Dynamic Identification of DSGE Models: Supplementary Results

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This supplementary document contains additional examples. Matlab code for constructing the $\Delta^{\text{S}}(\theta)$ and $\Delta^{\text{NS}}(\theta)$ matrices proposed in Komunjer and Ng (2011) is also provided to show that while the expression appears complex, the computation is simple. Once the minimal representation is obtained, $\Delta^{\text{S}}_{\Lambda}(\theta)$ and $\Delta^{\text{NS}}(\theta)$ can be computed using numerical differentiation. The $\Delta^{\text{S}}_{T}(\theta)$ and $\Delta^{\text{NS}}_{T}(\theta)$ only require specification of n_X, while $\Delta^{\text{S}}_{U}(\theta)$ only requires n_ϵ. Section 1 uses the model of An and Schorfheide (2007) to study the implications of (1) adding c_t to the observables, and (2) dropping variables to remove singularity. Section 2 analyzes the model in Smets and Wouters (2007). It is shown that putting the model into minimal state space representation reveals features about the model that are not otherwise transparent. In particular, the parameters in the policy rule, output, and potential output equations are not independent. Section 3 analyzes the model of Christiano, Eichenbaum, and Evans (2005). Section 4 considers the model of Cicco, Pancrazi, and Uribe (2009) that is identifiable without further restrictions. Matlab code for computing the $\Delta(\theta_0)$ matrix is given.

1 The An-Schorfheide Model

We first analyze the model with 4 observed variables, $Y_t = (r_t, y_t, \pi_t, c_t)'$, and the true value of $\theta = (\tau, \beta, \nu, \phi, \overline{\pi}, \psi_1, \psi_2, \rho_r, \rho_g, \rho_z, \sigma_r^2, \sigma_g^2, \sigma_z^2)$ as in Table 3 of An and Schorfheide (2007), $\theta_0 = (2, 0.9975, 0.1, 53.6797, 1.008, 1.5, 0.125, 0.75, 0.95, 0.9, 4\times 10^{-6}, 36\times 10^{-6}, 9\times 10^{-6})$. Let

$$\epsilon_t \equiv \begin{pmatrix} \epsilon_{zt} \\ \epsilon_{gt} \\ \epsilon_{rt} \end{pmatrix} \sim WN\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 4 \end{pmatrix} \times 10^{-6} \right).$$

This gives a state space solution of the form:

$$\widetilde{X}_{t+1} = \begin{pmatrix} z_{t+1} \\ g_{t+1} \\ r_{t+1} \\ y_{t+1} \\ \pi_{t+1} \\ c_{t+1} \\ E_{t+1}(\pi_{t+2}) \\ E_{t+1}(y_{t+2}) \end{pmatrix} = \underbrace{\begin{pmatrix} 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.95 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5450 & 0 & 0.5143 & 0 & 0 & 0 & 0 & 0 \\ 1.3377 & 0.9500 & -0.8258 & 0 & 0 & 0 & 0 & 0 \\ 1.3418 & 0 & -0.5596 & 0 & 0 & 0 & 0 & 0 \\ 1.3377 & 0 & -0.8258 & 0 & 0 & 0 & 0 & 0 \\ 0.9026 & 0 & -0.2878 & 0 & 0 & 0 & 0 & 0 \\ 0.7538 & 0.9025 & -0.4247 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\widetilde{A}(\theta)} \begin{pmatrix} z_t \\ g_t \\ r_t \\ y_t \\ \pi_t \\ c_t \\ E_t(\pi_{t+1}) \\ E_t(y_{t+1}) \end{pmatrix}$$

$$+ \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.6055 & 0 & 0.6858 \\ 1.4863 & 1 & -1.1011 \\ 1.4909 & 0 & -0.7462 \\ 1.4863 & 0 & -1.1011 \\ 1.0029 & 0 & -0.3838 \\ 0.8376 & 0.9500 & -0.5663 \end{pmatrix}}_{\widetilde{B}(\theta)} \begin{pmatrix} \epsilon_{z,t+1} \\ \epsilon_{g,t+1} \\ \epsilon_{r,t+1} \end{pmatrix}$$

$$Y_t = \begin{pmatrix} r_t \\ y_t \\ \pi_t \\ c_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{\widetilde{C}(\theta)} \widetilde{X}_t$$

This state space representation is not minimal, as we have that rank$\mathcal{C} = 3 < n_X = 8$ and rank$\mathcal{O} = 6 < n_X = 8$. We now show how the order of the above system can be reduced. Note that the above state space representation has the following structure:

$$\widetilde{X}_{t+1} = \begin{pmatrix} X_{1,t+1} \\ X_{2,t+1} \end{pmatrix} = \begin{pmatrix} \widetilde{A}_1(\theta) & 0 \\ \widetilde{A}_2(\theta) & 0 \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} + \begin{pmatrix} \widetilde{B}_1(\theta) \\ \widetilde{B}_2(\theta) \end{pmatrix} \epsilon_{t+1}$$
$$Y_{t+1} = \begin{pmatrix} \widetilde{C}_1(\theta) & \widetilde{C}_2(\theta) \end{pmatrix} \begin{pmatrix} X_{1,t+1} \\ X_{2,t+1} \end{pmatrix}$$

