistent with empirical findings of investment grade and speculative grade bonds. Jump risk leads to much lower optimal debt/equity ratios, helping to explain why some high tech companies (such as biotech and Internet companies) have almost no debt. The two-sided jumps lead to a variety of shapes for the implied volatility of equity options even for long-maturity options; and although in general credit spreads and implied volatility tend to move in the same direction for exogenous default, this may not be true in presence of endogenous default and jumps. Pricing formulae for credit and equity default swaps are provided.

There are several possible directions for future research. First, it will be of interest to study convertible bonds with jump risk, as convertible bonds provide a natural link between credit spreads and equity options. Second, the model in this paper is only a one-dimensional model. Extensions of the model to higher dimensions will be very useful, so that one can study correlated default events and pricing of collateralized debt obligations (CDO). References of these products can be found in, e.g., Sirbu and Shreve (2005) and Hurd and Kuznetsov (2005a, 2005b).

APPENDIX A: PROOF OF LEMMA 2.1

For any $\beta > 0$, consider the Laplace transform of the bond price:

$$\begin{aligned}\int_0^{+\infty} e^{-\beta T} B(V; V_B, T)\,dT &= \left(1-\frac{\rho}{r}\right)\int_0^{+\infty} e^{-(r+\beta)T} P[\tau \geq T]\,dT \\ &\quad + \frac{\alpha}{P}\frac{m+r}{m+\rho}\left(1-\frac{\rho}{r}\right)\int_0^{+\infty} e^{-(\beta+r)T} E[V_\tau \mathbf{1}_{\{\tau\leq T\}}]\,dT \\ &\quad + \frac{\alpha}{P}\frac{m+r}{m+\rho}\frac{\rho}{r}\int_0^{+\infty} e^{-\beta T} E\big[e^{-r\tau} V_\tau \mathbf{1}_{\{\tau\leq T\}}\big]\,dT + \frac{\rho}{r\beta} \\ &\quad - \frac{\rho}{r}\int_0^{+\infty} e^{-\beta T} E\big[e^{-r\tau}\mathbf{1}_{\{\tau\leq T\}}\big]\,dT.\end{aligned}$$

By Fubini's theorem, we can see that the four integral terms inside are

$$\begin{aligned}\int_0^{+\infty} e^{-(r+\beta)T} P[\tau \geq T]\,dT &= E\left[\int_0^{\tau} e^{-(r+\beta)T}\,dT\right] = \frac{1}{r+\beta}[1 - Ee^{-(r+\beta)\tau}],\\ \int_0^{+\infty} e^{-(\beta+r)T} E[V_\tau \mathbf{1}_{\{\tau\leq T\}}]\,dT &= E\left[V_\tau \int_\tau^{+\infty} e^{-(\beta+r)T}\,dT\right] = \frac{1}{\beta+r} E\big[V_\tau e^{-(\beta+r)\tau}\big],\\ \int_0^{+\infty} e^{-\beta T} E\big[e^{-r\tau} V_\tau \mathbf{1}_{\{\tau\leq T\}}\big]\,dT &= E\left[V_\tau e^{-r\tau}\int_\tau^{+\infty} e^{-\beta T}\,dT\right] = \frac{1}{\beta} E\big[V_\tau e^{-(\beta+r)\tau}\big],\\ \int_0^{+\infty} e^{-\beta T} E\big[e^{-r\tau}\mathbf{1}_{\{\tau\leq T\}}\big]\,dT &= E\left[e^{-r\tau}\int_0^{+\infty} e^{-\beta T}\mathbf{1}_{\{\tau\leq T\}}\,dT\right] = \frac{1}{\beta} E[e^{-(r+\beta)\tau}].\end{aligned}$$

In summary, we know that

$$\begin{aligned}\int_0^{+\infty} e^{-\beta T} B(V; V_B, T)\,dT &= \frac{\rho+\beta}{\beta(r+\beta)}[1 - E[e^{-(r+\beta)\tau}]] \\ &\quad + \frac{\alpha(m+r)}{P(m+\rho)}\frac{\rho+\beta}{\beta(r+\beta)} E\big[V_\tau e^{-(\beta+r)\tau}\big].\end{aligned}$$

