Option Pricing Under a Mixed-Exponential Jump Diffusion Model

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This paper aims to extend the analytical tractability of the Black–Scholes model to alternative models with arbitrary jump size distributions. More precisely, we propose a jump diffusion model for asset prices whose jump sizes have a mixed-exponential distribution, which is a weighted average of exponential distributions but with possibly negative weights. The new model extends existing models, such as hyperexponential and double-exponential jump diffusion models, because the mixed-exponential distribution can approximate *any* distribution as closely as possible, including the normal distribution and various heavy-tailed distributions. The mixed-exponential jump diffusion model can lead to analytical solutions for Laplace transforms of prices and sensitivity parameters for path-dependent options such as lookback and barrier options. The Laplace transforms can be inverted via the Euler inversion algorithm. Numerical experiments indicate that the formulae are easy to implement and accurate. The analytical solutions are made possible mainly because we solve a high-order integro-differential equation explicitly. A calibration example for SPY options shows that the model can provide a reasonable fit even for options with very short maturity, such as one day.

1. Introduction

1.1. Background
It is well known that empirically asset return distributions have heavier left and right tails than the normal distributions, as suggested in the classical Black–Scholes model. Jump diffusion models are among the most popular alternative models proposed to address this issue, and they are especially useful to price options with short maturities. For the background of alternative models, see, e.g., Hull (2005). This paper aims to further extend the analytical tractability of the Black–Scholes model to jump diffusion models with arbitrary jump size distributions. Indeed, we propose a jump diffusion model for asset prices whose jump sizes have a mixed-exponential distribution, which is a weighted average of exponential distributions but with possibly negative weights. The mixed-exponential distribution can approximate *any* distribution arbitrarily closely, including any discrete distribution, the normal distribution, and various heavy-tailed distributions such as Gamma, Weibull, and Pareto distributions. We show that the mixed-exponential jump diffusion model (MEM) can lead

to analytical solutions for Laplace transforms of prices and sensitivity measures (e.g., deltas) for path-dependent options, such as continuously monitored lookback and barrier options. These analytical solutions are made possible primarily because we solve a high-order integro-differential equation explicitly related to the first passage time problem.

The motivation of this paper is twofold. First, a key question for jump diffusion models is what jump size distributions will be used. The question is closely related to how heavy the tails of asset return distributions are. Although we know that asset return distributions have heavier tails than the normal distribution, it is not clear at all how heavy the tails may be. For example, empirically it may be difficult to identify how heavy the tails are based on 5,000 daily observations (approximately 20 years worth) (see Heyde and Kou 2004). Accordingly, we want the jump size distribution to be general enough to approximate *any* distribution, including various exponential- and power-tail distributions.

Second, analytical tractability is one of the challenges for alternative models to the Black–Scholes model. More precisely, although many alternative

models can lead to analytical solutions for European call and put options, unlike the Black–Scholes model, it is difficult to do so for path-dependent options such as lookback and barrier options. Even numerical methods for these derivatives are not easy. For example, the convergence rates of binomial trees and Monte Carlo (MC) simulation for path-dependent options are typically much slower than those for call and put options; for a survey, see, e.g., Boyle et al. (1997).

Therefore, it is desirable to have a class of jump diffusion models that allow jump size distributions that can approximate any distribution while remaining tractable enough to allow analytical solutions for path-dependent options. We shall show that this is possible if we consider an MEM.

1.2. Comparison with the Existing Literature

Two well-known jump diffusion models are Merton's model (1976) and the double-exponential jump diffusion model (see Kou 2002), in which the jump size distributions are normal and double exponential, respectively. One advantage of the double-exponential jump diffusion model is that it can lead to analytical tractability for path-dependent options, including lookback, barrier, Asian, and occupation-time-related options; see, e.g., Kou and Wang (2003, 2004), Cai and Kou (2011), and Cai et al. (2010). More general models, including the phase-type jump diffusion model (PHM) and the hyperexponential jump diffusion model (HEM), were also proposed; see, e.g., Asmussen et al. (2004), Boyarchenko (2006), Boyarchenko and Boyarchenko (2008), Boyarchenko and Levendorskiĭ (2009), Cai et al. (2009), Carr and Crosby (2010), Crosby et al. (2010), Jeannin and Pistorius (2010), and Lipton (2002). Here is a comparison between our MEM model and the existing HEM and PHM models.

(1) The HEM specifies the jump size distribution as a weighted average of exponential distributions, and the weights can only be nonnegative. Therefore, the HEM is a special case of our MEM model, because our weights can be negative. Compared with the HEM, which can only approximate jump diffusion models with *completely monotone* jump size distributions (see Appendix B for more details), our MEM can approximate jump diffusion models with *any* jump size distribution, because the mixed-exponential distribution is dense with respect to (w.r.t.) the class of all the distributions in the sense of weak convergence (see Botta and Harris 1986). In particular, the MEM may be used to approximate Merton's (1976) model, which cannot be approximated by the HEM because *the normal distribution is not completely monotone*. In §6, an example is provided to demonstrate that this approximation can lead to accurate prices and deltas for lookback and barrier options under the Merton model.

(2) The PHM, in which the jump sizes have a phase-type distribution, can also approximate jump diffusion models with any jump size distribution (see Botta and Harris 1986). One issue worth mentioning is that the class of the MEM and that of the PHM do not contain each other. Moreover, one advantage of the MEM might be that the representation of the mixed-exponential distribution is *unique*, whereas that of the phase-type distribution is *not unique* (see Botta and Harris 1986); i.e., for phase-type distributions different sets of parameters may lead to the same cumulative distribution function (cdf). The uniqueness is desirable for statistical procedures such as parameter estimation.

In terms of the related literature on pricing path-dependent options, Feng and Linetsky (2008) and Feng et al. (2007) showed how to price path-dependent options numerically, via extrapolation and variational methods, for jump diffusion models with general jump size distributions. Davydov and Linetsky (2001, 2003) provided analytical pricing formulae for lookback and barrier options under the CEV model. For option pricing under exponential Lévy models, see Carr et al. (2003), Cont and Tankov (2004), and Kijima (2002). The emphasis of the current paper is on explicit calculations for a particular exponential Lévy model, which are different from these results.

We point out that none of the exponential Lévy models can capture both short- and long-term behaviors of market options. In fact, the jump diffusion models are useful especially for short maturity options. In general, to get an excellent fit across all strikes and all option maturities, spatial inhomogeneity and/or stochastic volatilities may be used; see, e.g., Bates (1996), Bakshi et al. (1997), and Carr et al. (2003). Therefore, the formulae given in this paper are only meant to be a first step to price options analytically under more general models with jumps. However, the analytical formulae presented here may be useful for short-term options, and can also provide a useful benchmark for more complicated models, for which one perhaps has to resort to simulation or other numerical procedures.

This paper is organized as follows. Section 2 gives the basic setting of the MEM and provides motivation and intuition of our results. First passage times of the mixed-exponential jump diffusion process are studied in §3. Section 4 discusses pricing of European options and provides an example of calibration to a set of data of European options. Laplace transforms of prices and deltas for lookback and barrier options are given in §5, where numerical results are also provided. In §6, a numerical example is given to illustrate an approximation to Merton's (1976) model by the MEM, especially in terms of lookback and barrier

options. Section 7 concludes this paper. The proofs are deferred to the appendices or the e-companion.[1]

2. The Mixed-Exponential Jump Diffusion Model

Under the MEM, the dynamics of the asset price S_t under a risk-neutral measure[2] P to be used for option pricing is given by

$$\frac{dS_t}{S_{t-}} = r\,dt + \sigma\,dW_t + d\left(\sum_{i=1}^{N_t}(V_i - 1)\right), \qquad (1)$$

where r is the risk-free interest rate, σ the volatility, $\{N_t: t \geq 0\}$ a Poisson process with rate λ, $\{W_t: t \geq 0\}$ a standard Brownian motion, and $\{Y_i := \log(V_i): i = 1, 2 \ldots\}$ a sequence of independent and identically distributed mixed-exponential random variables with the probability density function (pdf) $f_Y(x)$. In this model all sources of randomness, N_t, W_t, and Y_i, are assumed to be independent.

More precisely, the pdf $f_Y(x)$ is given by

$$f_Y(x) = p_u \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x} I_{\{x \geq 0\}} + q_d \sum_{j=1}^{n} q_j \theta_j e^{\theta_j x} I_{\{x < 0\}}, \qquad (2)$$

where $p_u \geq 0$, $q_d = 1 - p_u \geq 0$,

$$p_i \in (-\infty, \infty) \quad \text{for all } i = 1, \ldots, m, \quad \sum_{i=1}^{m} p_i = 1,$$

$$q_j \in (-\infty, \infty) \quad \text{for all } j = 1, \ldots, n, \quad \sum_{j=1}^{n} q_j = 1,$$

$$\eta_i > 1 \quad \text{for all } i = 1, \ldots, m, \quad \text{and}$$

$$\theta_j > 0 \quad \text{for all } j = 1, \ldots, n.$$

Because p_i and q_j can be negative, the parameters should satisfy some conditions to guarantee that $f_Y(x)$ is always nonnegative and is a probability density function. A necessary condition for $f_Y(x)$ to be a probability density function is $p_1 > 0$, $q_1 > 0$, $\sum_{i=1}^{m} p_i \eta_i \geq 0$, and $\sum_{j=1}^{n} q_j \theta_j \geq 0$. A simple sufficient condition is $\sum_{i=1}^{k} p_i \eta_i \geq 0$ for all $k = 1, \ldots, m$ and $\sum_{j=1}^{l} q_j \theta_j \geq 0$ for all $l = 1, \ldots, n$. For alternative conditions, see Bartholomew (1969). A special case of the mixed-exponential distribution is the hyperexponential distribution, where all the p_i and q_j must be nonnegative.