In other words, there exists a subvector X_{2t} of the state vector $\widetilde{X}_t$ whose dynamics is entirely driven by the remaining subvector X_{1t} and ϵ_{t+1}. Then, the above system is equivalent to:

$$X_{1,t+1} = \underbrace{\widetilde{A}_1(\theta)}_{A(\theta)} X_{1t} + \underbrace{\widetilde{B}_1(\theta)}_{B(\theta)} \epsilon_{t+1}$$
$$Y_{t+1} = \underbrace{\left(\widetilde{C}_1(\theta)\widetilde{A}_1(\theta) + \widetilde{C}_2(\theta)\widetilde{A}_2(\theta)\right)}_{C(\theta)} X_{1t} + \underbrace{\left(\widetilde{C}_1(\theta)\widetilde{B}_1(\theta) + \widetilde{C}_2(\theta)\widetilde{B}_2(\theta)\right)}_{D(\theta)} \epsilon_{t+1}.$$

In particular, in An and Schorfheide's (2007) model, consider partitioning the state space vector $\widetilde{X}_t$ into $X_{1t} \equiv \left(z_t, g_t, r_t\right)'$ and $X_{2t} \equiv \left(y_t, \pi_t, c_t, E_t(\pi_{t+1}), E_t(y_{t+1})\right)'$. Then we have:

$$X_{1,t+1} = \begin{pmatrix} z_{t+1} \\ g_{t+1} \\ r_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0.5450 & 0 & 0.5143 \end{pmatrix}}_{A(\theta)} X_{1t} + \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.6055 & 0 & 0.6858 \end{pmatrix}}_{B(\theta)} \epsilon_{t+1}$$

$$Y_{t+1} = \begin{pmatrix} r_{t+1} \\ y_{t+1} \\ \pi_{t+1} \\ c_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0.5450 & 0 & 0.5143 \\ 1.3377 & 0.95 & -0.8258 \\ 1.3418 & 0 & -0.5596 \\ 1.3377 & 0 & -0.8258 \end{pmatrix}}_{C(\theta)} X_{1t} + \underbrace{\begin{pmatrix} 0.6055 & 0 & 0.6858 \\ 1.4863 & 1 & -1.1011 \\ 1.4909 & 0 & -0.7462 \\ 1.4863 & 0 & -1.1011 \end{pmatrix}}_{D(\theta)} \epsilon_{t+1}$$

The new system is minimal state, i.e. it is controllable as $\text{rank}\mathcal{C} = 3 = n_X$ as well as observable with $\text{rank}\mathcal{O} = 3 = n_X$.

For completeness, we report results both for the minimal as well as the non-minimal representations of the model. The results in Tables 1 below are qualitatively the same as those in Tables 1 of the paper; in other words, adding c_t to the observables does not affect any conclusions regarding the identification of the model.

Table 1: Full Model: $n_\theta = 13$

τ	β	ν	ϕ	$\overline{\pi}$	ψ_1	ψ_2	ρ_r	ρ_g	ρ_z	$100\sigma_r$	$100\sigma_g$	$100\sigma_z$
2	0.9975	0.1	53.6797	1.008	1.5	0.125	0.75	0.95	0.9	.2	.6	.3

Minimal State Space Representation

$$X_{t+1} = \begin{pmatrix} z_{t+1} \\ g_{t+1} \\ r_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0.5450 & 0 & 0.5143 \end{pmatrix}}_{A(\theta)} X_{1t} + \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.6055 & 0 & 0.6858 \end{pmatrix}}_{B(\theta)} \underbrace{\begin{pmatrix} \epsilon_{zt+1} \\ \epsilon_{gt+1} \\ \epsilon_{rt+1} \end{pmatrix}}_{\epsilon_{t+1}}$$

$$Y_{t+1} = \begin{pmatrix} r_{t+1} \\ y_{t+1} \\ \pi_{t+1} \\ c_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} .5450 & 0 & 0.5143 \\ 1.3377 & 0.95 & -0.8258 \\ 1.3418 & 0 & -0.5596 \\ 1.3377 & 0 & -0.8258 \end{pmatrix}}_{C(\theta)} X_{1t} + \underbrace{\begin{pmatrix} 0.6055 & 0 & 0.6858 \\ 1.4863 & 1 & -1.1011 \\ 1.4909 & 0 & -0.7462 \\ 1.4863 & 0 & -1.1011 \end{pmatrix}}_{D(\theta)} \epsilon_{t+1}$$

