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ON PRICING OF DISCRETE BARRIER OPTIONS

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Abstract: A barrier option is a derivative contract that is activated or extinguished when the price of the underlying asset crosses a certain level. Most models assume continuous monitoring of the barrier. However in practice most, if not all, barrier options traded in markets are discretely monitored. Unlike their continuous counterparts, there is essentially no closed form solution available, and even numerical pricing is difficult. This paper extends an approximation by Broadie, Glasserman and Kou (1997) for discretely monitored barrier options by covering more cases and giving a simpler proof. The techniques used here come from sequential analysis, particularly Siegmund and Yuh (1982) and Siegmund (1985).

Key words and phrases: Girsanov theorem, level crossing probabilities, Siegmund's corrected diffusion approximation.

1. Introduction

A barrier option is a financial derivative contract that is activated (knocked in) or extinguished (knocked out) when the price of the underlying asset (which could be a stock, an index, an exchange rate, an interest rate, etc.) crosses a certain level (called a barrier). For example, an up-and-out call option gives the option holder the payoff of a European call option if the price of the underlying asset does not reach a higher barrier level before the expiration date. More complicated barrier options may have two barriers (double barrier options), and may have the final payoff determined by one asset and the barrier level determined by another asset (two-dimensional barrier options). Taken together, they are among the most popular path-dependent options traded in exchanges worldwide and also in over-the-counter markets. This paper focuses exclusively on one-dimensional, single barrier options, which include eight possible types: up (down)-and-in (out) call (put) options.

An important issue of pricing barrier options is whether the barrier crossing is monitored in continuous time. Most models assume continuous monitoring of the barrier. In other words, in the models a knock-in or knock-out occurs if the barrier is reached at any instant before the expiration date, mainly because this leads to analytical solutions; see, for example, Merton (1973), Heynen and Kat (1994a, 1994b) and Kunitomo and Ikeda (1992) for various formulae for continuously monitored barrier options under the classical Brownian motion framework; see Kou and Wang (2001) for continuously monitored barrier options under a jump-diffusion framework.

However in practice most, if not all, barrier options traded in markets are discretely monitored. In other words, they specify fixed times for monitoring of the barrier (typically daily closings). Besides practical implementation issues, there are some legal and financial reasons why discretely monitored barrier options are preferred to continuously monitored barrier options. For example, some discussions in trader's literature ("*Derivatives Week*", May 29th, 1995) voice concern that, when the monitoring is continuous, extraneous barrier breach may occur in less liquid markets while the major western markets are closed, and may lead to certain arbitrage opportunities.

Although discretely monitored barrier options are popular and important, pricing them is not as easy as that of their continuous counterparts for three reasons. (1) There are essentially no closed solutions, except using m-dimensional normal distribution functions (m is the number of monitoring points), which can hardly be computed easily if, for example, $m > 5$; see Reiner (2000). (2) Direct Monte Carlo simulation or standard binomial trees may be difficult, and can take hours or even days to produce accurate results; see Broadie, Glasserman and Kou (1999). (3) Although the Central Limit Theorem asserts that, as $m \to \infty$, the difference between the discretely and continuously monitored barrier options should be small, it is well known in the trader's literature that numerically the difference can be surprisingly large, even for large m.

To deal with these difficulties, Broadie, Glasserman and Kou (1997) propose a continuity correction for the discretely monitored barrier option, and justify the correction both theoretically and numerically (Chuang (1996) independently suggests the approximation in a heuristic way). The resulting approximation, which only relies on a simple correction to the Merton (1973) formula (thus trivial to implement), is nevertheless quite accurate and has been used in practice; see, for example, the textbook by Hull (2000). The idea goes back to a classical technique in "sequential analysis," in which corrections to normal approximation are used to adjust for the "overshoot" effects when a discrete random walk crosses a barrier; see, for example, Chernoff (1965), Siegmund (1985a) and Woodroofe (1982). Therefore, from a statistical point of view, it is an interesting application of sequential analysis to a real life problem.