In addition, the condition $\eta_i > 1$, for all $i = 1, \ldots, m$, is imposed above to ensure that the stock price S_t has a finite expectation. By solving the stochastic differential Equation (1), we obtain that under the MEM the return process $X_t := \log(S_t/S_0)$ is given by

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad X_0 = 0,$$

where

$$\mu = r - \frac{\sigma^2}{2} - \lambda \zeta, \quad \text{and}$$

$$\zeta := E[e^{Y_1}] - 1 = p_u \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - 1} + q_d \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + 1} - 1.$$

Simple algebra yields that the moment generating function of X_t is

$$E[e^{xX_t}] = e^{G(x)t}, \quad \text{for any } t \geq 0 \text{ and } x \in (-\theta_1, \eta_1),$$

where $G(x)$, called the exponent of the Lévy process X_t, is defined as

$$G(x) = \frac{\sigma^2}{2} x^2 + \mu x + \lambda \left(p_u \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + \sum_{j=1}^{n} q_d \frac{q_j \theta_j}{\theta_j + x} - 1 \right). \qquad (3)$$

For more information about exponents of this type, we refer to Hirshman and Widder (1955). Besides, the infinitesimal generator of $\{X_t\}$ is given by

$$(Lu)(x) = \frac{\sigma^2}{2} u''(x) + \mu u'(x)$$
$$+ \lambda \int_{-\infty}^{+\infty} [u(x+y) - u(x)] f_Y(y)\,dy, \qquad (4)$$

where $u(x)$ is any twice continuously differentiable function, and $f_Y(\cdot)$ is given by (2).

The difficulty in distinguishing tail behaviors (see Heyde and Kou 2004) motivates us to consider the MEM, whose jump size distribution is general enough to approximate *any* jump size distribution, no matter which ones we prefer. In fact, the mixed-exponential distribution can approximate any distribution in the sense of weak convergence (see Botta and Harris 1986).

We shall provide several examples of approximating some heavy-tailed distributions numerically with the mixed-exponential distribution, including (a) Gamma (1.2, 0.5), i.e., the Gamma distribution with shape parameter 1.2 and scale parameter 0.5; (b) Gamma (0.8, 0.85); (c) Pareto (2, 25), i.e., the Pareto distribution with shape parameter 2 and scale parameter 25; and (d) Weibull (0.025, 0.5), i.e., the Weibull distribution with scale parameter 0.025 and shape parameter 0.5. Note that (b)–(d) are completely monotone, but (a) is not, and hence cannot be

Figure 1   Approximate Heavy-Tailed Distributions (Gamma, Pareto, and Weibull) with Mixed-Exponential Distributions
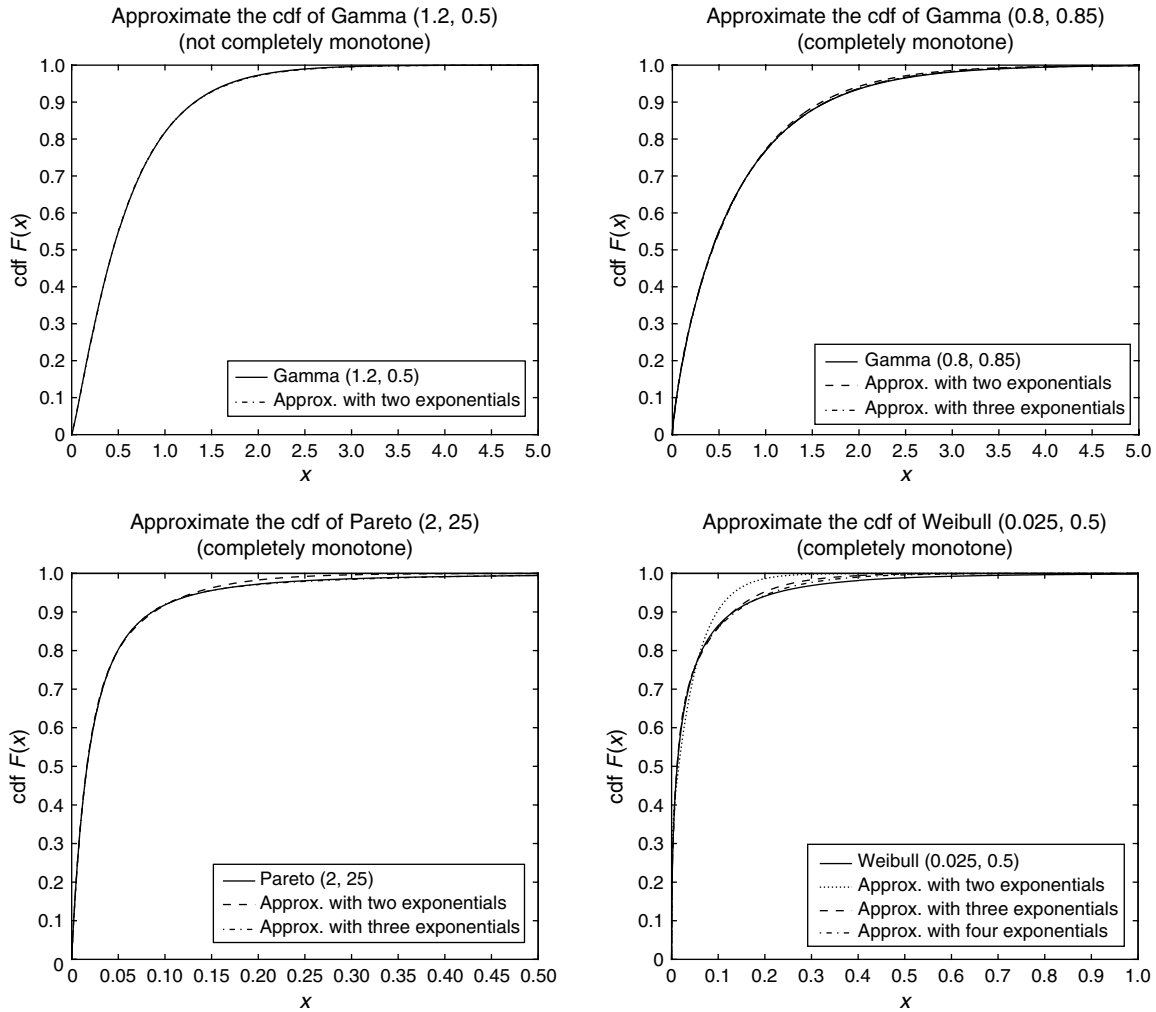
Notes. This figure suggests that it seems possible to use a mixture of two, three, three, and five exponential distributions to fit Gamma (1.2, 0.5), Gamma (0.8, 0.85), Pareto (2, 25), and Weibull (0.025, 0.5), respectively. Note that Gamma (1.2, 0.5) is not completely monotone and hence cannot be approximated by the hyperexponential distribution. For Gamma (1.2, 0.5), a plotted approximation is $1.1424(1 - e^{-1.8401x}) - 0.1424(1 - e^{-7.5316x})$. For Gamma (0.8, 0.85), a plotted approximation is $0.8435(1 - e^{-1.2937x}) + 0.1305(1 - e^{-5.4092x}) + 0.0260(1 - e^{-70.0207x})$. For Pareto (2, 25), a plotted approximation is $0.0841(1 - e^{-5.7140x}) + 0.5165(1 - e^{-27.5604x}) + 0.3994(1 - e^{-86.7169x})$. For Weibull (0.025, 0.5), a plotted approximation is $0.1411(1 - e^{-5.1891x}) + 0.1604(1 - e^{-14.5982x}) + 0.2519(1 - e^{-29.4403x}) + 0.2734(1 - e^{-135.0813x}) + 0.1732(1 - e^{-2000x})$.

approximated by the hyperexponential distributions. Besides, although theoretically phase-type distributions can approximate (a)–(d), the numerical fitting might not be easy because the representation of a phase-type distribution is not unique (see Botta and Harris 1986).

In our examples, we first fix the number m of components of the mixed-exponential distribution, whose cdf is denoted by $\mathrm{MExp}_m(x)$. Then an approximation to the target cdf $H(x)$ by a mixture of m exponential distributions can be found by minimizing $\sum_{i=1}^{N}(\mathrm{MExp}_m(x_i) - H(x_i))^2$, where $x_1, \ldots, x_N$ are grid points on some interval. Figure 1 suggests that it seems possible to use a mixture of two, three, three, and five exponential distributions to fit Gamma (1.2, 0.5), Gamma (0.8, 0.85), Pareto (2, 25), and Weibull (0.025, 0.5), respectively.

3.   First Passage Times

To price lookback and barrier options, it is pivotal to study the first passage times τ_b that the process crosses a flat boundary with a level b, where

$$\tau_b := \inf\{t \geq 0: X_t \geq b\}, \quad b > 0, \tag{5}$$

and the infimum of an empty set is defined as $+\infty$ and $X_{\tau_b} := \limsup_{t \to +\infty} X_t$ on the set $\{\tau_b = +\infty\}$. When a jump diffusion process crosses the boundary, sometimes it hits the boundary exactly, and sometimes it incurs an "overshoot," $X_{\tau_b} - b$, over the boundary.[3]

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[3] If the jump size distribution is one sided, one can solve the overshoot problems by either using renewal equations or fluctuation identities for Lévy processes; see, e.g., Avram et al. (2002) and Rogers (2000). However, for two-sided jumps, because of the

The overshoot presents several problems if one wants to compute the distribution of the first passage time analytically. First, one needs the exact distribution of the overshoot, $X_{\tau_b} - b$, particularly $P(X_{\tau_b} - b = 0)$ and $P(X_{\tau_b} - b > x)$ for $x > 0$. Second, one needs to know the dependence structure between the overshoot, $X_{\tau_b} - b$, and the first passage time τ_b.

