SIEO W3600 Solution to Homework 7

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Chapter 5

22) a) **Solution:**

$$\begin{aligned} E(X+Y) &= \sum_x \sum_y (x+y)p(x,y) \\ &= (0+0)(0.02)+(0+5)(0.06)+...+(10+15)(0.01) \\ &= 14.10 \end{aligned}$$

b) **Solution:**

$$\begin{aligned} E[max(X,Y)] &= \sum_x \sum_y max(x+y)p(x,y) \\ &= (0)(0.02)+(5)(0.06)+...+(15)(0.01) \\ &= 0.0124 \end{aligned}$$

27) **Solution:** The amount of time that they have to wait for each other is $h(X,Y)=|X-Y|$, so

$$\begin{aligned} E[h(X,Y)] &= \int_0^1 \int_0^1 |x-y| \cdot f(x,y)dxdy \\ &= \int_0^1 \int_0^1 |x-y| \cdot 6x^2 ydxdy \\ &= \int_0^1 \int_0^x (x-y) \cdot 6x^2 ydxdy + \int_0^1 \int_x^1 (y-x) \cdot 6x^2 ydxdy \\ &= \frac{1}{6}+\frac{1}{12} \\ &= \frac{1}{4} \end{aligned}$$

31) a) **Solution:**

$$\begin{aligned} E[Y] = E[X] &= \int_{20}^{30} x f_x(x)dx = \int_{20}^{30} x[10Kx^2+0.05]dx \\ &= 25.329 \\ E[XY] &= \int_{20}^{30}\int_{20}^{30} xy \cdot K(x^2+y^2)dxdy \\ &= 641.447 \end{aligned}$$

So, $\quad Cov(X,Y)=E[XY]-E[X]E[Y]=641.447-(25.329)^2=-0.111$

b) **Solution:**

$$E[Y^2]=E[X^2]=\int_{20}^{30} x^2(10Kx^2+0.05)dx = 649.8246$$

So, $\quad Var(Y)=Var(X)=E(X^2)-[E(X)]^2=649.8246-(25.329)^2=8.2664$

Thus,

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-0.111}{8.2664} = -0.0134$$

36) **Solution:**

$$Cov(X,Y) = Cov(X, aX+b) = E[X \cdot (aX+b)] - E(X)E(aX+b) = aVar(X)$$

$$Corr(X,Y) = \frac{aVar(X)}{\sqrt{Var(X)Var(Y)}} = \frac{aVar(X)}{\sqrt{Var(X) \cdot a^2 Var(X)}} = \begin{cases} 1 & \text{if} \quad a > 0 \\ -1 & \text{if} \quad a < 0 \end{cases}$$

58) a) **Solution:**

$$\begin{aligned} & E(27X_1 + 125X_2 + 512X_3) = 27E(X_1) + 125E(X_2) + 512E(X_3) \\ = \; & 27(200) + 125(250) + 512(100) \\ = \; & 87,850 \\ & Var(27X_1 + 125X_2 + 512X_3) = 27^2 Var(X_1) + 125^2 Var(X_2) + 512^2 Var(X_3) \\ = \; & 27^2(10)^2 + 125^2(12)^2 + 512^2(8)^2 \\ = \; & 19,100,116 \end{aligned}$$

b) **Solution:** If the $X_i's$ are not independent, then the expected value calculated above is still correct, but the variance is not because we have to include the covariance into variance.

60) **Solution:** Note that $Y = \frac{X_1+X_2}{2} - \frac{X_3+X_4+X_5}{3}$, and each $X_i's$ are independent and normally distributed. So Y is normally distributed with mean $\mu_Y = \frac{1}{2}(\mu_1+\mu_2) - \frac{1}{3}(\mu_3+\mu_4+\mu_5) = -1$ and variance $\sigma_Y^2 = \frac{1}{4}\sigma_1^2 + \frac{1}{4}\sigma_2^2 + \frac{1}{9}\sigma_3^2 + \frac{1}{9}\sigma_4^2 + \frac{1}{9}\sigma_5^2 = 3.167$ and $\sigma_Y = 1.7795$
Thus,

$$P(0 \le Y) = P\left(\frac{0-(-1)}{1.7795} \le Z\right) = P(0.56 \le Z) = 0.2877$$

$$P(-1 \le Y \le 1) = P\left(0 \le Z \le \frac{2}{1.7795}\right) = P(0 \le Z \le 1.12) = 0.3686$$