The goal of the current paper is twofold. (1) A new and shorter proof of Broadie, Glasserman and Kou (1997) is given, which makes the link between the sequential analysis and the barrier correction formula more transparent. (2) The new proof covers all eight cases of the barrier options, while the proof in the original paper covers only four of them. This is made possible by the results of

Siegmund and Yuh (1982) and Siegmund (1985a, pp. 220-224), and a change of measure argument via a simple discrete Girsanov theorem.

While this paper was under the review in 2001, an independent work by Hörfelt (2003) was brought to the author's attention, in which a similar method is proposed. However, the two methods lead to slightly different barrier correction formulae.

The mathematical formulation of the problem and the main result are stated in the next section, while the proof is deferred to Section 3. Discussion is given in the last section.

2. Main Result

We assume the price of the underlying asset $S(t)$, $t \geq 0$, satisfies

$$S(t) = S(0) \exp \{\mu t + \sigma B(t)\},$$

where under the risk-neutral probability P^*, the drift is $\mu = r - \sigma^2/2$, r is the risk-free interest rate and $B(t)$ is a standard Brownian motion under P^*. In the continuously monitored case, standard finance theory implies that the price of a barrier option will be the expectation, taken with respect to the risk-neutral measure P^*, of the discounted (with the discount factor being e^{-rT}, T the expiration date of the option) payoff of the option. For example, the price of a continuous up-and-out call option is given by

$$V(H) = \mathsf{E}^*(e^{-rT}(S(T) - K)^+ I(\tau(H,S) > T)),$$

where $K \geq 0$ is the strike price, $H > S(0)$ is the barrier and, for any process $Y(t)$, the notation $\tau(x,Y)$ means that $\tau(x,Y) := \inf\{t \geq 0 : Y(t) \geq x\}$; the price of a continuous down-and-in put option is given by

$$V(H) = \mathsf{E}^*(e^{-rT}(K - S(T))^+ I(\tilde{\tau}(H,S) \leq T)),$$

where $H < S(0)$ is the barrier, and $\tilde{\tau}(x,Y) = \inf\{t \geq 0 : Y(t) \leq x\}$. The other six types of the barrier options can be priced similarly. In the Brownian motion framework, all eight types of the barrier options can be priced in closed forms; see Merton (1973).

In the discretely monitoring case, under the risk neutral measure P^*, at the n-th monitoring point, $n\Delta t$, with $\Delta t = T/m$, the asset price is given by

$$S_n = S(0) \exp\left\{\mu n \Delta t + \sigma\sqrt{\Delta t}\sum_{i=1}^{n} Z_i\right\} = S(0)\exp(W_n \sigma \sqrt{\Delta t}), \quad n = 1, \ldots, m,$$

where the random walk W_n is defined by

$$W_n := \sum_{i=1}^{n}\left(Z_i + \frac{\mu}{\sigma}\sqrt{\Delta t}\right),$$

the drift is given by $\mu = r - \sigma^2/2$, and the Z_i's are independent standard normal random variables. By analogy, the price of the discrete up-and-out-call option is given by

$$\begin{aligned} V_m(H) &= \mathsf{E}^*(e^{-rT}(S_m - K)^+ I(\tau'(H,S) > m)) \\ &= \mathsf{E}^*\{e^{-rT}(S_m - K)^+ I\{\tau'(a/(\sigma\sqrt{T}), W) > m\}\}, \end{aligned}$$

where $a := \log(H/S(0)) > 0$, $\tau'(H,S) = \inf\{n \geq 1 : S_n \geq H\}$, $\tau'(x,W) = \inf\{n \geq 1 : W_n \geq x\sqrt{m}\}$. Note that in this case, we have a first passage problem for the random walk W_n, with small drift ($(\mu/\sigma)\sqrt{\Delta t} \to 0$, as $m \to \infty$), to cross a high boundary ($a\sqrt{m}/(\sigma\sqrt{T}) \to \infty$, as $m \to \infty$). The other seven types of discrete barrier options can be represented similarly. Since there is essentially no closed form solution for the discrete barrier options, the following result provides an approximation for the prices.

Theorem 2.1. *Let $V(H)$ be the price of a continuous barrier option, and $V_m(H)$ be the price of an otherwise identical barrier option with m monitoring points. Then for any of the eight discrete monitored regular barrier options, we have the approximation*

$$V_m(H) = V(He^{\pm\beta\sigma\sqrt{T/m}}) + o(1/\sqrt{m}), \tag{2.1}$$

with $+$ for an up option and $-$ for a down option, where the constant $\beta = -(\zeta(1/2)/\sqrt{2\pi}) \approx 0.5826$, ζ the Riemann zeta function.