Define $X_t = \ln(V_t/V)$, which means $X_t = (r - \delta - \frac{1}{2}\sigma^2 - \lambda\xi)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$, and consider $\tilde{X}_t = -X_t$, $\tilde{\tau} \equiv \inf\{t \geq 0 : \tilde{X}_t \geq -\ln(\frac{V_B}{V})\}$. It is easy to see that $\tilde{\tau} = \tau$, so that we only need the exact forms of $E[e^{-(\beta+r)\tilde{\tau}+X_{\tilde{\tau}}}]$ and $E[e^{-(r+\beta)\tilde{\tau}}]$, which are all given in Kou and Wang (2003). In particular, with the notations of $G(\cdot)$, $\gamma_{1,\beta}, \gamma_{2,\beta}, -\gamma_{3,\beta}, -\gamma_{4,\beta}$ and $c_{1,\beta}, c_{2,\beta}, d_{1,\beta}, d_{2,\beta}$, we have

$$\begin{aligned}\int_0^{+\infty} e^{-\beta T} B(V; V_B, T)\, dT &= \frac{\rho+\beta}{\beta(r+\beta)} - \frac{\rho+\beta}{\beta(r+\beta)} E[e^{-(r+\beta)\tilde{\tau}}] \\ &\quad + \frac{\alpha(m+r)}{P(m+\rho)} \frac{\rho+\beta}{\beta(r+\beta)} V E[e^{-(\beta+r)\tilde{\tau}-\tilde{X}_{\tilde{\tau}}}] \\ &= \frac{\rho+\beta}{\beta(r+\beta)} - \frac{\rho+\beta}{\beta(r+\beta)} \{d_{1,\beta} e^{\gamma_{1,\beta}\ln(V_B/V)} + d_{2,\beta} e^{\gamma_{2,\beta}(\ln(V_B/V))}\} \\ &\quad + \frac{\alpha(m+r)}{P(m+\rho)} \frac{\rho+\beta}{\beta(r+\beta)} V e^{\ln(V_B/V)} \\ &\quad \times \left\{c_{1,\beta} e^{\gamma_{1,\beta}\ln(V_B/V)} + c_{2,\beta} e^{\gamma_{2,\beta}(\ln(V_B/V))}\right\},\end{aligned}$$

from which the conclusion follows. The debt value follows readily by letting $\beta = m$. Next,

$$v(V; V_B) = V + \frac{P\kappa\rho}{r}\{1 - E[e^{-r\tilde{\tau}}]\} - (1-\alpha) V E[e^{-r\tilde{\tau}-\tilde{X}_{\tilde{\tau}}}],$$

from which the result follows. □

APPENDIX B: PROOF OF THEOREM 3.1 AND THE LOCAL CONVEXITY

LEMMA B.1. *Consider the function* $f(x) = Ax^{\alpha_1} + Bx^{\beta_1} - Cx^{\alpha_2} - Dx^{\beta_2}$, $0 \leq x \leq 1$. *Note that* $f(1) = A + B - C - D$. *In the case of* $0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$, *if* $A + B \geq C + D$ *and* $A \geq C$, *then* $f(x) \geq 0$ *for all* $0 \leq x \leq 1$.

Proof. Simply note that

$$f(x) \geq Ax^{\alpha_2} + Bx^{\beta_2} - Cx^{\alpha_2} - Dx^{\beta_2} = x^{\alpha_2}\{(A-C) - (D-B)x^{\beta_2-\alpha_2}\} \geq 0.$$ □

LEMMA B.2. *We have*

$$\epsilon \geq \frac{\frac{\rho+m}{r+m} d_{1,m}\gamma_{1,m} - \kappa\frac{\rho}{r} d_{1,0}\gamma_{1,0}}{\alpha c_{1,m}(\gamma_{1,m}+1) + (1-\alpha)c_{1,0}(\gamma_{1,0}+1)}. \tag{B.1}$$