These difficulties can be resolved if one can solve the following ordinary integro-differential equation (OIDE) explicitly:

$$
\begin{cases}
(Lu)(x) - \alpha u(x) = \dfrac{\sigma^2}{2} u''(x) + \mu u'(x) - (\lambda + \alpha) u(x) \\
\qquad + \lambda \displaystyle\int_{-\infty}^{+\infty} u(x+y) f_Y(y)\, dy = 0 \quad \text{if } x < x_0, \\
u(x) = g(x) \qquad\qquad\qquad\qquad\qquad \text{if } x \ge x_0,
\end{cases}
\tag{6}
$$

where $\alpha > 0$, L is the infinitesimal generator of $\{X_t\}$ given by (4), and $g(x)$ is a given function. Note that the main challenge is that although this OIDE exists only when $x < x_0$, it does involve the information of the function $u(x)$ for $x \ge x_0$ because the integral inside the generator L depends on the values of u on both regions. To emphasize the above particularity of the OIDE, we call it a *forced OIDE*, meaning that the OIDE has a forcing term defined by $g(x)$. In this section, this OIDE will be solved explicitly, leading to an analytical solution of the joint distribution of the first passage time τ_b and X_t. Intuitively, the solution is available analytically because the exponential function has some very nice properties, such as that the product of exponential functions is still an exponential function, and the derivatives of exponential functions are still exponential functions.

It is worth noting that our argument requires neither the Wiener–Hopf factorization nor a more general theory about Markov processes. We prove the main results by solving the OIDE (6) explicitly and by using a martingale method. More specifically, we will achieve the objective in four steps: (i) show that $G(x) = \alpha$ has only real roots for any sufficiently large $\alpha > 0$; (ii) use the roots to solve the OIDE (6) explicitly by transforming the OIDE into a high-order homogeneous linear ordinary differential equation (ODE) (some indications of possible reduction of an OIDE to a high-order ODE are also given in Mayo 2008, Carr et al. 2004); (iii) derive $E^x[e^{-\alpha \tau_b + \theta X_{\tau_b}}]$ via a martingale method based on the solution of the OIDE, where the superscript x means $X_0 = x$; (iv) obtain the double

ladder-variable problems, generally speaking, the renewal equations are not available, and the fluctuation identities for arbitrary distributions become too complicated for explicit computation; see, e.g., the discussion in Siegmund (1985) and Rogers (2000). See also Boyarchenko and Levendorskiĭ (2002) and Kyprianou and Pistorius (2003) for some representations related to the overshoot problems for general Lévy processes.

Figure 2   Plot of the Function $G(x)$ Under the MEM with $m = n = 3$
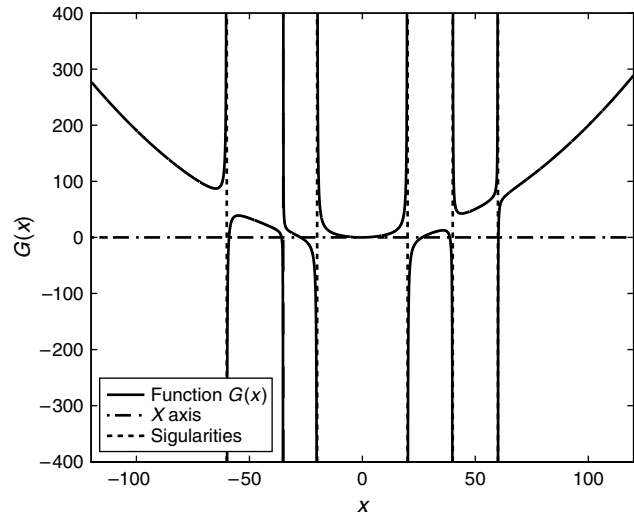
Notes. The parameters are $\mu = 0.05$, $\sigma = 0.2$, $\lambda = 5$, $(\eta_1, \eta_2, \eta_3) = (20, 40, 60)$, $(\theta_1, \theta_2, \theta_3) = (20, 35, 60)$, $p_u = q_d = 0.5$, $(p_1, p_2, p_3) = (1.2, -0.3, 0.1)$, and $(q_1, q_2, q_3) = (1.3, 0.1, -0.4)$. The vertical dotted lines represent the six singularities $-\theta_3$, $-\theta_2$, $-\theta_1$, η_1, η_2, and η_3. It is easily seen that for sufficiently large $\alpha > 0$, $G(x) = \alpha$ has eight (i.e., $m + n + 2$) distinctive real roots, among which four (i.e., $m + 1$) are positive and four (i.e., $n + 1$) are negative.

Laplace transform of the joint distribution of τ_b and X_t using the result in Step (iii).

(i) *Roots of the equation $G(x) = \alpha$.*

THEOREM 3.1. *For sufficiently large $\alpha > 0$, the equation $G(x) = \alpha$ has $(m + n + 2)$ roots that are all real and are distinct. Specifically, we have $(m + 1)$ positive roots, $\beta_{1,\alpha}, \ldots, \beta_{m+1,\alpha}$, and $(n + 1)$ negative roots, $\gamma_{1,\alpha}, \ldots, \gamma_{n+1,\alpha}$, as follows:*

$$
-\infty < \gamma_{n+1,\alpha} < \cdots < \gamma_{2,\alpha} < \gamma_{1,\alpha} < 0 < \beta_{1,\alpha}
$$
$$
< \beta_{2,\alpha} < \cdots < \beta_{m+1,\alpha} < +\infty.
\tag{7}
$$

The proof is given in §A of the e-companion. Figure 2 illustrates the function $G(x)$, from which we can see how the roots behave for sufficiently large $\alpha > 0$.

(ii) *Solving the OIDE (6) explicitly.* A technical contribution of the current paper is that we solve explicitly the forced OIDE (6) by transforming it into a homogeneous linear ODE.

THEOREM 3.2. *Assume that $\alpha > 0$ is sufficiently large such that $G(x) = \alpha$ has $(m + n + 2)$ real roots, satisfying (7). Then any solution $u(x)$ of OIDE (6) is also a solution of an $(m + n + 2)$-order homogeneous linear ODE with constant coefficients, whose characteristic equation is given by $(G(x) - \alpha) \prod_{i=1}^{m} (x - \eta_i) \prod_{j=1}^{n} (x + \theta_j) = 0$. Thus, any solution of OIDE (6) is of the form*

$$
u(x) = \sum_{i=1}^{m+1} c_i e^{\beta_{i,\alpha} x} + \sum_{j=1}^{n+1} d_j e^{\gamma_{j,\alpha} x},
\tag{8}
$$

where $c_1, c_2, \ldots, c_{m+1}, d_1, d_2, \ldots, d_{n+1}$ are undetermined constants.

PROOF. See §B in the e-companion. □

(iii) *Joint distribution of the first passage time τ_b and the overshoot $X_{\tau_b} - b$.*

THEOREM 3.3. *For any sufficiently large $\alpha > 0$, $\theta < \eta_1$ and $x, b \in \mathbb{R}$, we have*

$$E^x[e^{-\alpha \tau_b + \theta X_{\tau_b}}] = \begin{cases} e^{\theta x} & \text{if } x \geq b, \\ \sum_{l=1}^{m+1} w_l e^{\beta_{l,\alpha} x} & \text{if } x < b, \end{cases} \quad (9)$$

where $x = X_0$ and $\beta_{1,\alpha}, \ldots, \beta_{m+1,\alpha}$ are the $m+1$ positive roots of the equation $G(x) = \alpha$ such that $0 < \beta_{1,\alpha} < \beta_{2,\alpha} < \cdots < \beta_{m+1,\alpha}$. Here $w := (w_1, w_2, \ldots, w_{m+1})'$ is a vector uniquely determined by the following linear system

$$ABw = J, \quad (10)$$

where A is an $(m+1) \times (m+1)$ nonsingular matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \dfrac{\eta_1}{\eta_1 - \beta_{1,\alpha}} & \dfrac{\eta_1}{\eta_1 - \beta_{2,\alpha}} & \cdots & \dfrac{\eta_1}{\eta_1 - \beta_{m+1,\alpha}} \\ \dfrac{\eta_2}{\eta_2 - \beta_{1,\alpha}} & \dfrac{\eta_2}{\eta_2 - \beta_{2,\alpha}} & \cdots & \dfrac{\eta_2}{\eta_2 - \beta_{m+1,\alpha}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\eta_m}{\eta_m - \beta_{1,\alpha}} & \dfrac{\eta_m}{\eta_m - \beta_{2,\alpha}} & \cdots & \dfrac{\eta_m}{\eta_m - \beta_{m+1,\alpha}} \end{pmatrix}, \quad (11)$$

B is an $(m+1) \times (m+1)$ diagonal matrix, and J is an $(m+1)$-dimensional vector

$$B = \text{diag}\{e^{\beta_{1,\alpha} b}, e^{\beta_{2,\alpha} b}, \ldots, e^{\beta_{m+1,\alpha} b}\},$$
$$J = e^{\theta b}\left(1, \frac{\eta_1}{\eta_1 - \theta}, \frac{\eta_2}{\eta_2 - \theta}, \ldots, \frac{\eta_m}{\eta_m - \theta}\right)'. \quad (12)$$

In particular, with $\theta = 0$, we have, for sufficiently large $\alpha > 0$,

$$E^x[e^{-\alpha \tau_b}] = \begin{cases} 1 & \text{if } x \geq b, \\ \sum_{l=1}^{m+1} c_l e^{\beta_{l,\alpha} x} & \text{if } x < b. \end{cases} \quad (13)$$

Here $c := (c_1, c_2, \ldots, c_{m+1})'$ is a positive vector uniquely determined by the linear system

$$ABc = \mathbf{1}, \quad (14)$$

where $\mathbf{1} = (1, 1, \ldots, 1)'$.

