Lévy processes in Asset Pricing

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1 Empirical Motivation

The main empirical motivation of using Lévy processes in finance comes from fitting asset return distributions. Consider the daily (either continuous or simple) returns of S&P 500 index (SPX) from Jan 2, 1980 to Dec 31, 2005. We plot the histogram of normalized (mean zero and variance one) daily simple returns in Figure 1, along with the standard normal density function. The max and min (which all occurred in 1987) of the normalized daily returns are about 7.9967 and -21.1550. Note that for a standard normal random variable Z, $P(Z < -21.1550) \approx 1.4 * 10^{-107}$; as a comparison the whole universe is believed to have existed for 15 billion years or $5 * 10^{17}$ seconds.

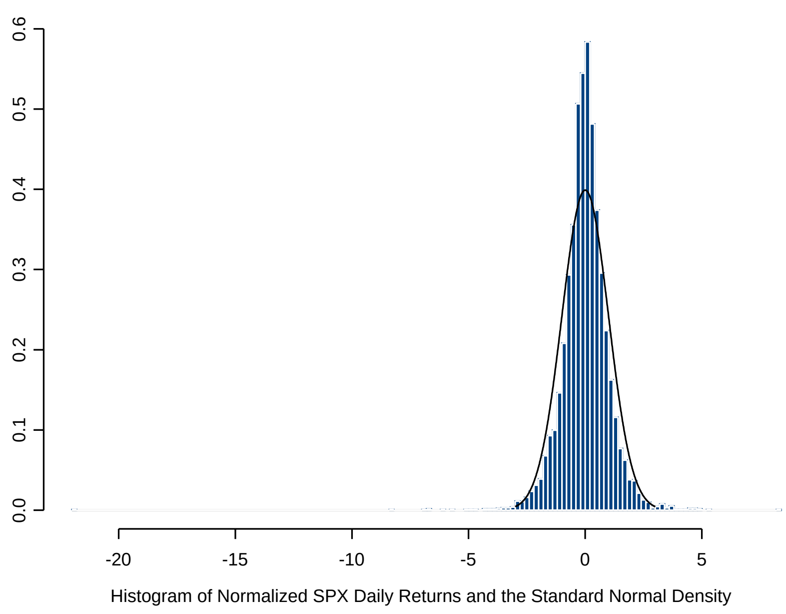

Figure 1: Comparison of the histogram of the normalized daily returns of S&P 500 index (from Jan 2, 1980 to Dec 31, 2005) and the density of N(0,1). The feature of a high peak and two heavy tails (i.e. the leptokurtic feature) is quite evident.

Clearly the histogram of SPX displays a high peak and two asymmetric heavy tails. This is not only true for SPX but also for almost all financial asset prices, e.g. world wide stock indices, individual stocks, foreign exchange rates, interest rates. In fact it is so evident that a name "leptokurtic distribution" is given, which means the kurtosis of the distribution is large. More precisely, the kurtosis and skewness are defined as $K = E\left(\frac{(X-\mu)^4}{\sigma^4}\right)$, $S = E\left(\frac{(X-\mu)^3}{\sigma^3}\right)$. For the standard normal density $K = 3$, and if $K > 3$ then the distribution is called leptokurtic and the distribution will have a higher peak and two heavier tails than those of the normal distribution. For the SPX data, the sample kurtosis is about 42.23. The skewness is about -1.73; the negative skewness indicates that the left tail is heavier than the right tail.

The classical geometric Brownian motion model, which models the stock price as $S(t) = S(0)e^{\mu t + \sigma W_t}$ with W_t being the standard Brownian motion, is inconsistent with this feature, because in this model the return, $\ln(S(t)/S(0))$, has a normal distribution. Lévy processes, among other processes, have been proposed to incorporate the leptokurtic feature.

2 Overview

A stochastic process X_t is a Lévy process if it has independent and stationary increments and has a stochastically continuous sample path, i.e. for any $\varepsilon > 0$, $\lim_{h \downarrow 0} P(|X_{t+h} - X_t| > \varepsilon) \to 0$. Therefore, Lévy processes provide a natural generalization of the sum of independent and identically distributed (i.i.d.) random variables. The simplest possible Lévy processes are the standard Brownian motion $W(t)$, Poisson processes $N(t)$, and compound Poisson processes $\sum_{i=1}^{N(t)} Y_i$, where Y_i are i.i.d. random variables.