Proof. By the definitions of $d_{1,m}, d_{2,m}, d_{1,0}, d_{2,0}$ and $c_{1,m}, c_{2,m}, c_{1,0}, c_{2,0}$, and the fact that $\gamma_{1,m} > \gamma_{1,0}$ and $\gamma_{2,m} > \gamma_{2,0}$, we have

$$
\begin{aligned}
&d_{2,m}\gamma_{2,m}c_{1,0}(\gamma_{1,0}+1) - d_{1,m}\gamma_{1,m}c_{2,0}(\gamma_{2,0}+1)\\
&\quad = \frac{\gamma_{1,m}\gamma_{2,m}(\gamma_{1,0}+1)(\gamma_{2,0}+1)}{\eta_d(\eta_d+1)(\gamma_{2,m}-\gamma_{1,m})(\gamma_{2,0}-\gamma_{1,0})}\\
&\qquad \times [(\eta_d-\gamma_{1,0})(\gamma_{2,m}-\eta_d) - (\eta_d-\gamma_{1,m})(\gamma_{2,0}-\eta_d)] \geq 0;\\
&d_{1,0}\gamma_{1,0}c_{2,m}(\gamma_{2,m}+1) - d_{2,0}\gamma_{2,0}c_{1,m}(\gamma_{1,m}+1)\\
&\quad = \frac{\gamma_{1,0}\gamma_{2,0}(\gamma_{1,m}+1)(\gamma_{2,m}+1)}{\eta_d(\eta_d+1)(\gamma_{2,m}-\gamma_{1,m})(\gamma_{2,0}-\gamma_{1,0})}\\
&\qquad \times [(\eta_d-\gamma_{1,m})(\gamma_{2,0}-\eta_d) - (\eta_d-\gamma_{1,0})(\gamma_{2,m}-\eta_d)] \leq 0.
\end{aligned}
$$

These inequalities, along with the fact that

$$
\begin{aligned}
d_{2,m}\gamma_{2,m}c_{1,m}(\gamma_{1,m}+1) &= d_{1,m}\gamma_{1,m}c_{2,m}(\gamma_{2,m}+1),\\
d_{2,0}\gamma_{2,0}c_{1,0}(\gamma_{1,0}+1) &= d_{1,0}\gamma_{1,0}c_{2,0}(\gamma_{2,0}+1),
\end{aligned}
$$

yield

$$
\frac{\frac{\rho+m}{r+m}d_{2,m}\gamma_{2,m} - \frac{\kappa\rho}{r}d_{2,0}\gamma_{2,0}}{(1-\alpha)c_{2,0}(\gamma_{2,0}+1)+\alpha c_{2,m}(\gamma_{2,m}+1)} \geq \frac{\frac{\rho+m}{r+m}d_{1,m}\gamma_{1,m} - \frac{\kappa\rho}{r}d_{1,0}\gamma_{1,0}}{(1-\alpha)c_{1,0}(\gamma_{1,0}+1)+\alpha c_{1,m}(\gamma_{1,m}+1)},
$$

from which the conclusion follows as $a/b > c/d$ if and only if $\frac{a+b}{c+d} > \frac{c}{d}$. □

LEMMA B.3. *For any* $V \geq V_B \geq \epsilon P$, *we have* $H \leq 0$, *where*

$$
\begin{aligned}
H :=\ & \frac{(\rho+m)P}{(r+m)V_B}\left\{d_{1,m}\gamma_{1,m}\left(\frac{V_B}{V}\right)^{\gamma_{1,m}} + d_{2,m}\gamma_{2,m}\left(\frac{V_B}{V}\right)^{\gamma_{2,m}}\right\}\\
& - \frac{P\kappa\rho}{rV_B}\left\{d_{1,0}\gamma_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} + d_{2,0}\gamma_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}}\right\}\\
& - (1-\alpha)\left\{c_{1,0}(\gamma_{1,0}+1)\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} + c_{2,0}(\gamma_{2,0}+1)\left(\frac{V_B}{V}\right)^{\gamma_{2,0}}\right\}\\
& - \alpha\left\{c_{1,m}(\gamma_{1,m}+1)\left(\frac{V_B}{V}\right)^{\gamma_{1,m}} + c_{2,m}(\gamma_{2,m}+1)\left(\frac{V_B}{V}\right)^{\gamma_{2,m}}\right\}.
\end{aligned}
$$