	Non-Minimal Model				Minimal Model						
Tol	Δ^S_Λ	Δ^S_U	$\Delta^S_{\Lambda U}$	pass	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda T}$	$\Delta^S_{\Lambda U}$	Δ^S	pass
e-02	11	9	19	No	11	9	9	20	19	28	No
e-03	11	9	19	No	11	9	9	20	19	28	No
e-04	11	9	19	No	11	9	9	20	19	28	No
e-05	11	9	19	No	11	9	9	20	19	28	No
e-06	11	9	19	No	11	9	9	20	19	28	No
e-07	12	9	21	No	11	9	9	20	20	29	No
e-08	12	9	21	No	11	9	9	20	20	29	No
e-09	12	9	21	No	11	9	9	20	20	29	No
e-10	12	9	21	No	12	9	9	21	21	29	No
e-11	12	9	21	No	12	9	9	21	21	29	No
default	13	9	22	1	12	9	9	21	21	30	No
Required	13	9	22		13	9	9	22	22	31	

Full Minimal Model with Restrictions: Tol=1e-3

Restriction			Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda,T}$	$\Delta^S_{\Lambda,U}$	Δ^S	pass
ν	-	-	12	9	9	21	20	29	No
ν	ϕ	-	13	9	9	22	21	30	No
ϕ	$\overline{\pi}$	-	13	9	9	22	21	30	No
ν	$\overline{\pi}$	-	13	9	9	22	21	30	No
β	ϕ	-	12	9	9	21	20	29	No
ϕ	ρ_g	-	12	9	9	21	20	29	No
β	ν	ϕ	13	9	9	22	21	30	No
β	ψ_1	ψ_2	11	9	9	20	20	29	No
ν	ϕ	ψ_1	13	9	9	22	22	31	Yes
ν	ϕ	ψ_2	13	9	9	22	22	31	Yes
τ	ψ_1	ψ_2	11	9	9	20	20	29	No
Required			13	9	9	22	22	31	

Dropping variables (which could make a singular system non-singular) can have an impact on identification. As an example, consider the reparameterized version of An and Schorfheide's model with 11 parameters and 3 shocks. Table 2 on page 27 of Komunjer and Ng (2011) shows that under an additional restriction on ψ_1, the (singular) model is identified from the second moments of $Y_t = (r_t, y_t, \pi_t, c_t)'$.

Suppose we drop one variable at a time so that $n_Y = 3 = n_\epsilon$. The order condition $n_\theta \leqslant 24$ is clearly satisfied. It remains to check the rank condition on $\overline{\Delta}^{\mathrm{S}}(\theta_0)$.

Case 1: r_t is dropped so $Y_t = (y_t, \pi_t, c_t)'$. The new state space system after deleting the first row of the measurement equation (see Table 1) is

$$X_{t+1} = \begin{pmatrix} z_{t+1} \\ g_{t+1} \\ r_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0.5450 & 0 & 0.5143 \end{pmatrix}}_{A(\theta)} X_t + \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.6055 & 0 & 0.6858 \end{pmatrix}}_{B(\theta)} \epsilon_{t+1}$$

$$Y_{t+1} = \begin{pmatrix} y_{t+1} \\ \pi_{t+1} \\ c_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1.3377 & 0.95 & -0.8258 \\ 1.3418 & 0 & -0.5596 \\ 1.3377 & 0 & -0.8258 \end{pmatrix}}_{C(\theta)} X_t + \underbrace{\begin{pmatrix} 1.4863 & 1 & -1.1011 \\ 1.4909 & 0 & -0.7462 \\ 1.4863 & 0 & -1.1011 \end{pmatrix}}_{D(\theta)} \epsilon_{t+1}$$

The square system is still minimal state. Yet, using the tolerance 1e-3 as in Table 2, $\mathrm{rank}\Delta^{\mathrm{S}}_R(\theta_0) = 27 < 29 = n_\theta + n_X^2 + n_\epsilon^2$. Thus θ_0 cannot be identified from the second moments of $\{(y_t, \pi_t, c_t)'\}$ alone.

Case 2: drop y_t from the observables so $Y_t = (r_t, \pi_t, c_t)'$. The new square system is no longer observable as $\mathrm{rank}\mathcal{O} = 2 < n_X = 3$. Without minimality, the rank of $\Delta_{\Lambda U}$ remains necessary for identification. As $\mathrm{rank}\Delta_{\Lambda U}$ is at most 21 and $n_\theta + n_\epsilon^2 = 22$, θ_0 is not identified.

Case 3: drop π_t from the observables so $Y_t = (r_t, y_t, c_t)'$. The new square system is both controllable and observable. But $\mathrm{rank}\Delta^{\mathrm{S}}_R(\theta_0) = 27 < n_\theta + n_X^2 + n_\epsilon^2$ so θ_0 cannot be identified from the second moments of $\{(r_t, y_t, c_t)'\}$.