Remark. The above result was proposed in Broadie, Glasserman and Kou (1997), where it is proved for four cases: down-and-in call, down-and-out call, up-and-in put, and up-and-out put. We cover all eight cases with a simpler proof. It may be worth commenting briefly why this is possible. To compute the option price, one can approximate the expectation directly via an integration of some asymptotic expansion of the probability, as Broadie, Glasserman and Kou (1997) have done. However, the asymptotic expansion breaks down when the integration is performed towards (rather than away from) the boundary, as in the four cases not covered in that paper; see Siegmund (1985b) for another discussion of this technical difficulty in the context of sequential analysis. Here we avoid the problem by using a discrete Girsanov theorem to transform the computation of the expectation to the computation of a probability under a new measure; in fact, no integration is required after the transformation.

To give a feel for the accuracy of the approximation, Table 2.1 is taken from Broadie, Glasserman and Kou (1997). The numerical results suggest that, even for daily monitored discrete barrier options, there can still be big differences between the discrete prices and the continuous prices. The improvement from using the approximation, which shifts the barrier from H to $He^{\pm\beta\sigma\sqrt{T/m}}$ in the continuous time formulae, is significant.

Table 2.1. Up-and-Out Call Option Price Results, $m = 50$ (daily monitoring). This table is taken from Table 2.6 in Broadie, Glasserman and Kou (1997). The option parameters are $S(0) = 110$, $K = 100$, $\sigma = 0.30$ per year, $r = 0.1$, and $T = 0.2$ year, which represents roughly 50 trading days.

Barrier	Continuous Barrier	Corrected Barrier eq. (2.1)	True	Relative error of eq. (2.1) (in percent)
155	12.775	12.905	12.894	0.1
150	12.240	12.448	12.431	0.1
145	11.395	11.707	11.684	0.2
140	10.144	10.581	10.551	0.3
135	8.433	8.994	8.959	0.4
130	6.314	6.959	6.922	0.5
125	4.012	4.649	4.616	0.7
120	1.938	2.442	2.418	1.0
115	0.545	0.819	0.807	1.5

3. Proof of Theorem 2.1

We first prove the case of the discrete up-and-out call option case (of course with $H \geq K$ and $H > S(0)$). Note that this case is not covered by the theorem in Broadie, Glasserman and Kou (1997). The following results are needed in the proof.

Proposition 3.1. (Discrete Girsanov Theorem). *For any probability measure* P*, let* $\hat{\mathsf{P}}$ *be defined by*

$$\frac{d\hat{\mathsf{P}}}{d\mathsf{P}} = \exp\left\{\sum_{i=1}^{m} a_i Z_i - \frac{1}{2}\sum_{i=1}^{m} a_i^2\right\},$$

where the a_i*,* $i = 1, \ldots, n$*, are arbitrary constants, and the* Z_i*'s are standard normal random variables under the probability measure* P*. Then under the probability measure* $\hat{\mathsf{P}}$*, for every* $1 \leq i \leq m$*,* $\hat{Z}_i := Z_i - a_i$ *is a standard normal random variable.*

The proof of this follows easily by checking the likelihood ratio identity; see Karatzas and Shreve (1991, p.190).

Proposition 3.2. (Rescaling Property). *For Brownian motions with drifts* $\alpha\mu$ *and* μ*, and standard deviation* 1*,*

$$\mathsf{P}\left(W_{\alpha\mu}(1) \geq x,\, \tau(c, W_{\alpha\mu}) > 1\right) = \mathsf{P}\left(W_\mu(\alpha^2) \geq x\alpha,\, \tau(\alpha c, W_\mu) > \alpha^2\right),$$

where the notation $W_c(t)$ means a Brownian motion with drift c and standard deviation 1.