Proof. Note that

$$
H \leq C_2\left(\frac{V_B}{V}\right)^{\gamma_{1,m}} + D_2\left(\frac{V_B}{V}\right)^{\gamma_{2,m}} - A_2\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} - B_2(\gamma_{2,0}+1)\left(\frac{V_B}{V}\right)^{\gamma_{2,0}},
$$

where

$$
\begin{aligned}
A_2 &= \frac{P\kappa\rho}{rV_B}d_{1,0}\gamma_{1,0} + \alpha c_{1,m}(\gamma_{1,m}+1) + (1-\alpha)c_{1,0}(\gamma_{1,0}+1),\\
B_2 &= \frac{P\kappa\rho}{rV_B}d_{2,0}\gamma_{2,0} + \alpha c_{2,m}(\gamma_{2,m}+1) + (1-\alpha)c_{2,0}(\gamma_{2,0}+1),\\
C_2 &= \frac{(\rho+m)P}{(r+m)V_B}d_{1,m}\gamma_{1,m}, \qquad D_2 = \frac{(\rho+m)P}{(r+m)V_B}d_{2,m}\gamma_{2,m}.
\end{aligned}
$$

Since $0 < \gamma_{1,0} \leq \gamma_{1,m} < \gamma_{2,0} < \gamma_{2,m}$, by Lemma B.1, we only need to show $A_2 + B_2 \geq C_2 + D_2$ and $A_2 \geq C_2$. To do this, note that, since $c_{1,m} + c_{2,m} = 1$ and $c_{1,0} + c_{2,0} = 1$, we have

$$\epsilon P = \frac{C_2 + D_2 - \frac{\kappa\rho P}{r V_B}(d_{1,0}\gamma_{1,0} + d_{2,0}\gamma_{2,0})}{A_2 + B_2 - \frac{\kappa\rho P}{r V_B}(d_{1,0}\gamma_{1,0} + d_{2,0}\gamma_{2,0})} V_B.$$

The fact $V_B \geq \epsilon P$ implies that $A_2 + B_2 \geq C_2 + D_2$. By (B.1),

$$\frac{V_B}{P} \geq \epsilon \geq \frac{\frac{\rho + m}{r + m} d_{1,m}\gamma_{1,m} - \kappa \frac{\rho}{r} d_{1,0}\gamma_{1,0}}{\alpha c_{1,m}(\gamma_{1,m} + 1) + (1 - \alpha)c_{1,0}(\gamma_{1,0} + 1)}.$$

Therefore, $A_2 \geq C_2$, and the conclusion follows. □

Now we are in a position to prove that the optimal $V_B^* = \epsilon P$. The proof is based on four facts:

Fact (i): The optimal V_B must satisfy $V_B \geq \epsilon P$. To show this, note that for all $V' > V_B$, $0 \leq S(V'; V_B)$, which is equivalent to saying that V_B must satisfy the constraints that for all $0 < x(=V_B/V') < 1$,

$$\frac{V_B}{x} + \frac{P\kappa\rho}{r}\left\{1 - (d_{1,0}x^{\gamma_{1,0}} + d_{2,0}x^{\gamma_{2,0}})\right\} - \frac{(\rho + m)P}{r + m}\left\{1 - (d_{1,m}x^{\gamma_{1,m}} + d_{2,m}x^{\gamma_{2,m}})\right\}$$
$$-(1 - \alpha)V_B\left\{c_{1,0}x^{\gamma_{1,0}} + c_{2,0}x^{\gamma_{2,0}}\right\} - \alpha V_B\left\{c_{1,m}x^{\gamma_{1,m}} + c_{2,m}x^{\gamma_{2,m}}\right\} \geq 0.$$