This example illustrates that dropping some variables from the system can cause identification to fail. It also shows that certain variables can be dropped without altering the identifiability of the model; however, it is not clear how such variables can be chosen a priori.

2 The Smets and Wouters Model

The model estimated by Smets and Wouters (2007) (SW) is widely cited. The sticky price model has real and nominal rigidities. The endogenous variables are output (y_t), consumption (c_t), investment

(i_t), capital services (k^s_t), installed capital (k_t), capacity utilization (z_t), rental rate (r^k_t), Tobin's q (q_t), price markup (μ^p_t), wage markup (μ^w_t), inflation (π_t), real wage (w_t), hours worked (l_t), and the nominal interest rate r_t). The monetary policy rule is specified as

$$r_t = \rho_r r_{t-1} + (1-\rho_r)\big(r_\pi \pi_t + r_y(y_t - y^f_t)\big) + r_{\Delta y}\big((y_t - y^f_t) - (y_{t-1} - y^f_{t-1})\big) + e^r_t$$

where y^f_t is output of the flexible price economy. Thus variables for the flexible price economy (such as consumption) are also relevant, and these have superscript f. The steady state values are defined for inflation ($\overline{\pi}$), output growth ($\overline{\gamma}$), level of hours worked ($\bar{l}$), and the nominal interest rat $\overline{r}$)e. The model has AR(1) shocks: productivity (e^a_t), investment (e^i_t), government spending (e^g_t), risk premium (e^b_t), monetary policy, (e^r_t), and two ARMA(1,1) shocks: wage markup (e^w_t) and price markup (e^p_t). There unknown parameter vector θ is of dimension $n_\theta = 41$. The observables used in estimation are $\Delta y_t, \Delta i_t, \Delta c_t, \Delta w_t, r_t, \pi_t, l_t$.

We take the GENSYS code written by Iskrev (2010) from the JME web site. His implementation of the model consists of 40 equations. Iskrev's specification of the ARMA shocks is different after GENSYS solves the model. We rewrite these two exogenous processes such that it is not altered by GENSYS. Specifically, an arbitrary ARMA(1,1) process y_t with AR parameter ρ and MA parameter θ has state space representation:

$$\begin{aligned} y_{t+1} &= a_t + \eta_{t+1} \\ a_{t+1} &= \rho a_t + (\rho+\theta)\eta_{t+1}. \end{aligned}$$

If y_t were observed, the dimension of the state vector is one, which is smaller than the usual max(p,q+1) formulation as in Harvey (1989), for example.

The minimal state vector is the smallest number of exogenous and endogenous variables necessary to describe the dynamics of the model. To facilitate isolating this vector, the state variables are always ordered first in GENSYS. The Smets-Wouters model has 18 such variables. Thus the first nine equations are for the five AR(1) shocks and two ARMA(1,1) shocks. These are followed by $c_t, i_t, k^s_t, \pi_t, w_t, r_t, c^f_t, i^f_t, k^{sf}_t, y_t, y^f_t$. The remaining 40-18=22 equations (such as l_t) then follow. This solution is determinate. Letting ϵ_t be the 7 innovations, GENSYS gives

$$\begin{aligned} \widetilde{X}_{t+1} = \begin{pmatrix} \widetilde{X}_{1,t+1} \\ \widetilde{X}_{2,t+1} \end{pmatrix} &= \begin{pmatrix} A_1(\theta) & 0 \\ A_2(\theta) & 0 \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} + \begin{pmatrix} B_1(\theta) \\ B_2(\theta) \end{pmatrix} \epsilon_{t+1} \\ Y_{t+1} = \begin{pmatrix} C_1(\theta) & C_2(\theta) \end{pmatrix} &\begin{pmatrix} \widetilde{X}_{1,t+1} \\ \widetilde{X}_{2,t+1} \end{pmatrix} \end{aligned}$$

where $\widetilde{X}_{1t}$ is 18×1. When A_1 is full rank, the minimal state vector can be usually be obtained by finding the columns of zeros in the A matrix, and the rows of zeros in the B matrix. (See the CEE example below).

The Smets and Wouters model is somewhat more complicated because the 18×18 matrix A_1 only has rank 16. An analysis of the null space of A_1 reveals that there is a dependence between row for interest rate r_t, output y_t, and output of the flexible price economy y_t^f. This is not surprising in view of the monetary policy rule which can be rewritten as

$$\begin{aligned} r_t &= \rho \widetilde{r}_{t-1} + (1-\rho)(r_\pi \pi_t + r_y(y_t - y_t^f)) + r_{\Delta y}(y_t - y_t^f) + e_t^r \\ \widetilde{r}_t &= r_t - \frac{r_{\Delta y}}{\rho}(y_t - y_t^f) \end{aligned}$$

The rank deficiency arises because the state vector $(\widetilde{X}_{1,t+1}, \widetilde{X}_{2,t+1})$ is really a function of $\widetilde{r}_t$ instead of (r_t, y_t, y_t^f). To resolve this problem, a new 16×1 state vector X_{1t} is defined from $\widetilde{X}_{1t}$ that removes this dependency. Let $Y_t = (y_t, i_t, c_t, w_t, r_t, \pi_t, l_t)$. The state space system defined for $X_t = X_{1t}$ and Y_t is minimal with a state vector that is of dimension 16. The system has 7 equations and 7 shocks and is hence full rank. Both Propositions 2-S and 2-NS apply.