Proof. This holds because the process $W(t) = \alpha\mu t + B(t)$ has the same joint distribution as that of the process

$$\alpha\mu t + \frac{B(\alpha^2 t)}{\alpha} = \frac{\alpha^2\mu t + B(\alpha^2 t)}{\alpha}.$$

For a standard Brownian motion $B(t)$, define some stopping times for discrete random walk and for continuous-time Brownian motion as

$$\tau'(b,U) := \inf\{n \geq 1 : U_n \geq b\sqrt{m}\}, \quad \tilde{\tau}'(b,U) := \inf\{n \geq 1 : U_n \leq b\sqrt{m}\},$$
$$\tau(b,U) := \inf\{t \geq 0 : U(T) \geq b\}, \quad \tilde{\tau}(b,U) := \inf\{t \geq 0 : U(T) \leq b\}.$$

Here $U(t) := vt + B(t)$ and U_n is a random walk with a small drift (as $m \to \infty$),

$$U_n := \sum_{i=1}^{n} \left(Z_i + \frac{v}{\sqrt{m}} \right),$$

where the Z_i's are independent standard normal random variables.

Theorem 3.1 (Siegmund-Yuh (1982), Siegmund (1985a, pp. 220-224)). *For any constants $b \geq y$ and $b > 0$, as $m \to \infty$,*

$$\mathsf{P}\left(U_m < y\sqrt{m},\, \tau'(b,U) \leq m\right) = \mathsf{P}(U(1) \leq y,\, \tau(b+\beta/\sqrt{m}, U) \leq 1) + o(1/\sqrt{m}), \quad (3.1)$$

where $\beta = -(\zeta(1/2)/\sqrt{2\pi})$.

The constant β was calculated in Chernoff (1965).

Corollary 3.1. *For any constants $b \geq y$ and $b > 0$,*

$$\mathsf{P}\left(U_m \geq y\sqrt{m},\, \tau'(b,U) > m\right) = \mathsf{P}(U(1) \geq y,\, \tau(b+\beta/\sqrt{m}, U) > 1) + o(1/\sqrt{m}). \quad (3.2)$$

Note that the range of U_m in (3.2) includes the boundary b, while the range of U_m in (3.1) excludes the boundary b.

Proof. Simple algebra yields

$$\begin{aligned}
&\mathsf{P}\left(U_m \geq y\sqrt{m},\, \tau'(b,U) > m\right) \\
&= \mathsf{P}\left(\tau'(b,U) > m\right) - \mathsf{P}\left(U_m < y\sqrt{m},\, \tau'(b,U) > m\right) \\
&= \mathsf{P}\left(U_m < b\sqrt{m},\, \tau'(b,U) > m\right) - \mathsf{P}\left(U_m < y\sqrt{m},\, \tau'(b,U) > m\right) \\
&= \mathsf{P}\left(U_m < b\sqrt{m}\right) - \mathsf{P}\left(U_m < b\sqrt{m},\, \tau'(b,U) \leq m\right) - \mathsf{P}\left(U_m < y\sqrt{m}\right) \\
&\quad + \mathsf{P}\left(U_m < y\sqrt{m},\, \tau'(b,U) \leq m\right).
\end{aligned}$$

Since by Theorem 3.1,

$$\mathsf{P}\left(U_m < b\sqrt{m},\, \tau'(b,U) \le m\right) = \mathsf{P}\left(U(1) \le b,\, \tau(b+\beta/\sqrt{m},U) \le 1\right) + o(1/\sqrt{m}),$$
$$\mathsf{P}\left(U_m < y\sqrt{m},\, \tau'(b,U) \le m\right) = \mathsf{P}\left(U(1) \le y,\, \tau(b+\beta/\sqrt{m},U) \le 1\right) + o(1/\sqrt{m}),$$

we have

$$\begin{aligned}
&\mathsf{P}\left(U_m \ge y\sqrt{m},\, \tau'(b,U) > m\right)\\
&= \mathsf{P}\left(U(1) \le b\right) - \mathsf{P}\left(U(1) \le b,\, \tau(b+\beta/\sqrt{m},U) \le 1\right) - \mathsf{P}\left(U(1) \le y\right)\\
&\quad + \mathsf{P}\left(U(1) \le y,\, \tau(b+\beta/\sqrt{m},U) \le 1\right) + o(1/\sqrt{m})\\
&= \mathsf{P}\left(\tau(b+\beta/\sqrt{m},U) > 1\right) - \mathsf{P}\left(U(1) \le y,\, \tau(b+\beta/\sqrt{m},U) > 1\right) + o(1/\sqrt{m})\\
&= \mathsf{P}(U(1) \ge y,\, \tau(b+\beta/\sqrt{m},U) > 1) + o(1/\sqrt{m}),
\end{aligned}$$

from which the corollary is proved.