Rearranging the terms, we have, for all $0 < x < 1$,

$$V_B \geq \frac{\frac{(\rho + m)P}{r + m}\left\{1 - (d_{1,m}x^{\gamma_{1,m}} + d_{2,m}x^{\gamma_{2,m}})\right\} - \frac{P\kappa\rho}{r}\left\{1 - (d_{1,0}x^{\gamma_{1,0}} + d_{2,0}x^{\gamma_{2,0}})\right\}}{\frac{1}{x} - (1 - \alpha)\left\{c_{1,0}x^{\gamma_{1,0}} + c_{2,0}x^{\gamma_{2,0}}\right\} - \alpha\left\{c_{1,m}x^{\gamma_{1,m}} + c_{2,m}x^{\gamma_{2,m}}\right\}}.$$

In particular,

$$V_B \geq \lim_{x\to 1} \frac{\frac{(\rho + m)P}{r + m}\left\{1 - (d_{1,m}x^{\gamma_{1,m}} + d_{2,m}x^{\gamma_{2,m}})\right\} - \frac{P\kappa\rho}{r}\left\{1 - (d_{1,0}x^{\gamma_{1,0}} + d_{2,0}x^{\gamma_{2,0}})\right\}}{\frac{1}{x} - (1 - \alpha)\left\{c_{1,0}x^{\gamma_{1,0}} + c_{2,0}x^{\gamma_{2,0}}\right\} - \alpha\left\{c_{1,m}x^{\gamma_{1,m}} + c_{2,m}x^{\gamma_{2,m}}\right\}}$$
$$= P \cdot \frac{\frac{\rho + m}{r + m}(d_{1,m}\gamma_{1,m} + d_{2,m}\gamma_{2,m}) - \frac{\kappa\rho}{r}(d_{1,0}\gamma_{1,0} + d_{2,0}\gamma_{2,0})}{(1 - \alpha)(c_{1,0}\gamma_{1,0} + c_{2,0}\gamma_{2,0}) + \alpha(c_{1,m}\gamma_{1,m} + c_{2,m}\gamma_{2,m}) + 1} = \epsilon P,$$

thanks to L'Hospital's rule.

Fact (ii): The solution of $\frac{\partial S(V; V_B)}{\partial V}|_{V=V_B} = 0$ is given by $V_B = \epsilon P$. Indeed, we have

$$\frac{\partial}{\partial V} S(V; V_B) = 1 + \frac{P\kappa\rho}{r}\frac{1}{V}\left\{d_{1,0}\gamma_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} + d_{2,0}\gamma_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}}\right\}$$
$$- \frac{(\rho + m)P}{r + m}\frac{1}{V}\left\{d_{1,m}\gamma_{1,m}\left(\frac{V_B}{V}\right)^{\gamma_{1,m}} + d_{2,m}\gamma_{2,m}\left(\frac{V_B}{V}\right)^{\gamma_{2,m}}\right\}$$
$$+ \alpha\frac{V_B}{V}\left\{c_{1,m}\gamma_{1,m}\left(\frac{V_B}{V}\right)^{\gamma_{1,m}} + c_{2,m}\gamma_{2,m}\left(\frac{V_B}{V}\right)^{\gamma_{2,m}}\right\}$$
$$+ (1 - \alpha)\frac{V_B}{V}\left\{c_{1,0}\gamma_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} + c_{2,0}\gamma_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}}\right\}.$$

Thus,

$$\frac{\partial S(V;V_B)}{\partial V}|_{V=V_B} = 1 + \frac{1}{V_B}\frac{P\kappa\rho}{r}\{\gamma_{1,0}d_{1,0}+\gamma_{2,0}d_{2,0}\} + (1-\alpha)\{c_{1,0}\gamma_{1,0}+c_{2,0}\gamma_{2,0}\}$$
$$-\frac{1}{V_B}\frac{(\rho+m)P}{r+m}\{d_{1,m}\gamma_{1,m}+d_{2,m}\gamma_{2,m}\} + \alpha\{c_{1,m}\gamma_{1,m}+c_{2,m}\gamma_{2,m}\},$$

which shows (ii).