Our results are still necessary even without minimality. In such a case, $\Delta^{\mathrm{S}}_{\Lambda\,T}(\theta_0) \equiv (\Delta^{\mathrm{S}}_{\Lambda}(\theta_0)\ \ \Delta^{\mathrm{S}}_{T}(\theta_0))$ and $\Delta^{\mathrm{S}}_{T}(\theta_0)$ will be rank deficient and not useful to analyze. However, full rank of $\Delta^{\mathrm{S}}_{\Lambda\,U}(\theta_0) \equiv (\Delta^{\mathrm{S}}_{\Lambda}(\theta_0)\ \ \Delta^{\mathrm{S}}_{U}(\theta_0))$ is still required for identification. While re-arranging the model to the minimal representation helps understand the properties of the model, a case can be made that one checks the necessary condition before spending the effort to assure sufficiency. For this reason, results for both the minimal and non-minimal models are reported.

We evaluate θ_0 at the posterior mean reported in Smets and Wouters. The results are as follows:

Smets and Wouters (2007)

	Minimal Model					Non-Minimal Model			
Tol	$\Delta^{\mathrm{S}}_{\Lambda}$	Δ^{S}_{T}	Δ^{S}_{U}	Δ^{S}	pass	$\Delta^{\mathrm{S}}_{\Lambda}$	Δ^{S}_{U}	$\Delta^{\mathrm{S}}_{\Lambda\,U}$	pass
1.000000e-03	36	256	49	341	No	36	49	85	No
1.000000e-04	36	256	49	341	No	36	49	85	No
1.000000e-05	36	256	49	341	No	36	49	85	No
1.000000e-06	36	256	49	341	No	36	49	85	No
1.000000e-07	38	256	49	341	No	38	49	87	No
1.000000e-08	39	256	49	343	No	39	49	88	No
1.000000e-09	39	256	49	344	No	39	49	88	No
1.000000e-10	39	256	49	344	No	39	49	88	No
1.000000e-11	39	256	49	344	No	39	49	88	No
1.000000e-12	39	256	49	344	No	39	49	88	No
3.973355e-11	39	256	49	344	No	39	49	88	No
Required	41	256	49	346		41	49	90	

Results for the minimal model in the left panel. The model is not identified from the second moments of y_t at any tolerance. At TOL=1e-3, $\Delta^{\mathrm{S}}_{\Lambda}(\theta_0)$ is rank deficient by 5, even though $\Delta^{\mathrm{S}}_{T}(\theta_0)$ and $\Delta^{\mathrm{S}}_{U}(\theta_0)$ are full rank. The results in the right panel show that $\Delta^{\mathrm{S}}_{\Lambda\,U}(\theta_0)$ of the non-minimal model is also reduced rank. For a range of values of TOL, $\Delta^{S}(\theta_0)$ of the minimal model and $\Delta^{NS}_{\Lambda U}(\theta_0)$ of the non-minimal are both short rank by 5. This suggests five restrictions are necessary.

To isolate the restrictions, we study the null space of $\Delta^{S}(\theta_0)$ of the minimal model. The seven smallest entries in the null space correspond to steady state hours ($\bar{l}$), steady state inflation, ($\overline{\pi}$), the discount factor (β), elasticity of capital utilization adjustment cost (ϕ), steady state output growth ($\overline{\gamma}$), price curvature (ϵ^p), and wage curvature (ϵ^w). Obviously, some of these parameters are identifiable from the mean which we can incorporate via $\varphi(\theta) = 0$.