Corollary 3.2. *For any constants $b \le y$ and $b < 0$, as $m \to \infty$,*

$$\mathsf{P}\left(U_m > y\sqrt{m},\, \tilde{\tau}'(b,U) \le m\right) = \mathsf{P}(U(1) \ge y,\, \tilde{\tau}(b-\beta/\sqrt{m},U) \le 1) + o(1/\sqrt{m}), \quad (3.3)$$
$$\mathsf{P}\left(U_m \le y\sqrt{m},\, \tilde{\tau}'(b,U) > m\right) = \mathsf{P}(U(1) \le y,\, \tilde{\tau}(b-\beta/\sqrt{m},U) > 1) + o(1/\sqrt{m}). \quad (3.4)$$

Proof. This follows easily by using $-U$ and $-U_m$ in (3.1) and (3.2).

Proof of Theorem 2.1 for the Case of the Up-and-Out Call Option. Note that

$$\begin{aligned}
&\mathsf{E}^*(e^{-rT}(S_m - K)^+ I(\tau'(H,S) > m))\\
&= \mathsf{E}^*(e^{-rT}(S_m - K) I(S_m \ge K,\, \tau'(H,S) > m))\\
&= \mathsf{E}^*(e^{-rT} S_m I(S_m \ge K,\, \tau'(H,S) > m)) - Ke^{-rT}\mathsf{P}^*(S_m \ge K,\, \tau'(H,S) > m)\\
&= I - Ke^{-rT} \cdot II \quad \text{(say)}.
\end{aligned}$$

Using the discrete Girsanov theorem in Proposition 3.1, with $a_i = \sigma\sqrt{\Delta t}$, we have that the first term is given by

$$\begin{aligned}
I &= \mathsf{E}^*\left(e^{-rT} S(0) \exp\left\{\mu m \Delta t + \sigma\sqrt{\Delta t}\sum_{i=1}^m Z_i\right\} I(S_m \ge K,\, \tau'(H,S) > m)\right)\\
&= S(0)\mathsf{E}^*\left[\exp\left\{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{\Delta t}\sum_{i=1}^m Z_i\right\} I(S_m \ge K,\, \tau'(H,S) > m)\right]\\
&= S(0)\hat{\mathsf{E}}(I(S_m \ge K, \tau'(H,S) > m))\\
&= S(0)\hat{\mathsf{P}}(S_m \ge K,\, \tau'(H,S) > m).
\end{aligned}$$

Under $\hat{\mathsf{P}}$, $\log S_m$ has a mean $\mu m\Delta t + \sigma\sqrt{\Delta t}\cdot m\sigma\sqrt{\Delta t} = (\mu+\sigma^2)T$ instead of μT under the measure P^*. Therefore, the price of a discrete up-and-out-call option is given by

$$V_m(H) = S(0)\hat{\mathsf{P}}\left(W_m \geq \frac{\log(K/S(0))}{\sigma\sqrt{\Delta t}},\ \tau'(a/(\sigma\sqrt{T}), W) > m\right)$$
$$-Ke^{-rT}\mathsf{P}^*\left(W_m \geq \frac{\log(K/S(0))}{\sigma\sqrt{\Delta t}},\ \tau'(a/(\sigma\sqrt{T}), W) > m\right),$$

where under $\hat{\mathsf{P}}$, $W_m = \sum_{i=1}^m (\hat{Z}_i + \{\{(\mu+\sigma^2)/\sigma\}\sqrt{T}/m\})$ and under P^*, $W_m = \sum_{i=1}^m (Z_i + \{(\mu/\sigma)\sqrt{T}/m\})$, where $\hat{Z}_i$ and Z_i being standard normal random variables under $\hat{\mathsf{P}}$ and P^*, respectively.