63) a) **Solution:** From the joint pmf of $X_1$and X_2, it is easy to calculate

$$E(X_1) = 1.7 \quad E(X_2) = 1.55\text{and} \quad E(X_1X_2) = \sum_{x_1}\sum_{x_2} x_1x_2p(x_1,x_2) = 3.33$$

Thus, $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = 3.33 - 2.635 = 0.695$

b) **Solution:**

$$V(X_1+X_2) = V(X_1) + V(X_2) + 2Cov(X_1,X_2) = 1.59 + 1.0875 + 2(0.695) = 4.0675$$

77) a) **Solution:**

$$\begin{aligned} 1 = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy \\ = & \int_0^{20}\int_{20-x}^{30-x} kxydxdy + \int_{20}^{30}\int_0^{30-x} kxydxdy \\ = & \frac{81,250}{3} \end{aligned}$$

Thus, $k = \frac{3}{81,250}$

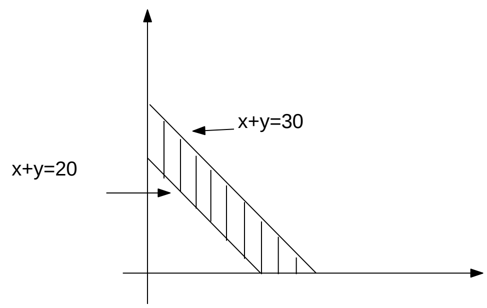

Figure 1: The region of positive denstiy

b) **Solution:** For $0 \le x \le 20$,we have

$$f_X(x) = \int_{20-x}^{30-x} kxydy = k(250x - 10x^2)$$

For $20 \le x \le 30$, we get

$$f_X(x) = \int_{0}^{30-x} kxydy = k(450x - 30x^2 + \frac{1}{2}x^3)$$

By symmetry of X and Y, $f_Y(y)$ is obtained by substituting y for x in $f_X(x)$. Clearly, $f_X(x) \cdot f_Y(y) \neq f(x, y)$, so X and Y are not independent.

c) **Solution:**

$$\begin{aligned} P(X+Y \le 25) &= \int_0^{20} \int_{20-x}^{25-x} kxydxdy + \int_{20}^{25} \int_{0}^{25-x} kxydxdy \\ &= k \cdot \frac{230,625}{24} = \frac{3}{81,250} \cdot (9609.375) \\ &= 0.355 \end{aligned}$$

d) **Solution:**

$$\begin{aligned} & E(X+Y) = E(X) + E(Y) = 2E(X) \\ = & \ 2\left[\int_0^{20} x \cdot k(250x - 10x^2)dx + \int_{20}^{30} x \cdot k(450x - 30x^2 + \frac{1}{2}x^3)dx\right] \\ = & \ 2 \cdot \frac{3}{81,250} \cdot (352,666.67) = 25.969 \end{aligned}$$

e) **Solution:**

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \cdot f(x,y)dxdy \\ &= \int_0^{20} \int_{20-x}^{30-x} kx^2y^2dydx + \int_{20}^{30} \int_{0}^{30-x} kx^2y^2dydx \\ &= \frac{k}{3} \cdot \frac{33,250,000}{3} = 136.4103 \end{aligned}$$

And, from part(d), we know $E(X) = \frac{25.969}{2} = 12.9845$. So,

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 136.4103 - (12.9845)^2 = -32.19$$

$$E(Y^2) = E(X^2) = \int_0^{20} x^2 \cdot k(250x - 10x^2)dx + \int_{20}^{30} x^2 \cdot k(450x - 30x^2 + \frac{1}{2}x^3)dx = 204.6154$$

Thus, $\sigma_y^2 = \sigma_x^2 = 204.6154 - (12.9845)^2 = 36.0182$
and

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-32.19}{36.0182} = -0.894$$

f) **Solution:**

$$Var(X+Y) = var(X) + Var(Y) + 2Cov(X,Y) = 2 \times 36.0182 + 2 \times (-32.19) = 7.66$$