Rank conditions with Restrictions: Tol=1e-3

Restriction					$\overline{\Delta}^{S}_{\Lambda}$	Δ^{S}_{T}	$\overline{\Delta}^{S}_{U}$	$\overline{\Delta}^{S}_{\Lambda T}$	$\overline{\Delta}^{S}_{\Lambda U}$	$\overline{\Delta}^{S}$	pass
δ	μ^w	$\overline{g}$	ϵ^p	ϵ^w	36	256	49	292	85	341	No
$\bar{l}$	$\overline{\pi}$	ϵ^p	ϵ^w		39	256	49	295	88	344	No
$\bar{l}$	$\overline{g}$	$\overline{\pi}$	β	$\overline{\mu}^w$	40	256	49	295	88	344	No
$\bar{l}$	$\overline{\pi}$	β	ϵ^p	ϵ^w	40	256	49	296	89	345	No
$\bar{l}$	$\overline{\pi}$	$\overline{\mu}^w$	ϵ^p	ϵ^w	40	256	49	296	89	345	No
$\bar{l}$	$\overline{\pi}$	$\overline{g}$	ϵ^p	ϵ^w	40	256	49	296	89	345	No
$\bar{l}$	$\overline{\pi}$	β	$\overline{\mu}^w$	ϵ^w	40	256	49	296	89	345	No
$\bar{l}$	$\overline{\pi}$	$\overline{\gamma}$	ϵ^p	ϵ^w	41	256	49	297	90	346	Yes
$\bar{l}$	$\overline{\pi}$	β	ϵ^p	ϵ^w	41	256	49	297	90	346	Yes
$\bar{l}$	$\overline{\pi}$	δ	ϵ^p	ϵ^w	41	256	49	297	90	346	Yes
$\bar{l}$	$\overline{\pi}$	ϕ	ϵ^p	ϵ^w	41	256	49	297	90	346	Yes
$\bar{l}$	$\overline{\pi}$	λ	ϵ^p	ϵ^w	41	256	49	297	90	346	Yes
Required					41	256	49	297	90	346	

Row 1 are results for the five restrictions imposed by Smets and Wouters: depreciation (δ), steady state markup ($\overline{\mu}^w$) in the labor market, exogenous spending ($\overline{g}$), price curvature (ϵ^p) and wage curvature and (ϵ^w). These restrictions evidently do not yield an identifiable model. While the mean restrictions $\bar{l}, \overline{\pi}$ help identification, they are not sufficient. Restricting ϵ^w without restricting ϵ^p will not enable identification. All identifiable models involve restricting parameters suggested by the null space of $\Delta^{S}(\theta_0)$.

We also see if the model is identified at the prior means used in Smets and Wouters (2007). The results are the same as the ones reported above for the posterior mean: restrictions on $\bar{l}, \overline{\pi}, \epsilon^p, \epsilon^w$ and one of $\beta, \overline{\gamma}, \delta, \phi, \lambda$. Non-identification of this model is purely a consequence of parameter dependency. Similarity transformations leading to identical transfer functions play no role here. These results thus agree with Iskrev (2010).

Removing labor supply from the observables would result in more shocks than observables. Hence, only the results for non-singular models apply here. We obtain the following results:

Rank conditions with Restrictions (without l_t): Tol=1e-3

Restriction					$\overline{\Delta}_\Lambda^{\text{NS}}$	$\overline{\Delta}_T^{\text{NS}}$	$\overline{\Delta}^{\text{NS}}$	pass
-	-	-	-	-	36	256	292	No
δ	$\overline{\mu}^w$	$\overline{g}$	ϵ^p	ϵ^w	39	256	295	No
$\bar{l}$	$\overline{\pi}$	ϵ^p	ϵ^w		40	256	296	No
$\bar{l}$	$\overline{\pi}$	$\overline{\mu}^w$	ϵ^p	ϵ^w	40	256	296	No
$\bar{l}$	$\overline{\pi}$	$\overline{g}$	ϵ^p	ϵ^w	40	256	296	No
$\bar{l}$	$\overline{\pi}$	β	ϵ^p	ϵ^w	41	256	297	Yes
$\bar{l}$	$\overline{\pi}$	$\overline{\gamma}$	ϵ^p	ϵ^w	41	256	297	Yes
$\bar{l}$	$\overline{\pi}$	δ	ϵ^p	ϵ^w	41	256	297	Yes
$\bar{l}$	$\overline{\pi}$	ϕ	ϵ^p	ϵ^w	41	256	297	Yes
$\bar{l}$	$\overline{\pi}$	λ	ϵ^p	ϵ^w	41	256	297	Yes
Required					41	256	297	

The results are exactly the same as when labor supply was used. Further analysis reviews that the analysis holds up when additional variables are dropped. However, θ_0 is not identifiable when $n_Y \leq 3$.

3 The CEE Model

The model of Christiano, Eichenbaum, and Evans (2005) has many features as the SW model, but the CEE has only two shocks, technology (z_t) and government spending (g_t). Schmitt-Grohe and Uribe (2004) used a version of the CEE model to assess welfare effects using higher order solution methods. We use their code available at `http://www.columbia.edu/~mu2166/cee/cee.html` to symbolically obtain a first order linear approximation. Their function then solves the model by qz-decomposition and returns H and G, where $\widetilde{X}_{t+1} = H\widetilde{X}_t$ and $Z_t = G\widetilde{X}_t$, $\widetilde{X}_t$ is a 11×1 state vector declared by the user, and Z_t is also 11×1. One advantage of this rational expectations model solver (which is a version of Klein's code) is that the user needs to specify the dimension of the state vector, and the output matrices are 'almost' what is required for our analysis. The missing step is to find the matrices that characterize the impact response of $\widetilde{X}_t$ and Z_t to ϵ_t. These matrices were derived in Klein (2000) and also explained in Anderson (2008). We verify that when applied to the An-Schorfheide model, the code agrees with the GENSYS and DYNARE output provided by the authors. Given G, H and the two impact matrices, simple rearranging gives A_1, B_1, C_1, D_1 which allows us to proceed to test minimality.