PROOF. See §C in the e-companion. □

(iv) *Joint distribution of τ_b and X_t.* Without loss of generality, we assume $X_0 = 0$. The joint distribution of τ_b and X_t, i.e.,

$$P^0(X_t \geq a, \tau_b \leq t) = P^0(X_t \geq -\hat{a}, \tau_b \leq t), \quad (15)$$

for some fixed numbers $a \equiv -\hat{a} \leq b$ and $b > 0$, has a variety of applications, including pricing barrier options.

THEOREM 3.4. *Assume $X_0 = 0$. Denote by $\mathcal{L}(\alpha, \theta)$ the double Laplace transform of $P^0(X_t \geq -\hat{a}, \tau_b \leq t)$ w.r.t. t and $\hat{a}$, respectively, i.e.,*

$$\mathcal{L}(\alpha, \theta) = \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\theta \hat{a} - \alpha t} P^0(X_t \geq -\hat{a}, \tau_b \leq t)\, d\hat{a}\, dt.$$

Then for any $\theta \in (0, \eta_1)$ and sufficiently large $\alpha > \max(G(\theta), 0)$, we have

$$\mathcal{L}(\alpha, \theta) = \frac{\sum_{l=1}^{m+1} \hat{d}_l e^{-\beta_{l,\alpha} b}}{\theta(\alpha - G(\theta))}, \quad (16)$$

where $\beta_{1,\alpha}, \ldots, \beta_{m+1,\alpha}$ are the $m+1$ positive roots of the equation $G(x) = \alpha$ such that $0 < \beta_{1,\alpha} < \beta_{2,\alpha} < \cdots < \beta_{m+1,\alpha}$. Here $\hat{d} := (\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_{m+1})'$ solves the linear system $A\hat{d} = J$, where A and J are the same as in (11) and (12).

PROOF. See §D in the e-companion. □

4. Pricing European Options Under the MEM and a Calibration Example

4.1. Pricing European Options Under the MEM

Under the risk-neutral measure P, the value of a European call option with a fixed strike K and maturity T is given by $e^{-rT} E[(S_T - K)^+]$. To apply the two-sided Euler inversion (EI) algorithm proposed by Petrella (2004), we introduce a scaling factor $X > K$, which ensures that the Euler inversion algorithm converges quickly. The call option value can then be expressed as

$$C_T(k_c) = e^{-rT} X \cdot E\left[\left(\frac{S_T}{X} - e^{-k_c}\right)^+\right],$$

where $k_c = \log(X/K)$. The same proof as that for Lemma 1 in Kou et al. (2005), which is based on Carr and Madan (1999), leads to the Laplace transform of $C_T(k_c)$ w.r.t. k_c:

$$\hat{C}(\psi) := \int_{-\infty}^{+\infty} e^{-\psi k_c} C_T(k_c)\, dk_c = e^{-rT} \frac{S_0^{\psi+1} e^{G(\psi+1)T}}{\psi(\psi+1) X^\psi},$$
$$\text{for any } \psi \in (0, \eta_1 - 1), \quad (17)$$

where $G(\cdot)$ is given by (3). As an immediate result, we can obtain the closed-form Laplace transform

Table 1   **The Euler Inversion (EI Value or BA Value) vs. Monte Carlo Simulation (MC Value) for Calculating the Prices and Deltas of European Options Under the MEM**

Prices of European call options under the MEM

		$\sigma = 0.2$				$\sigma = 0.3$			
η_1	λ	EI value	BA value	MC value	Std. err.	EI value	BA value	MC value	Std. err.
20	1	10.97472	10.97472	10.97339	0.01923	14.59752	14.59752	14.59520	0.02776
	3	11.94485	11.94485	11.92641	0.02172	15.29993	15.29993	15.27560	0.02944
	5	12.83076	12.83076	12.87446	0.02407	15.96677	15.96677	15.99681	0.03121
40	1	10.57572	10.57572	10.56959	0.01807	14.31636	14.31636	14.30866	0.02703
	3	10.82050	10.82050	10.80522	0.01865	14.48475	14.48475	14.46554	0.02740
	5	11.05846	11.05846	11.07213	0.01934	14.65079	14.65079	14.65308	0.02790

Deltas of European call options under the MEM

		$\sigma = 0.2$				$\sigma = 0.3$			
S_0	K	EI value	BA value	MC value	Std. err.	EI value	BA value	MC value	Std. err.
100	90	0.79027	0.79027	0.78973	0.00080	0.74249	0.74249	0.74283	0.00082
	100	0.63645	0.63645	0.63709	0.00091	0.62633	0.62633	0.62610	0.00091
	110	0.47150	0.47150	0.47112	0.00100	0.50940	0.50940	0.50895	0.00099
102	90	0.81396	0.81396	0.81368	0.00077	0.76198	0.76198	0.76237	0.00080
	100	0.66848	0.66848	0.66869	0.00089	0.64953	0.64953	0.64899	0.00089
	110	0.50648	0.50648	0.50494	0.00098	0.53414	0.53414	0.53450	0.00097

Notes. For the "price" part, the default choices for unvarying parameters are $r = 0.05$, $\theta_1 = \eta_1$, $\eta_2 = \theta_2 = 50$, $p_u = 0.4$, $q_d = 0.6$, $p_1 = 1.2$, $p_2 = -0.2$, $q_1 = 1.3$, $q_2 = -0.3$, $S_0 = 100$, $K = 100$, and $t = 1$. For the "delta" part, the default choices of unvarying parameters are $r = 0.05$, $\lambda = 3$, $\eta_1 = 30$, $\eta_2 = 40$, $\theta_1 = 20$, $\theta_2 = 30$, $p_u = 0.4$, $q_d = 0.6$, $p_1 = 1.2$, $p_2 = -0.2$, $q_1 = 1.3$, $q_2 = -0.3$, and $t = 1$. Parameters for the Euler inversion method are $A = 18$, $n = 30$, and $X = 10{,}000$. The MC values and the associated standard errors are obtained by simulating 100,000 sample paths and by setting the step size to be 0.00005. Here S_T is used as a control variate to achieve variance reduction. We can see that all the EI values stay within the 95% confidence intervals of the associated MC values. In addition, the BA values denote the European call option prices or deltas obtained by calculating prices or deltas of up-and-in call barrier options with barrier $H = S_0$. It is easily seen that all the BA values agree with the EI values to five decimal points. The CPU times to generate one EI value, one BA value, and one MC value are less than 0.01 seconds, approximately 5 seconds, and approximately 50 seconds, respectively.

of the European option delta $\Delta_T(k_c) := \partial C_T(k_c)/\partial S_0$ w.r.t. k_c:

$$\hat{\Delta}(\psi) := \int_{-\infty}^{+\infty} e^{-\psi k_c} \Delta_T(k_c) \, dk_c = e^{-rT} \frac{S_0^\psi e^{G(\psi+1)T}}{\psi X^\psi},$$

$$\text{for any } \psi \in (0, \eta_1 - 1), \quad (18)$$

where the interchange of derivatives and integrals can be justified by Theorem A.12 of Schiff (1999, pp. 203–204).

Inverting $\hat{C}(\psi)$ and $\hat{\Delta}(\psi)$ via the two-sided Euler inversion method[4] yields numerical results of European call option prices and deltas under the MEM,

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[4] Note that Petrella's (2004) algorithm is faster and more stable than the original Euler inversion (see Abate and Whitt 1992) because of the introduction of a scaling factor. In the implementation of the one-sided, one-dimensional (two-sided, two-dimensional) Euler inversion method, some parameters n and A (n_1, n_2, A_1, and A_2) are involved. The parameter n (n_1 and n_2) is used in the Euler transformation to accelerate the computation of some alternating series in the inversion formula, whereas A (A_1 and A_2) is used to control the discretization errors. For more information about these parameters and the inversion formulae, see Petrella (2004), Abate and Whitt (1992), and Cai et al. (2007).

which are provided in Table 1. It can be seen that all of our numerical results (denoted by EI values) stay within the 95% confidence intervals of the associated Monte Carlo simulation estimates (denoted by MC values), which are obtained by using S_T as a control variate to reduce variances. All the computations in this paper are conducted on a laptop with a Duo 2.50 GHz central processing unit (CPU).

Because a European call option is the same as a degenerated up-and-in call barrier option with barrier $H = S_0$, the column "BA value" in Table 1 also reports the results using the degenerated barrier options by numerically inverting the double Laplace transforms (22) and (23) in Theorem 5.2. We can see that the numerical European option prices (and deltas) obtained in this way agree with those by inverting $\hat{C}(\psi)$ in (17) (and $\hat{\Delta}(\psi)$ in (18)) up to five decimal points.

4.2.   A Calibration Example

In general, model calibration is an important and yet difficult problem that involves various optimization and numerical pricing techniques. In this subsection, we give an example to illustrate the calibration of our

MEM to a set of market data. For a more comprehensive discussion on the calibration, we refer readers to Cont and Tankov (2004). In our example, the data set obtained from Morningstar Inc. consists of the closing prices (i.e., the averages of bid and ask prices) of 47 SPY (S&P 500 ETF stock) European call options on March 29, 2010, with three maturities (1 day, 18 days, and 53 days) and various strike prices. Our goal is to calibrate the model to these option prices across different maturities and different strikes using only one set of parameters. The calibration is especially interesting because it is well known that it is difficult to calibrate models to options with *very short maturity* such as one day.

We shall minimize the objective function $\sum_{i=1}^{47}((C_i(\pi) - \tilde{C}_i(\pi))^2)/\text{Vega}^2(IV_i)$ over the set of unknown parameters $\pi = (\sigma, \lambda, \eta_1, \ldots, \eta_m, \theta_1, \ldots, \theta_n, p_1, \ldots, p_m, q_1, \ldots, q_n)$, where C_i and $\tilde{C}_i$ represent the calibrated European option price and the market price, respectively, and IV_i is the market implied volatility for the ith option. This objective function for calibration is suggested by Cont and Tankov (2004, p. 439). For simplicity, we use an MEM with an upward jump distribution that is exponential and a downward jump distribution that is a mixture of two exponentials (i.e., $m = 1$ and $n = 2$). To solve the optimization problem, we first select 100 best starting points from approximately 20,000 grid points and then search the optimal solution for each of these 100 starting points. The best one is chosen to be our final optimal solution.