Fact (iii): For all $V > V_B \geq \epsilon P$, we have $\frac{\partial S(V;V_B)}{\partial V} > 0$. To show this, note that

$$\begin{aligned}\frac{\partial}{\partial V}S(V;V_B) &= -\frac{V_B}{V}H + 1 - \alpha\frac{V_B}{V}\left\{c_{1,m}\left(\frac{V_B}{V}\right)^{\gamma_{1,m}} + c_{2,m}\left(\frac{V_B}{V}\right)^{\gamma_{2,m}}\right\}\\ &\quad -(1-\alpha)\frac{V_B}{V}\left\{c_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} + c_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}}\right\}\\ &\geq 1-\alpha\frac{V_B}{V}\{c_{1,m}+c_{2,m}\} - (1-\alpha)\frac{V_B}{V}\{c_{1,0}+c_{2,0}\}\\ &= 1-\frac{V_B}{V} > 0,\end{aligned}$$

via Lemma B.3 and the facts that $c_{1,m}+c_{2,m}=1$ and $c_{1,0}+c_{2,0}=1$.

Fact (iv): We have $S(V;y_1) \geq S(V;y_2)$, if $\epsilon P \leq y_1 \leq y_2 \leq V$. Indeed, for any fixed V, we have $\frac{\partial}{\partial V_B}S(V;V_B) = H \leq 0$ for all $0 \leq V_B/V \leq 1$.

With the above four facts, we can show that ϵP is the optimal solution if $\epsilon P < V$ and optimal $\tau = 0$ if $\epsilon P \geq V$. Indeed, when $\epsilon P \geq V$, by fact (i), the optimal $V_B^* \geq \epsilon P \geq V$, which implies the process should be stopped at time 0, i.e., $\tau = 0$. When $\epsilon P < V$, first, ϵP satisfies the constraints that $S(V';\epsilon P) \geq 0$ for all $V' \geq \epsilon P$, because $S(\epsilon P, \epsilon P) = 0$ and S is increasing in V by (iii); second, any $V_B \in (\epsilon P, V]$ cannot be better, as by (iv) $S(V;\epsilon P) \geq S(V;V_B)$; and any V_B less than ϵP is ruled out by (i).

For the second-stage optimization problem, the bondholders face two options: either choose P satisfying $\epsilon P < V$ or choose P satisfying $\epsilon P \geq V$.

Case 1. Bondholders choose P such that $\epsilon P \geq V$. By the first-stage optimization problem, the equity holder will choose $\tau = 0$ and the firm value v would be αV.

Case 2. Bondholders choose P such that $\epsilon P < V$. The equity holder will choose $V_B^* = \epsilon P$, the firm value would be

$$\begin{aligned}v(V;\epsilon P) &= V + \frac{P\kappa\rho}{r}\left\{1 - d_{1,0}\left(\frac{\epsilon P}{V}\right)^{\gamma_{1,0}} - d_{2,0}\left(\frac{\epsilon P}{V}\right)^{\gamma_{2,0}}\right\}\\ &\quad -(1-\alpha)V\left\{c_{1,0}\left(\frac{\epsilon P}{V}\right)^{\gamma_{1,0}+1} + c_{2,0}\left(\frac{\epsilon P}{V}\right)^{\gamma_{2,0}+1}\right\}\\ &= V\left\{1 + \frac{\kappa\rho}{r}\left(\frac{P}{V}\right) - B_3\left(\frac{P}{V}\right)^{\gamma_{1,0}+1} - C_3\left(\frac{P}{V}\right)^{\gamma_{2,0}+1}\right\},\end{aligned}$$

$$B_3 := (1-\alpha)c_{1,0}\epsilon^{\gamma_{1,0}+1} + \frac{\kappa\rho}{r}d_{1,0}\epsilon^{\gamma_{1,0}}, \quad C_3 := (1-\alpha)c_{2,0}\epsilon^{\gamma_{2,0}+1} + \frac{\kappa\rho}{r}d_{2,0}\epsilon^{\gamma_{2,0}}.$$