Of course, one can combine the above processes to form other Lévy processes. For example, an important class of Lévy processes is the jump-diffusion process given by $\mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$, where μ and σ are constants. Interestingly the famous Lévy-Itô decomposition says that the converse is also true. More precisely, any Lévy process can be written as a drift term μt, a Brownian motion with variance and covariance matrix A, and a possibly infinite sum of independent compound Poisson processes which are related to a intensity measure $v(dx)$. This implies that a Lévy process can be approximated by jump-diffusion processes. This has important numerical applications in finance, as jump-diffusion models are widely used in finance.

The triplet (μ, A, v) is also linked to the Lévy-Khinchin representation which states that the characteristic function of a Lévy process X_t can be written in terms of (μ, A, v) as

$$\frac{1}{t}\log E\left[e^{iz'X_t}\right] = -\frac{1}{2}z'Az + i\mu z + \int_{R^d}(e^{iz'x} - 1 - iz'xI_{\{|x|\le 1\}})v(dx).$$

The representation suggests that it is easier to study Lévy processes via Laplace transforms, and then numerically invert Laplace transforms. For numerical algorithms of Laplace inversion, see Abate and Whitt (1992), Craddock, Heath, and Platen (2000), and Petrella (2004). Some of the infinite activity Lévy processes may not have an analytical form for the intensity measure v, although the probability density of X_t can be obtained explicitly; see, e.g., the Generalized hyperbolic model in Eberlein and Prause (2002).

There are two types of Lévy processes, jump-diffusion and infinite activity Lévy processes. In jump-diffusion processes jumps are considered rare events, and in any given finite interval there are only finite many jumps. Examples of jump-diffusion models in finance include Merton's (1976) model in which the jump size Y has a normal distribution, and the double exponential jump-diffusion model in Kou (2002), in which Y has a density $f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y<0\}}$. The double exponential jump-diffusion model leads to analytical solutions for path-dependent options, such as barrier and lookback options, and analytical approximations of finite-horizon American options, thanks to the memoryless property of exponential density; see Kou and Wang (2003, 2004), Kou, Petrella, and Wang (2005), and Huang and Kou (2006). The economic justification of the double exponential jump-diffusion model, including the rational expectations equilibrium and the consideration from behavioral finance, is discussed in Kou (2002).

For infinite activity Lévy processes, in any finite time interval there are infinitely many jumps. Many of these model can be constructed via Brownian subordination, i.e. $W(\tau(t))$ where $\tau(t)$ is a subordinator which is an increasing Lévy process. The subordinator must have no diffusion component (i.e. with only positive jumps and a positive drift); see Cont and Tankov (2004, p. 88). For example, if the subordinator $\tau(t)$ is a gamma process, then $W(\tau(t))$ leads to the variance gamma model; see Madan and Seneta (1990) and Madan, Carr, and Chang (1998). If the subordinator $\tau(t)$ is a inverse Gaussian process, then $W(\tau(t))$ leads to the normal inverse Gaussian model; see Barndorff-Nielsen (1998) and Rydberg (1997).

There is a large literature on Lévy processes in finance, including several excellent books, e.g. the books by Cont and Tankov (2004), Kijima (2002). The book by Cont and Tankov (2004) also discusses the issue of hedging in incomplete markets, as Lévy processes lead to incomplete markets and the complete replication of an option payoff is impossible. One can also use rational expectations in Lucas (1978) and Stokey and Lucas (1989) to choose a risk-neutral measure to price derivative as in Kou (2002). Here are some additional topics related to Lévy processes:

(1) In terms of computational issues, see Cont and Voltchkova (2005) and d'Haullin et al. (2003) on numerical methods via solving partial integro-differential equations, and Feng and Linetsky (2005) and Feng, Linetsky and Marcozzi (2004) on how to price path-dependent options numerically via variational methods and extrapolation.

(2) In terms of applications, see the references in Glasserman and Kou (2004) for applications of Lévy processes in fixed income derivatives and term structure models, and the references in Chen and Kou (2005) for applications in credit risk and credit derivatives.

(3) Statistical inference and econometric analysis for Lévy processes are discussed in the book by Singleton (2006).

3 Some Difficulties

3.1 The Volatility Clustering Effect

In addition to the leptokurtic feature, returns distributions also have an interesting dependent structure, called the volatility clustering effect; see Engle (1995). More precisely, the volatility of returns (which are related to the squared returns) are correlated, but asset returns themselves have almost no autocorrelation. In other words, a large movement in asset prices, either upside or downside, tends to generate large movements in the future asset prices, although the direction of the movements is unpredictable.