Figure 3 shows both observed market implied volatilities and calibrated implied volatilities. It is worth mentioning that although in general it is difficult to fit the implied volatilities for the options with very short maturity such as one day, it seems that our model can produce a close fit even to this sharp volatility skew.

5. Lookback and Barrier Options

5.1. Lookback Options
We shall only consider lookback put options because lookback call options can be obtained by symmetry. Under the risk-neutral measure P, the price of a lookback put option with the maturity T is given by

$$LP(T) = E\left[e^{-rT}\left(\max\left\{M, \max_{\{0 \le t \le T\}} S_t\right\} - S_T\right)\right]$$

$$= E\left[e^{-rT}\max\left\{M, \max_{\{0 \le t \le T\}} S_t\right\}\right] - S_0,$$

where $M \ge S_0$ is a fixed constant representing the prescribed maximum at time 0.

Figure 3   Calibrated Implied Volatilities vs. Observed Market Implied Volatilities
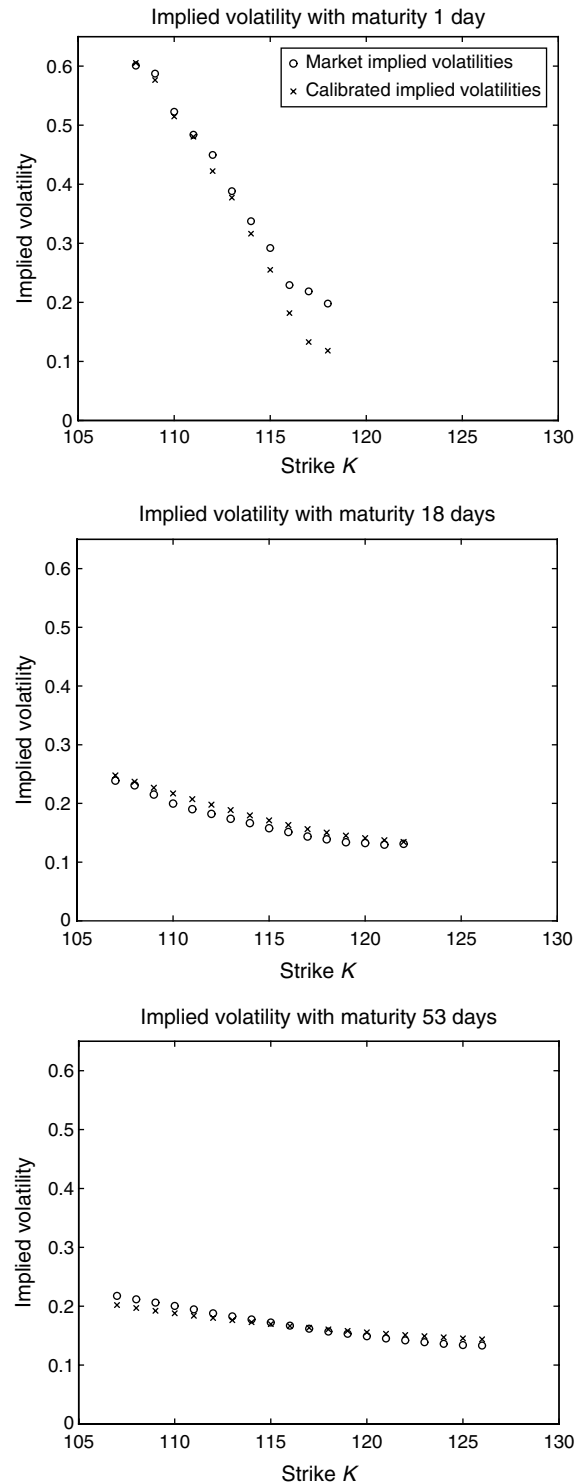
Notes. The initial stock price is 117.32. The risk-free interest rates corresponding to these three maturities (1 day, 18 days, and 53 days) are 0.0011, 0.0011, and 0.0012, respectively. Note that in general it is difficult to fit the implied volatilities for the options with very short maturity such as one day. However, it seems that our model can produce a close fit even to this sharp volatility skew. The parameters used in the calibrated model are $\hat{\sigma} = 0.10997$, $\hat{\lambda} = 6.19653$, $\hat{\eta}_1 = 202$, $\hat{\theta}_1 = 45.21588$, $\hat{\theta}_2 = 78.40339$, $\hat{p}_1 = 0.00077$, $\hat{q}_1 = 3.09202$, and $\hat{q}_2 = -2.09279$.

Table 2   Pricing Lookback and Barrier Options Under the MEM

		Pricing lookback options under the MEM					
		$\sigma = 0.1$			$\sigma = 0.2$		
M	λ	EI value	MC value	Std. err.	EI value	MC value	Std. err.
105	1	7.13774	7.14438	0.00717	15.11777	15.12439	0.00676
	3	8.38446	8.39709	0.00737	15.96876	15.97562	0.00679
	5	9.54401	9.55256	0.00747	16.79455	16.80643	0.00683
107	1	7.69091	7.69950	0.01046	15.48878	15.49762	0.01038
	3	8.91452	8.93285	0.01084	16.33422	16.34394	0.01043
	5	10.05471	10.06516	0.01108	17.15486	17.16885	0.01052

		Pricing up-and-in call barrier options under the MEM when K varies and $H = 115$					
		$\sigma = 0.1$			$\sigma = 0.2$		
λ	K	EI value	MC value	Std. err.	EI value	MC value	Std. err.
2	101	10.02579	10.03530	0.02077	13.94197	13.91411	0.02840
	105	8.35828	8.36193	0.02072	12.25099	12.24949	0.02935
	109	6.84158	6.84678	0.02073	10.70121	10.67303	0.02984

		Pricing up-and-in call barrier options under the MEM when H varies and $K = 102$					
		$\sigma = 0.1$			$\sigma = 0.2$		
λ	H	EI value	MC value	Std. err.	EI value	MC value	Std. err.
3	105	10.09973	10.10853	0.02004	13.75881	13.75839	0.02859
	110	10.04892	10.08167	0.02035	13.74182	13.78379	0.02877
	115	9.83030	9.83137	0.02117	13.66264	13.68834	0.02910

Notes. The Euler inversion (EI value) versus Monte Carlo simulation (MC value) is shown. Default parameters are $r = 0.05$, $S_0 = 100$, $\eta_1 = 30$, $\eta_2 = 50$, $\theta_1 = 30$, $\theta_2 = 40$, $p_u = 0.4$, $q_d = 0.6$, $p_1 = 1.2$, $p_2 = -0.2$, $q_1 = 1.3$, $q_2 = -0.3$, and $t = 1$. EI values are obtained using Euler inversion (related parameters are $A = 18$ and $n = 30$ for lookback options and $A_1 = A_2 = 18$, $n_1 = 30$, $n_2 = 50$, and $X = 1{,}000$ for barrier options). MC values are Monte Carlo simulation estimates by simulating 20,000 sample paths and using step size 0.00001 for lookback options, and by simulating 100,000 sample paths and using step size 0.00005 for barrier options. We can see that all the EI values stay within the 95% confidence intervals of the associated MC values. The CPU times for generating one EI value for lookback options, one MC value for lookback options, one EI value for barrier options, and one MC value for barrier options are approximately 0.04 seconds, 3 minutes, 6 seconds, and 2 minutes, respectively.

THEOREM 5.1. *For all sufficiently large $\alpha > 0$, the Laplace transforms of the lookback put option price $LP(T)$ and delta $\Delta(T) := \partial LP(T)/\partial S_0$ w.r.t. the maturity T are given by*

$$\int_0^{+\infty} e^{-\alpha T} LP(T)\, dT = \frac{S_0}{\alpha + r} \sum_{i=1}^{m+1} \frac{d_i}{\beta_{i,\alpha+r} - 1} \left(\frac{S_0}{M} \right)^{\beta_{i,\alpha+r} - 1}$$
$$+ \frac{M}{\alpha + r} - \frac{S_0}{\alpha} \qquad (19)$$

and

$$\int_0^{+\infty} e^{-\alpha T} \Delta(T)\, dT$$
$$= \frac{1}{\alpha + r} \sum_{i=1}^{m+1} \frac{d_i \beta_{i,\alpha+r}}{\beta_{i,\alpha+r} - 1} \left(\frac{S_0}{M} \right)^{\beta_{i,\alpha+r} - 1} - \frac{1}{\alpha}, \qquad (20)$$

respectively, where $\beta_{1,\alpha+r}, \beta_{2,\alpha+r}, \ldots, \beta_{m+1,\alpha+r}$ are the $(m+1)$ positive roots of the equation $G(x) = \alpha + r$, and $d := (d_1, d_2, \ldots, d_{m+1})'$ is the unique solution of the linear system $Ad = \mathbf{1}$, where A associated with $\alpha + r$ is defined in Theorem 3.3 and $\mathbf{1} = (1, 1, \ldots, 1)'$.

PROOF. See Appendix A. □

To invert the Laplace transform, we employ the one-sided, one-dimensional Euler inversion method (see Abate and Whitt 1992). The corresponding numerical results are given in the upper panels of Tables 2 and 3, where EI and MC values represent the results obtained via the Euler inversion method and the Monte Carlo simulation, respectively. "Std. err." is the associated standard error of the MC value. For Monte Carlo simulation, we use the running maximum $\max_{0 \leq t \leq T} S_t$, whose expectation can be computed analytically (see §E in the e-companion), as a control variate to achieve variance reduction. It can be seen that all the EI values stay within the 95% confidence intervals of the associated MC values. The CPU time is only around 0.04 seconds. In comparison, the CPU time to generate one MC value is approximately two to three minutes.[5]

[5] Furthermore, the numerical inversion is robust w.r.t. the inversion algorithm parameters. For example, any $A \in [15, 45]$ produces almost identical results with four-digit accuracy.