Since $B_3 > 0$, $C_3 > 0$, the function $v(V; \epsilon P)$ is concave in P. Meanwhile, when $\epsilon P < V$, the firm value becomes

$$\begin{aligned} v(V; \epsilon P) = &\ V + \frac{P\kappa\rho}{r}\left\{1 - d_{1,0}\left(\frac{\epsilon P}{V}\right)^{\gamma_{1,0}} - d_{2,0}\left(\frac{\epsilon P}{V}\right)^{\gamma_{2,0}}\right\} \\ &- (1-\alpha)V\left\{c_{1,0}\left(\frac{\epsilon P}{V}\right)^{\gamma_{1,0}+1} + c_{2,0}\left(\frac{\epsilon P}{V}\right)^{\gamma_{2,0}+1}\right\} \\ \geq &\ V - (1-\alpha)V\left\{c_{1,0} + c_{2,0}\right\} = \alpha V. \end{aligned}$$

Therefore, Case 2 is better than Case 1 in terms of maximizing the firm value v. In summary, the bondholder will choose P such that $\epsilon P < V$ and maximizes concave function $v(V; \epsilon P)$. □

REMARK B.1. In the above proof, the results (ii) and (iii) actually imply a local convexity at V_B. More precisely, for all $V \geq V_B \geq \epsilon P$, we have the local convexity

$$\left.\frac{\partial^2 S(V; V_B)}{\partial V^2}\right|_{V=V_B} \geq 0.$$

We can show this by contradiction. Suppose this is not the case. By (ii), we have

$$0 > \left.\frac{\partial^2 S(V; V_B)}{\partial V^2}\right|_{V=V_B} = \lim_{V'\downarrow V_B} \frac{\frac{\partial S(V'; V_B)}{\partial V} - \left.\frac{\partial S(V; V_B)}{\partial V}\right|_{V=V_B}}{V' - V_B} = \lim_{V'\downarrow V_B} \frac{\frac{\partial S(V'; V_B)}{\partial V}}{V' - V_B}.$$

Thus, there must be $\tilde{V} > V_B \geq \epsilon P$ such that $\frac{\partial S(\tilde{V}; V_B)}{\partial \tilde{V}} < 0$, which contradicts result (ii). The local convexity has been conjectured in Leland and Toft (1996, footnote 9) under the Brownian model, and in Hilberink and Rogers (2002) under a one-sided jump model; both papers verified it numerically. Here we are able to give a proof for the local convexity for the two-sided jump model mainly because we prove the local convexity indirectly by using Laplace transforms. It is difficult to verify the convexity directly without using Laplace transforms even in the case of Brownian motion, due to the difficulty in analyzing the monotonicity of various normal distribution functions.

APPENDIX C: PROOF OF THEOREM 3.2

First, by the conditional memoryless property of the overshoot distribution,

$$E[V_\tau \mid \tau \leq T] = V \cdot E\left[\exp\left(\log\left(\frac{V_\tau}{V}\right)\right)\middle|\, \tau \leq T\right] = V \cdot \frac{V_B}{V}\frac{\eta_d}{\eta_d + 1} + o(1),$$

as the probability of the default caused by the diffusion is $o(1)$. Second,

$$P[\tau \leq T] = \lambda T p_d \left(\frac{V_B}{V}\right)^{\eta_d} + o(T).$$

Therefore,

$$\begin{aligned} E[V_\tau \cdot \mathbf{1}_{\{\tau \leq T\}}]e^{-rT} &= E[V_\tau \mid \tau \leq T]P(\tau \leq T)e^{-rT} \\ &= \left\{V \cdot \frac{V_B}{V}\frac{\eta_d}{\eta_d + 1} + o(1)\right\}\left\{\lambda p_d T\left(\frac{V_B}{V}\right)^{\eta_d} + o(T)\right\}\{1 - rT + o(T)\}, \end{aligned}$$