In particular any model for stock returns with independent increments (such as Lévy processes) cannot incorporate the volatility clustering effect. However, one can combine Lévy processes with other processes (e.g. Duffie, Pan, Singleton, 2000, Barndorff-Nielsen and Shephard 2001) or consider time-changed Brownian motion and Lévy processes to incorporate the volatility clustering effect. More precisely, if $\tau(t)$ contains a diffusion component (i.e. not a subordinator), then $W(\tau(t))$ and $X(\tau(t))$ may have dependent increments and no longer be Lévy processes. See Carr, Madan, Geman, and Yor (2002, 2003) and Carr and Wu (2004).

3.2 Difficulties in Distinguishing Tail Behavior

Although the main empirical motivation for using Lévy processes in finance comes from the fact that asset return distributions tend to have tails heavier than those of normal distribution, it is not clear how heavy the tail distributions are, as some people favor power-type distributions others exponential-type distributions, although, as pointed out by Kou (2002, p. 1090), the power-type right tails cannot be used in models with continuous compounding as they lead to infinite expectation for the asset price. We will stress that, quite surprisingly, it is very difficult to distinguish power-type tails from exponential-type tails from empirical data unless one has extremely large sample size perhaps in the order of tens of thousands or even hundreds of thousands; see Heyde and Kou (2004). Therefore, it is very difficult to choose a good model based on the limited empirical data alone.

A good intuition may be obtained by simply looking at the quantiles for both standardized Laplace (with a symmetric density $f(x) = \frac{1}{2}e^{-x}I_{[x>0]} + \frac{1}{2}e^{x}I_{[x<0]}$) and standardized t distributions with mean zero and variance one. The right quantiles for the Laplace and normalized t densities with degrees of freedom from 3 to 7 are given in the following table.

prob.	Laplace	t7	t6	t5	t4	t3
1%	**2.77**	2.53	2.57	2.61	2.65	2.62
0.1%	**4.39**	4.04	4.25	**4.57**	5.07	5.90
0.01%	**6.02**	5.97	**6.55**	7.50	9.22	12.82
0.001%	**7.65**	**8.54**	9.82	12.04	16.50	27.67

This table shows that Laplace distribution may have higher tail probabilities than those of t distributions, even if asymptotically the Laplace distribution should have lighter tails than those of t distributions. For example, regardless of the sample size, the Laplace distribution may appear to be heavier tailed than a t-distribution with d.f. 6 or 7, up to the 99.99% percentile. To distinguish the distributions it is necessary to use quantiles with very low p values and correspondingly large samples. If the true quantiles have to be estimated from data, then the problem is even worse, as the sample standard deviations need to be considered, resulting in sample sizes typically in the tens of thousands or even hundreds of thousands necessary to distinguish power-type tails from exponential-type tails.

Of course, one can use intraday data to have more sample sizes. However, intraday data may involve market microstructures, resulting in totally different empirical methods. For example, there is no single price due to the bid-ask spread, and the order size information is also relevant; see O'Hara (1995).

3.3 Risk due to Model Selection

The difficulties in distinguishing tail distributions indicate that there is a risk associated with choosing a proper model, as whether one prefers to use power-type distributions or exponential-type distributions is a subjective issue, which cannot be easily justified empirically. This also has implications in risk management, which is sensitive to the choice of models.

For example, a controversy in axiomatic approaches to risk measures is whether one should use Value-at-Risk (or VaR), which is a measure based on quantiles, or the tail conditional expectation. The tail conditional expectation satisfies a set of axioms based on subadditivity (Artzner et al. 1999), while the VaR also satisfies a different set of axioms based on comonotonic subadditivity (Heyde, Kou, and Peng 2006), which is consistent with the prospect theory in behavioral finance. The intuition behind subadditivity is that merger should reduce risk, although the intuition may not be valid in the presence of the limited liability law. Furthermore, both VaR and the tail conditional expectation take into consideration of the loss beyond the threshold, because VaR at a higher quantile (e.g. 97.5%) can also be represented as tail conditional median (e.g. at 95%).

Whether one should measure risk by using VaR or the tail conditional expectation depends on whether the measure is used for internal or external risk management. The main advantage of VaR is that it is more robust against model assumptions and misspecifications, thus making the VaR more suitable to generate more consistent results for external risk regulations and law enforcement. Indeed, VaR is widely used in governmental regulation, e.g. the recent Basel (II) banking regulations. On the other hand both the tail conditional expectation and VaR may be suitable for internal risk management.

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