The objective of Schmitt-Grohe and Uribe (2004) was to perform a welfare analysis using a second order approximation of the CEE model. We only analyze the linear approximation to the model. A rank test finds that the 11×1 vector $\widetilde{X}_t$ declared by the user is not minimal. These 11 variables are $c_t, i_t, r_t, \pi_t, y_t, s_t, \widetilde{s}_t, w_{t-1}, k_t, g_t, z_t$. Inspection of A_1 reveals that columns 3 and 5 are zeros. Removing these variables from $\widetilde{X}_t$ leads to a 9×1 state vector $X_t = (c_t, \pi_t, r_t, q_t, s_t, \widetilde{s}_t, y_t, h_t, u_t)$.

As noted in Schmitt-Grohe and Uribe (2004), the variables s_t and $\widetilde{s}_t$ have no first order effects and are thus superfluous, implying that system expressed in terms of X_t is still not minimal. In particular, the system is not controllable. A minimal system can be obtained by removing $s_t, \widetilde{s}_t$ from the analysis altogether. However, our rank conditions are still necessary for identification.

The model has a total of 25 unknown parameters which are calibrated by Schmitt-Grohe and Uribe (2004). We fix the steady state share of government purchases in value-added, Tobin's Q, steady state productivity, steady state capacity utilization, and a parameter that scales the standard deviation of shocks. We also set the degree of wage indexation is set to the one. (The solution is not unique otherwise). We then proceed to assess identifiability of remaining 18 dimensional θ. As there are two shocks and four observables, Proposition 2-S applies.

Model CEE

Tol	Δ^S_Λ	Δ^S_T	Δ^S_U	$\Delta^S_{\Lambda\,T}$	$\Delta^S_{\Lambda\,U}$	Δ^S	pass
1.0e-02	14	81	4	94	18	96	No
1.0e-03	14	81	4	95	18	99	No
1.0e-04	14	81	4	95	18	99	No
1.0e-05	14	81	4	95	18	99	No
1.0e-06	14	81	4	95	18	99	No
1.0e-07	14	81	4	95	18	99	No
1.0e-08	15	81	4	96	19	99	No
1.0e-09	15	81	4	96	19	100	No
1.0e-10	15	81	4	96	19	100	No
1.0e-11	15	81	4	96	19	100	No
default	17	81	4	97	21	101	No
Required	18	81	4	99	22	103	

The 18 parameters in the model are not identified. The rank of $\Delta^S(\theta_0)$ suggests 4 restrictions. The smallest entries in the null space of $\Delta^S(\theta_0)$ are due to: steady state labor demand ($\overline{h}$), the cash in advance constraint parameter (ν), inflation target $\overline{\pi}$, labor elasticity of substitution η, a money demand parameter σ^m.

Rank conditions with Restrictions: Tol=1e-3

Restriction				$\overline{\Delta}^S_\Lambda$	$\overline{\Delta}^S_T$	$\overline{\Delta}^S_U$	$\overline{\Delta}^S_{\Lambda\,T}$	$\overline{\Delta}^S_{\Lambda\,U}$	$\overline{\Delta}^S$	pass
$\overline{h}$	η	σ^m		17	81	4	98	21	102	No
$\overline{h}$	ν	$\overline{\pi}$	σ^m	17	81	4	98	21	102	No
$\overline{h}$	ν	η	σ^m	18	81	4	99	22	103	Yes
$\overline{h}$	$\overline{\pi}$	η	σ^m	18	81	4	99	22	103	Yes
Required				18	81	4	99	22	103	

The conditional rank analysis shows that identification requires two mean restrictions on $\overline{h}, \eta, \sigma^m$ and either ν or $\overline{\pi}$. The results hold when more observables are used in the identification analysis.

4 Cicco-Pancrazi-Uribe

Cicco, Pancrazi, and Uribe (2009) consider two real business models for emerging countries, one with two shocks, and a more elaborate model with frictions that has five shocks. We focus on the big model with frictions.

The code available for download at `http://www.columbia.edu/~mu2166/rbc_emerging/rbc_emerging.html` produces $\widetilde{X}_t = H\widetilde{X}_{t-1}$, $Z_t = G\widetilde{X}_t$ where $\widetilde{X}_t$ is 11×1 vector and Z_t is 9×1. The first step is again to work out the impact matrices. The H matrix has 3 columns of zeros and the B_1 matrix has one row of zeros. Removing the associated variables yield a 7 dimensional state vector that along with four observables in the Z_t vector: consumption growth, output growth, investment growth, and the ratio of trade-balance to output, yield a minimal system.