Now using (3.2) in Corollary 3.1 with

$$y = \frac{\log(K/S(0))}{\sigma\sqrt{T}}, \quad b = \frac{a}{\sigma\sqrt{T}} = \frac{\log(H/S(0))}{\sigma\sqrt{T}} \geq y,$$

yields, as $m \to \infty$,

$$V_m(H) = S(0)\mathsf{P}\left(W_{\frac{(\mu+\sigma^2)\sqrt{T}}{\sigma}}(1) \geq \frac{\log(K/S(0))}{\sigma\sqrt{T}},\ \tau(b+\beta/\sqrt{m}, W_{\frac{(\mu+\sigma^2)\sqrt{T}}{\sigma}}) > 1\right)$$
$$-Ke^{-rT}\mathsf{P}\left(W_{\frac{\mu\sqrt{T}}{\sigma}}(1) \geq \frac{\log(K/S(0))}{\sigma\sqrt{T}},\ \tau(b+\beta/\sqrt{m}, W_{\frac{\mu\sqrt{T}}{\sigma}}) > 1\right)$$
$$+o(1/\sqrt{m}),$$

where the notation $W_c(t)$ means a Brownian motion with drift c and standard deviation 1. By Proposition 3.2, we get

$$V_m(H) = S(0)\mathsf{P}\left(W_{\frac{\mu+\sigma^2}{\sigma}}(T) \geq \frac{\log(K/S(0))}{\sigma},\ \tau(b\sqrt{T}+\beta\sqrt{T/m}, W_{\frac{\mu+\sigma^2}{\sigma}}) > T\right)$$
$$-Ke^{-rT}\mathsf{P}\left(W_{\frac{\mu}{\sigma}}(T) \geq \frac{\log(K/S(0))}{\sigma},\ \tau(b\sqrt{T}+\beta\sqrt{T/m}, W_{\frac{\mu}{\sigma}}) > T\right)$$
$$+o(1/\sqrt{m}).$$

Since $\tau(b\sqrt{T}+\beta\sqrt{T/m}, W) = \tau(a/\sigma+\beta\sqrt{T/m}, W) = \tau(He^{\beta\sigma\sqrt{T/m}}, S)$, we have

$$V_m(H) = S(0)\mathsf{P}\left(S(0)e^{(\mu+\sigma^2)T+\sigma B(T)} \geq K,\ \tau(He^{\beta\sigma\sqrt{T/m}}, S) > T\right)$$
$$-Ke^{-rT}\mathsf{P}\left(S(0)e^{\mu T+\sigma B(T)} \geq K, \tau(He^{\beta\sigma\sqrt{T/m}}, S) > T\right) + o(1/\sqrt{m}). \quad (3.5)$$

Similarly, by using the continuous time Girsanov theorem, the continuous time price $V(H)$ can be written as

$$V(H) = S(0)\mathsf{P}\left(S(0)e^{(\mu+\sigma^2)T+\sigma B(T)} \geq K,\ \tau(H,S) > T\right)$$
$$-Ke^{-rT}\mathsf{P}\left(S(0)e^{\mu T+\sigma B(T)} \geq K, \tau(H,S) > T\right). \quad (3.6)$$

Comparing (3.5) and (3.6) yields $V_m(H) = V(He^{\beta\sigma\sqrt{T/m}}) + o(1/\sqrt{m})$, from which the result of the up-and-out call option is proved.

The results for the other seven options can be derived easily. (1) The case of up-and-in put follows by using (3.1) directly in the above proof instead of using (3.2). (2) The case of down-and-in call follows by using (3.3) in the above proof. (3) The case of down-and-out put follows by using (3.4) in the above proof. (4) The cases of the up-and-in call, up-and-out put, down-and-out call, and down-and-in put follow readily, because the sum of two otherwise identical in- and out-put (call) options is a regular put (call) option.

Now all eight cases of barrier options have been proved.

4. Discussion

This paper simplifies the proof in Broadie, Glasserman and Kou (1997) paper, and generalizes the result to include more cases of discrete barrier options. The method used in this paper can be applied to study other problems. For example, in a forthcoming paper with Menghui Cao, the method here, along with those in Broadie, Glasserman and Kou (1997), is used to derive barrier correction formulae for two-dimensional barrier options and partial barrier options.

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