from which we have

$$E[V_\tau \cdot \mathbf{1}_{\{\tau \le T\}}]e^{-rT} = V_B \frac{\eta_d}{\eta_d + 1}\lambda p_d \left(\frac{V_B}{V}\right)^{\eta_d} T + o(T). \tag{C.1}$$

$$E[V_\tau \cdot \mathbf{1}_{\{\tau \le T\}}] = V_B \frac{\eta_d}{\eta_d + 1}\lambda p_d \left(\frac{V_B}{V}\right)^{\eta_d} T + o(T).$$

Furthermore, since

$$E[V_\tau \cdot \mathbf{1}_{\{\tau \le T\}}]e^{-rT} \le E[V_\tau e^{-r\tau} \cdot \mathbf{1}_{\{\tau \le T\}}] \le E[V_\tau \cdot \mathbf{1}_{\{\tau \le T\}}],$$

we also have

$$E[V_\tau e^{-r\tau} \cdot \mathbf{1}_{\{\tau \le T\}}] = V_B \frac{\eta_d}{\eta_d + 1}\lambda p_d \left(\frac{V_B}{V}\right)^{\eta_d} T + o(T). \tag{C.2}$$

Finally note that

$$\begin{aligned} 1 &= \lim_{T\to 0} \frac{1}{T} E\left[\int_0^T e^{-rs} ds\right] \ge \lim\sup_{T\to 0} \frac{1}{T} E\left[\int_0^{\tau\wedge T} e^{-rs} ds\right] \\ &\ge \lim\inf_{T\to 0} E\left[\frac{1}{T}\int_0^T e^{-rs} ds \cdot \mathbf{1}_{\{\tau \ge T\}}\right] = 1, \end{aligned}$$

by the dominated convergence theorem. Thus,

$$1 = \lim_{T\to 0} \frac{1}{T} E\left[\int_0^{\tau\wedge T} e^{-rs} ds\right]. \tag{C.3}$$

In summary, we have, via (C.1)–(C.3),

$$\begin{aligned} &B(V; V_B, T) \\ &= e^{-rT} P[\tau > T] + \frac{\alpha}{P}\frac{m+r}{m+\rho}\left(1 - \frac{\rho}{r}\right) E[V_\tau \cdot \mathbf{1}_{\{\tau \le T\}}]e^{-rT} \\ &\quad + \frac{\alpha}{P}\frac{m+r}{m+\rho}\frac{\rho}{r} E\left[V_\tau e^{-r\tau} \cdot \mathbf{1}_{\{\tau \le T\}}\right] + \rho E\left[\int_0^{\tau\wedge T} e^{-rs} ds\right] \\ &= (1 - rT)\left(1 - \lambda p_d T \left(\frac{V_B}{V}\right)^{\eta_d}\right) + \frac{\alpha V_B}{P}\frac{m+r}{m+\rho}\frac{\eta_d}{\eta_d+1} \cdot \lambda p_d \left(\frac{V_B}{V}\right)^{\eta_d} T + \rho T + o(T) \\ &= 1 - \left[r + \lambda p_d \left(\frac{V_B}{V}\right)^{\eta_d}\right] T + \frac{\alpha V_B}{P}\frac{m+r}{m+\rho}\frac{\eta_d}{\eta_d+1} \cdot \lambda p_d \left(\frac{V_B}{V}\right)^{\eta_d} T + \rho T + o(T). \end{aligned}$$

Thus, L'Hospital's rule leads to

$$\nu(0) = \lim_{T\to 0} \frac{1 - B(V, 0; V_B, T)}{T} + \rho = r + \lambda p_d \left(\frac{V_B}{V}\right)^{\eta_d} \left[1 - \frac{\alpha V_B}{P}\frac{m+r}{m+\rho}\frac{\eta_d}{\eta_d+1}\right],$$

from which the proof is terminated. □

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