Table 3 Deltas of Lookback and Barrier Options Under the MEM

		Deltas of lookback options under the MEM					
		$\lambda = 1$			$\lambda = 3$		
S_0	σ	EI value	MC value	Std. err.	EI value	MC value	Std. err.
100	0.2	−0.16998	−0.17434	0.00293	−0.15430	−0.15790	0.00294
	0.3	−0.00711	−0.01025	0.00296	0.00201	0.00181	0.00295
	0.4	0.12957	0.12923	0.00290	0.13616	0.13792	0.00289
102	0.2	−0.10240	−0.10499	0.00287	−0.08909	−0.09001	0.00286
	0.3	0.04420	0.04453	0.00279	0.05248	0.05264	0.00279
	0.4	0.17194	0.17204	0.00270	0.17815	0.18155	0.00267

		Deltas of up-and-in call barrier options under the MEM					
		$\sigma = 0.2$			$\sigma = 0.3$		
S_0	K	EI value	MC value	Std. err.	EI value	MC value	Std. err.
100	95	0.65316	0.64857	0.00243	0.66112	0.66375	0.00230
	100	0.60236	0.60289	0.00242	0.61359	0.61711	0.00237
	105	0.53793	0.53699	0.00247	0.56102	0.56205	0.00243
102	95	0.69626	0.69906	0.00235	0.68828	0.68779	0.00224
	100	0.64241	0.64539	0.00238	0.63967	0.63941	0.00233
	105	0.57555	0.57783	0.00243	0.58646	0.58203	0.00241

Notes. The Euler inversion (EI value) versus Monte Carlo simulation (MC value) is shown. The default choices of unvarying parameters are $r = 0.05$, $M = 110$ (for lookback options), $H = 110$ and $\lambda = 3$ (for barrier options), $\eta_1 = 30$, $\eta_2 = 50$, $\theta_1 = 30$, $\theta_2 = 40$, $p_u = 0.4$, $q_d = 0.6$, $p_1 = 1.2$, $p_2 = -0.2$, $q_1 = 1.3$, $q_2 = -0.3$, and $t = 1$. EI values are obtained using the Euler inversion with parameters $A = 18$ and $n = 30$ for lookback options and $A_1 = A_2 = 18$, $n_1 = 30$, $n_2 = 50$, and $X = 1{,}000$ for barrier options. All the MC values along with the associated standard errors are obtained by simulating 15,000 sample paths and by using step size 0.00001. We can see that all the EI values stay within the 95% confidence intervals of the associated MC values. The CPU times to generate one EI value for lookback option delta, one MC value for lookback option delta, one EI value for barrier option delta, and one MC value for barrier option delta are approximately 0.04 seconds, 2 minutes, 6 seconds, and 2 minutes, respectively.

5.2. Barrier Options

There are eight types of (one dimensional, single) barrier options: up (down)-and-in (out) call (put) options. Here, we only illustrate how to deal with the up-and-in call (UIC) barrier option because the other seven barrier options can be priced similarly. The price of a UIC with a fixed strike K and a maturity T under the risk-neutral measure P can be expresses as $E[e^{-rT} \cdot (S_T - K)^+ I_{\{\tau_b < T\}}]$, where $H > S_0$ is the barrier level, and $b = \log(H/S_0)$ is the barrier corresponding to the return process $X_t \equiv \log(S_t/S_0)$. To apply the two-sided Euler inversion method proposed by Petrella (2004), we introduce a scaling factor X so that the price of a UIC can be rewritten as

$$UIC(k, T) = E\left[Xe^{-rT}\left(\frac{S_T}{X} - e^{-k}\right)^+ I_{\{\tau_b < T\}}\right], \quad (21)$$

where $k = \log(X/K)$. Note that the scaling factor is crucial in our algorithm because it ensures that the Euler inversion method converges rapidly, making the algorithm accurate and efficient.

Define $\hat{f}_{UIC}(\alpha, \psi)$ and $\hat{\Delta}_{UIC}(\alpha, \psi)$ as the double Laplace transforms of the price $UIC(k, T)$ in (21) and the delta $\Delta_{UIC}(k, T) := \partial UIC(k, T)/\partial S_0$ w.r.t. T and k, respectively, i.e.,

$$\hat{f}_{UIC}(\alpha, \psi) := \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\psi k - \alpha T} UIC(k, T)\, dk\, dT,$$

$$\hat{\Delta}_{UIC}(\alpha, \psi) := \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\psi k - \alpha T} \Delta_{UIC}(k, T)\, dk\, dT.$$

THEOREM 5.2. *For any $\psi \in (0, \eta_1 - 1)$ and sufficiently large $\alpha > \max(G(\psi + 1) - r, 0)$,*

$$\hat{f}_{UIC}(\alpha, \psi) = \frac{S_0^{\psi+1} \sum_{l=1}^{m+1} w_l e^{-\beta_{l, \alpha+r} b}}{X^\psi \psi(\psi + 1)(r + \alpha - G(\psi + 1))}, \quad (22)$$

and

$$\hat{\Delta}_{UIC}(\alpha, \psi) = \frac{S_0^\psi \sum_{l=1}^{m+1} w_l e^{-\beta_{l, \alpha+r} b}}{X^\psi \psi(r + \alpha - G(\psi + 1))}, \quad (23)$$

where $\beta_{1, \alpha+r}, \beta_{2, \alpha+r}, \dots, \beta_{m+1, \alpha+r}$ are the $(m+1)$ positive roots of the equation $G(x) = \alpha + r$, and $w := (w_1, w_2, \dots, w_{m+1})'$ is the unique solution of the linear system $Aw = J$, where A associated with $\alpha + r$ is defined as in Theorem 3.3 and

$$J = e^{(\psi+1)b}\left(1, \frac{\eta_1}{\eta_1 - \psi - 1}, \frac{\eta_2}{\eta_2 - \psi - 1}, \dots, \frac{\eta_m}{\eta_m - \psi - 1}\right)'.$$

PROOF. See Appendix A. □

To compute the prices and deltas of barrier options, we apply a two-sided, two-dimensional Euler inversion method (see Petrella 2004). The numerical results

Table 4 Pricing European Options and Calculating Deltas Under Merton's (1976) Model by Approximating It with the MEM

σ	K	$\lambda = 1$			$\lambda = 3$		
		EI value	True value	Abs. err.	EI value	True value	Abs. err.
		Prices of European options under Merton's model					
0.2	95	13.35534	13.35476	0.00058	13.37306	13.37131	0.00175
	100	10.46061	10.45995	0.00066	10.48063	10.47866	0.00197
	105	8.03196	8.03126	0.00070	8.05312	8.05103	0.00209
0.3	95	16.80753	16.80711	0.00042	16.82015	16.81891	0.00124
	100	14.23802	14.23758	0.00044	14.25154	14.25021	0.00133
	105	11.98391	11.98345	0.00046	11.99796	11.99657	0.00139
		Deltas of European options under Merton's model					
0.2	95	0.72772	0.72773	−0.00001	0.72736	0.72740	−0.00004
	100	0.63676	0.63676	0.00000	0.63661	0.63662	−0.00001
	105	0.54228	0.54227	0.00001	0.54237	0.54237	0.00000
0.3	95	0.68706	0.68706	0.00000	0.68698	0.68699	−0.00001
	100	0.62425	0.62425	0.00000	0.62424	0.62424	0.00000
	105	0.56124	0.56124	0.00000	0.56131	0.56130	0.00001

Notes. Default parameters are $r = 0.05$, $S_0 = 100$, and $t = 1$. "EI" values and "true" values are obtained by using Euler inversion (related parameters are $A = 18$, $n = 30$, and $X = 10,000$) under the approximate MEM with the jump size pdf being (24) and under Merton's model, respectively.

are given in the lower panels of Tables 2 and 3, where "EI value," "MC value," and "Std. err." have the same meanings as for lookback options. For Monte Carlo simulation, we use S_T as a control variate to achieve variance reduction. All the EI values stay within the 95% confidence intervals of the associated MC values. The CPU time of computing one price via the Euler inversion is approximately 6.0 seconds, whereas the CPU time of generating one MC value is approximately two minutes.[6]

# 6. An Example of Approximating Merton's (1976) Model via the MEM

Although Merton's normal jump diffusion model (see Merton 1976) is very popular in finance, analytical pricing of path-dependent options under Merton's model remains challenging. We shall approximate Merton's model by the MEM, partly because of the denseness of the mixed-exponential distributions w.r.t. the class of all the distributions in the sense of weak convergence. In this section, our objective is not to discuss how to approximate Merton's model optimally, which by itself is an interesting open problem. Rather, we shall provide a simple example to illustrate that this approximation may lead to quite accurate European, lookback, and barrier option prices and deltas for Merton's model.

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[6] The numerical inversion is also insensitive to the change of the algorithm parameters A_1, A_2, and X. Indeed, any $A_1 \in [15, 55]$, $A_2 \in [16, 45]$, and $X \in [300, 4,000]$ can produce almost identical results with four-digit accuracy.

For simplicity, we intend to approximate Merton's (1976) model with the jump size distribution $N(0, 0.01^2)$, the normal distribution with mean 0 and standard deviation 0.01, using the MEM with the pdf of the jump size given by

$$
\begin{aligned}
f_Y(x) = 0.5\big(&8.7303 \times 213.0215 e^{-213.0215|x|} \\
&+ 2.1666 \times 236.0406 e^{-236.0406|x|} \\
&- 10 \times 237.1139 e^{-237.1139|x|} \\
&+ 0.0622 \times 939.7441 e^{-939.7441|x|} \\
&+ 0.0409 \times 939.8021 e^{-939.8021|x|}\big). \quad (24)
\end{aligned}
$$

Figure 4 Approximate the Normal Distribution $N(0, 0.01^2)$ Using the Mixed-Exponential Distribution with the pdf Given by (24)

Approximate the cdf of normal$(0, 0.01^2)$
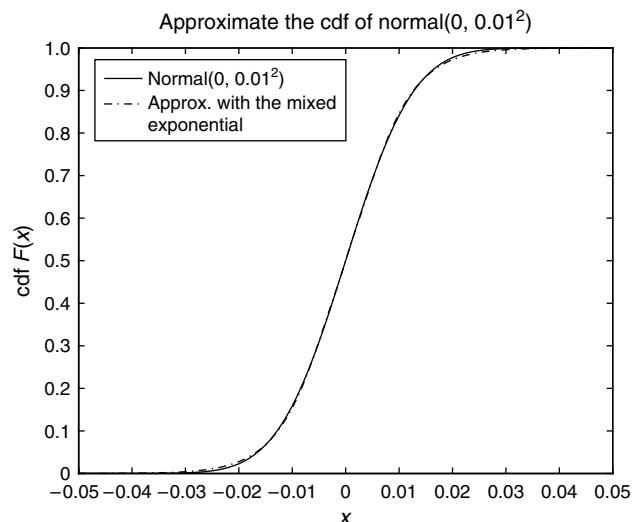
Table 5 **Pricing Lookback and Barrier Options Under Merton's (1976) Model by Approximating It with the MEM**

		Pricing lookback options under Merton's model					
		$\lambda = 1$			$\lambda = 5$		
σ	M	EI value	MC value	Std. err.	EI value	MC value	Std. err.
0.1	105	6.51320	6.50105	0.01049	6.66681	6.65129	0.01077
	107	7.07855	7.06909	0.01041	7.22694	7.21430	0.01068
	109	7.84251	7.83448	0.01026	7.98273	7.97271	0.01051
0.2	105	14.70418	14.67465	0.02287	14.79288	14.77413	0.02311
	107	15.07787	15.05112	0.02270	15.16551	15.15086	0.02293
	109	15.57228	15.54649	0.02253	15.65840	15.64581	0.02275

		Pricing up-and-in call barrier options under Merton's model when K varies and $H = 115$					
		$\sigma = 0.2$			$\sigma = 0.3$		
λ	K	EI value	MC value	Std. err.	EI value	MC value	Std. err.
1	101	9.54594	9.54397	0.01988	13.63140	13.64003	0.02774
	105	7.88753	7.87605	0.01966	11.93684	11.93757	0.02855
	109	6.38154	6.39107	0.01952	10.38581	10.41109	0.02922

		Pricing up-and-in call barrier options under Merton's model when H varies and $K = 102$					
		$\sigma = 0.2$			$\sigma = 0.3$		
λ	H	EI value	MC value	Std. err.	EI value	MC value	Std. err.
3	105	9.45243	9.44670	0.01839	13.31261	13.29997	0.02754
	110	9.39394	9.38877	0.01863	13.29437	13.30942	0.02754
	115	9.14142	9.12743	0.01979	13.20890	13.25942	0.02804

Notes. The Euler inversion (EI value) versus Monte Carlo simulation (MC value) is shown. Default parameters are $r = 0.05$, $S_0 = 100$, and $t = 1$. EI values are obtained using the Euler inversion (related parameters are $A = 18$ and $n = 30$ for lookback options, and $A_1 = A_2 = 18$, $n_1 = 30$, $n_2 = 50$, and $X = 1,000$ for barrier options) under the approximate MEM with the jump size pdf being (24). MC values are Monte Carlo simulation estimates under Merton's model by simulating 100,000 sample paths and by using step sizes 0.00001 for lookback options and 0.00005 for barrier options. We can see that all the EI values obtained using the approximate MEM stay within the 95% confidence intervals of the MC values obtained under Merton's model. The CPU times to generate one EI value for the lookback option price, one MC value for the lookback option price, one EI value for the barrier option price, and one MC value for the barrier option price are approximately 0.04 seconds, 10 minutes, 6 seconds, and 2 minutes, respectively.

These parameters are obtained by minimizing the sum of the square differences between cdf values of $N(0, 0.01^2)$ and the mixed-exponential distribution over the grid points on the interval $[-0.035, 0.035]$. We first select 100 best starting points among approximately 1,600,000 points and then minimize the objective function by starting from each of these 100 points. The final solution is the best one among the 100 optimal solutions. Figure 4 demonstrates the close fit of the mixed-exponential distribution (24) to the cdf of $N(0, 0.01^2)$.

Using the MEM with jump size pdf being (24), we shall price European, lookback, and barrier options as well as calculate associated deltas approximately under Merton's model.

Table 4 provides the approximation to the European option prices and deltas. Our approximation appears to be reasonably good, because the maximum absolute errors between our approximate values (denoted by EI values) and "true values" are quite small.

Tables 5 and 6 provide approximate lookback and barrier option prices and deltas under Merton's model, respectively. All of our numerical approximations (denoted by EI values) obtained using the approximate MEM stay within the 95% confidence intervals of the Monte Carlo simulation estimates (denoted by MC values) obtained under Merton's model. For Monte Carlo simulation, we use S_T as a control variate to achieve variance reduction. In addition, our approximation method is very fast in that it takes only approximately 0.04 and 6 seconds to produce one EI value for lookback options and barrier options, respectively.

7. Conclusion

We propose a jump diffusion model for option pricing whose jump sizes have the mixed-exponential distribution, which can approximation any jump size distribution. The Laplace transforms of option prices and

Table 6 **Deltas of Lookback and Barrier Options Under Merton's (1976) Model by Approximating It with the MEM with the Jump Size pdf Being (24)**

		Deltas of lookback options under Merton's model					
		$\lambda = 3$			$\lambda = 5$		
S_0	σ	El value	MC value	Std. err.	El value	MC value	Std. err.
100	0.2	−0.17675	−0.17610	0.00406	−0.17579	−0.18082	0.00407
	0.3	−0.01101	−0.01092	0.00419	−0.01051	−0.01062	0.00418
	0.4	0.12675	0.13043	0.00431	0.12709	0.12681	0.00431
102	0.2	−0.10808	−0.10785	0.00389	−0.10725	−0.10582	0.00387
	0.3	0.04068	0.04341	0.00393	0.04114	0.04203	0.00392
	0.4	0.16929	0.17165	0.00402	0.16961	0.17135	0.00400

		Deltas of up-and-in call barrier options under Merton's model					
		$\sigma = 0.2$			$\sigma = 0.3$		
S_0	K	El value	MC value	Std. err.	El value	MC value	Std. err.
100	95	0.65201	0.65094	0.00095	0.66127	0.66061	0.00090
	100	0.60053	0.60019	0.00094	0.61262	0.61145	0.00092
	105	0.53291	0.53228	0.00096	0.55839	0.55790	0.00095
102	95	0.69917	0.69815	0.00092	0.68963	0.69016	0.00087
	100	0.64418	0.64356	0.00092	0.63978	0.63937	0.00090
	105	0.57368	0.57410	0.00094	0.58483	0.58536	0.00093

Notes. The Euler inversion (EI value) versus Monte Carlo simulation (MC value) is shown. Default parameters are $M = 110$ (for lookback options), $H = 110$ and $\lambda = 3$ (for barrier options), $r = 0.05$, and $t = 1$. Parameters for the Euler inversion methods are $A = 18$ and $n = 30$ for the lookback options and $A_1 = A_2 = 18$, $n_1 = 30$, $n_2 = 50$, and $X = 1,000$ for the barrier options. MC values along with the associated standard errors are obtained by simulating 100,000 sample paths and using 20,000 steps for barrier options, and by simulating 10,000 sample paths and using 150,000 steps for lookback options. We can see that all the EI values obtained using the approximate MEM stay within the 95% confidence intervals of the MC values obtained under Merton's model. The CPU times to generate one EI value for the lookback option delta, one MC value for the lookback option delta, one EI value for the barrier option delta, and one MC value for the barrier option delta are approximately 0.04 seconds, 2 minutes, 6 seconds, and 2 minutes, respectively.

deltas for some path-dependent options such as lookback and barrier options are obtained. These Laplace transforms can be inverted easily via the Euler inversion method, and numerical examples indicate that the method is accurate and efficient. In addition, we show that the mixed-exponential jump diffusion model may be used to approximate Merton's (1976) normal jump diffusion model. Open problems for future research include pricing of sequential barrier options and finite-horizon American options under the mixed-exponential jump diffusion model, as well as extensions to more general models, e.g., the models with stochastic interest rates.

8. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

Appendix A. Some Proofs

PROOF OF THEOREM 5.1. First of all, define $M_X(t) := \max_{0 \le u \le t} X_u$. Then for any $t > 0$, we shall prove that

$$\lim_{y \to +\infty} e^y P[M_X(t) \ge y] = 0. \qquad (A1)$$

Indeed, note that the process $\{e^{\theta X_t - G(\theta)t}: t \ge 0\}$ is a martingale for any $\theta \in (-\theta_1, \eta_1)$, because $G(\theta)$ is the exponent of the Lévy process $\{X_t: t \ge 0\}$. Fix $\theta \in (1, \eta_1)$ such that $G(\theta) > 0$. This θ must exist because $G(\eta_1-) = +\infty$, and $G(\theta)$ is continuous on the interval $(1, \eta_1)$. Note that

$$e^{\theta y} P(\tau_y \le t) \le E[e^{\theta X_{t \wedge \tau_y}}] \le e^{G(\theta)t} E[e^{\theta X_{t \wedge \tau_y} - G(\theta)(t \wedge \tau_y)}] \le e^{G(\theta)t},$$

where the last inequality holds thanks to the optional sampling theorem. So for any $y > 0$,

$$e^y P[M_X(t) \ge y] = e^{(1-\theta)y} e^{\theta y} P[M_X(t) \ge y]$$
$$= e^{(1-\theta)y} e^{\theta y} P(\tau_y \le t) \le e^{(1-\theta)y} e^{G(\theta)t}.$$

Note that $\theta > 1$, so letting y go to infinity completes the proof of (A1).

Next, define

$$L(S_0, M, T) := E\left[e^{-rT} \max\left\{M, \max_{\{0 \le t \le T\}} S_t\right\}\right]$$

$$= E[e^{-rT} \max\{M, S_0 e^{M_X(T)}\}],$$

and $z := \log(M/S_0) \ge 0$. Then we have

$$L(S_0, M, T) = S_0 E[e^{-rT} \max\{e^z, e^{M_X(T)}\}]$$

$$= S_0 e^{-rT} E[(e^{M_X(T)} - e^z) I_{\{M_X(T) \ge z\}}] + S_0 e^{z-rT}$$

$$= S_0 e^{-rT} E[(e^{M_X(T)} - e^z) I_{\{M_X(T) \ge z\}}] + M e^{-rT}.$$

On the other hand, we can obtain

$$E[(e^{M_X(T)} - e^z) I_{\{M_X(T) \ge z\}}] = \int_0^{+\infty} (e^y - e^z) I_{\{y \ge z\}} f_{M_X(T)}(y)\, dy$$

$$= -\int_z^{+\infty} (e^y - e^z)\, dP(M_X(T) \ge y)$$

$$= \int_z^{+\infty} e^y P(M_X(T) \ge y)\, dy,$$

where $f_{M_X(T)}$ is the pdf of $M_X(T)$, and the third equality holds because of (A1). Therefore,

$$L(S_0, M, T) = S_0 e^{-rT} \int_z^{+\infty} e^y P(M_X(T) \ge y)\, dy + M e^{-rT}.$$

For any $\alpha > 0$, the Laplace transform of $L(S_0, M, T)$ w.r.t. T is given by

$$\int_0^{+\infty} e^{-\alpha T} L(S_0, M, T)\, dT$$

$$= S_0 \int_0^{+\infty} e^{-\alpha T} e^{-rT} \int_z^{+\infty} e^y P(M_X(T) \ge y)\, dy\, dT + \frac{M}{\alpha + r}$$

$$= S_0 \int_z^{+\infty} e^y \left[\int_0^{+\infty} e^{-(\alpha+r)T} P(M_X(T) \ge y)\, dT\right] dy + \frac{M}{\alpha + r}. \quad \text{(A2)}$$

Note that for any $y > z \ge 0$, integration by parts leads to

$$\int_0^{+\infty} e^{-(\alpha+r)T} P(M_X(T) \ge y)\, dT$$

$$= \frac{1}{\alpha + r} \int_0^{+\infty} e^{-(\alpha+r)T}\, dP(M_X(T) \ge y)$$

$$= \frac{1}{\alpha + r} \int_0^{+\infty} e^{-(\alpha+r)T}\, dP(\tau_y \le T) = \frac{1}{\alpha + r} E[e^{-(\alpha+r)\tau_y}].$$

Applying (13) with $x = 0$, we have that, for sufficiently large $\alpha > 0$,

$$\int_0^{+\infty} e^{-(\alpha+r)T} P(M_X(T) \ge y)\, dT = \frac{1}{\alpha + r} \sum_{i=1}^{m+1} d_i e^{-\beta_{i,\alpha+r} y}. \quad \text{(A3)}$$

Plugging (A3) into (A2) yields

$$\int_0^{+\infty} e^{-\alpha T} L(S_0, M, T)\, dT$$

$$= S_0 \int_z^{+\infty} e^y \left[\frac{1}{\alpha + r} \sum_{i=1}^{m+1} d_i e^{-\beta_{i,\alpha+r} y}\right] dy + \frac{M}{\alpha + r}$$

$$= \frac{S_0}{\alpha + r} \sum_{i=1}^{m+1} d_i \int_z^{+\infty} e^{-(\beta_{i,\alpha+r} - 1)y}\, dy + \frac{M}{\alpha + r}.$$

Note that $\beta_{1,\alpha+r} > \beta_{1,r} = 1$ and $\beta_{i,\alpha+r} > \eta_1 > 1$ for any $i = 2, \ldots, m+1$. So we have that

$$\int_0^{+\infty} e^{-\alpha T} L(S_0, M, T)\, dT$$

$$= \frac{S_0}{\alpha + r} \sum_{i=1}^{m+1} \frac{d_i}{\beta_{i,\alpha+r} - 1} e^{-(\beta_{i,\alpha+r} - 1)z} + \frac{M}{\alpha + r}$$

$$= \frac{S_0}{\alpha + r} \sum_{i=1}^{m+1} \frac{d_i}{\beta_{i,\alpha+r} - 1} \left(\frac{S_0}{M}\right)^{\beta_{i,\alpha+r} - 1} + \frac{M}{\alpha + r},$$

which leads to (19), because $LP(T) = L(S_0, M, T) - S_0$.

Then (20) can be obtained by interchanging derivatives and integrals based on Theorem A.12 of Schiff (1999, pp. 203–204). □

PROOF OF THEOREM 5.2. Note that $\hat{f}_{UIC}(\alpha, \psi)$ can be expressed as follows:

$$\hat{f}_{UIC}(\alpha, \psi)$$

$$= X \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\psi k - (r+\alpha)T} E\left[\left(\frac{S_T}{X} - e^{-k}\right)^+ I_{\{\tau_b < T\}}\right] dk\, dT$$

$$= X E\left[\int_0^{+\infty} e^{-(r+\alpha)T} I_{\{\tau_b < T\}} \int_{-\log(S_T/X)}^{+\infty} e^{-\psi k} \left(\frac{S_T}{X} - e^{-k}\right)^+ dk\, dT\right]$$

$$= \frac{1}{\psi(\psi + 1)X^\psi} E\left[\int_0^{+\infty} e^{-(r+\alpha)T} I_{\{\tau_b < T\}} S_T^{\psi+1}\, dT\right]$$

$$= \frac{1}{\psi(\psi + 1)X^\psi} E\left[\int_{\tau_b}^{+\infty} e^{-(r+\alpha)T} S_T^{\psi+1}\, dT\right]$$

$$= \frac{1}{\psi(\psi + 1)X^\psi} E\left[e^{-(r+\alpha)\tau_b} \int_0^{+\infty} e^{-(r+\alpha)t} S_{t+\tau_b}^{\psi+1}\, dt\right],$$

where the last equality holds because of a change of variables $T = t + \tau_b$. On the other hand, the strong Markov property of the return process $\{X_t\}$ implies that for any $\alpha > G(\psi + 1) - r$,

$$E\left\{\left[e^{-(r+\alpha)\tau_b} \int_0^{+\infty} e^{-(r+\alpha)t} S_{t+\tau_b}^{\psi+1}\, dt\right] \middle| \mathcal{F}_{\tau_b}\right\}$$

$$= e^{-(r+\alpha)\tau_b} S_{\tau_b}^{\psi+1} \int_0^{+\infty} e^{-(r+\alpha)t} E[e^{(\psi+1)X_t}]\, dt$$

$$= e^{-(r+\alpha)\tau_b} S_{\tau_b}^{\psi+1} \int_0^{+\infty} e^{-(r+\alpha - G(\psi+1))t}\, dt$$

$$= \frac{S_0^{\psi+1} e^{-(r+\alpha)\tau_b + (\psi+1)X_{\tau_b}}}{r + \alpha - G(\psi + 1)},$$

where $G(\cdot)$ is the exponent of $\{X_t\}$. Combining them together and applying (9) with $x = 0$ yields (22) immediately. Then (23) can be obtained by interchanging derivatives and integrals based on Theorem A.12 of Schiff (1999, pp. 203–204). □

Appendix B. Hyperexponential Distributions and Completely Monotone Distributions

A distribution with the pdf $h(x)$ for $x \ge 0$ is completely monotone if the function $h(x)$ is completely monotone, i.e., $h^{(k)}(x)$ exists for any $k \ge 1$ and $(-1)^k h^{(k)}(x) \ge 0$ for any $x > 0$ and $k \ge 1$ (see, e.g., Feldmann and Whitt 1998). A distribution with the pdf $h(x)$ for $x \in (-\infty, +\infty)$ is completely

monotone if the two functions $h(x)I_{\{x \geq 0\}}$ and $h(-x)I_{\{x \geq 0\}}$ are both completely monotone. Without loss of generality, from now on we assume that the supports of all the cdf's are $(0, +\infty)$. For any completely monotone distribution with the cdf $F(x)$, there must exist a sequence of hyperexponential distributions that converge to $F(x)$ weakly (see, e.g., Feldmann and Whitt 1998, p. 256). The following proposition shows the converse under some conditions.

PROPOSITION B.1. *Consider a sequence of hyperexponential distributions (with the cdf's $\{F_n(x)\}$ and the pdf's $\{f_n(x)\}$), which converge to a continuous distribution (with the cdf $F(x)$ and the pdf $f(x)$) weakly, namely, $\lim_{n \to +\infty} F_n(x) = F(x)$ for any $x > 0$. Assume that (i) $f^{(k)}(x)$ exists for any $x > 0$ and $k \geq 1$, and (ii) $\lim_{n \to +\infty} f_n^{(k)}(x) = f^{(k)}(x)$ for any $x > 0$ and $k \geq 1$. Then $f(x)$ is completely monotone.*

PROOF. Because the hyperexponential distribution is completely monotone, it follows that $(-1)^k f_n^{(k)}(x) \geq 0$ for any $x > 0$ and $k, n \geq 1$. Then by assumption (ii), we have $(-1)^k f^{(k)}(x) = \lim_{n \to +\infty} (-1)^k f_n^{(k)}(x) \geq 0$ for any $x > 0$ and $k \geq 1$, which implies that $f(x)$ is completely monotone. $\quad\square$

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