The authors estimated 13 parameters, including ten autoregressive parameters and standard deviations for the five mutually uncorrelated shocks. There are five shocks and four observables. Hence the model is non-singular and Proposition 2-NS applies.

Model Frictions

Tol	$\Delta^{\text{NS}}_{\Lambda}$	Δ^{NS}_{T}	Δ^{NS}	pass
1.00e-02	12	48	58	No
1.00e-03	13	48	61	No
1.00e-04	13	49	62	Yes
1.00e-05	13	49	62	Yes
1.00e-06	13	49	62	Yes
1.00e-07	13	49	62	Yes
default	13	49	62	Yes
Required	13	49	62	

The model is identified at TOL ¡= 1e-4. The fact that $\Delta^{\text{NS}}_{T}(\theta_0)$ and $\Delta^{\text{NS}}(\theta_0)$ are short rank when TOL=1e-3 suggests the possibility of similar transfer functions. However, the null space of Δ^{NS} is empty. Thus, we view the model as identified at θ_0.

```
% Given solv_sw07  solves the model by gensys and returns ABCD.
% delta_sw07 computes the four Delta matrices

function [Delta,Delta_lambda,Delta_T,Delta_U] = delta_sw07(theta,A,B,C,D,Sigma)

lambda = [vec(A); vec(B); vec(C); vec(D); vec(Sigma)];
n_x = size(A,1);
n_eps = size(B,2);
n_y = size(C,1);

% compute numerical derivatives with respect to theta
Delta_lambda = zeros(size(lambda,1),size(theta,1));
for i=1:1:size(theta)
    delta_theta = zeros(size(theta)); delta_theta(i) = theta(i)*1e-3;
    if delta_theta(i) ==0; delta_theta(i)=1e-3; end;
    theta_p = theta + delta_theta;
     [minA,minB,minC,minD,Sigma]=solv_sw07(theta_p,flex);
    lambda_p = [vec(minA); vec(minB); vec(minC); vec(minD); vec(Sigma)];

    theta_m = theta - delta_theta;
    [minA,minB,minC,minD,Sigma]=solv_sw07(theta_m,flex);
   lambda_m = [vec(minA); vec(minB); vec(minC); vec(minD); vec(Sigma)];

    Delta_lambda(:,i) = (lambda_p - lambda_m)/(2*delta_theta(i));
end;

% computes the permutation matix T
T = [];
for j=1:1:n_eps
    ind_j = zeros(n_eps,1); ind_j(j) = 1;
    T = [T, kron(eye(n_eps,n_eps),ind_j)];
end

% computes Delta_T
Delta_T = [kron(A',eye(n_x)) - kron(eye(n_x),A);
          kron(B',eye(n_x));
          -1*kron(eye(n_x),C);
          zeros(n_y*n_eps,n_x^2);
          zeros(n_eps^2,n_x^2)];

%computes Delta_U
Delta_U = [zeros(n_x^2,n_eps^2);
           kron(eye(n_eps),B);
           zeros(n_y*n_x,n_eps^2);
           kron(eye(n_eps),D);
           -1*(eye(n_eps^2) + T)*kron(Sigma,eye(n_eps))];

Delta = [Delta_lambda, Delta_T, Delta_U];

Delta_orth=null(Delta,'r')  % computes the null space of Delta
```

```
function [K,S] = dare_kn(A,B,C,D,Sigma,TolCV)

% dare_kn.m
%
%  This program solves the Riccati matrix difference equations associated
%  with the Kalman filter by iterating until the tolerance TolCV is reached.
%
%  Inputs:  A is n x n, B is n_y x n_e, C is n_y x n, D is n_y x n_e, TolCV
%  is a scalar.
%  Outputs: steady state Kalman gain K is n_y x n_y, stationary covariance
%           matrix S of the one-step ahead errors a(t+1) in forecasting y(t+1)
%           is n_y x n_y.
%
%  The program creates the Kalman filter for the following system:
%       x(t+1) = A*x(t) + B*e(t+1)
%       y(t+1) = C*x(t) + D*e(t+1).
%
%  The program creates an innovations representation:
%        xx(t+1) = A*xx(t) + K*a(t+1)
%         y(t+1) = C*xx(t) + a(t+1),
%  where K is the (steady state) Kalman gain, S is the covariance matrix of
%  the one-step-ahead forecast error S = E[a(t)*a(t)'], and
%  a(t+1) = y(t+1) - E[y(t+1)| y(t), y(t-1), ... ], and xx(t)=E[x(t)|y(t),...].

% Initialization
Q = B*Sigma*B'; % Q is n x n
R = D*Sigma*D'; % R is n_y x n_y
P = B*Sigma*D';
g0 = Q;
dd = 1;

% Iterating until steady state
while dd > TolCV
    b0 = A*g0*C' + P;
    s0 = C*g0*C' + R;
    k0 = b0/s0;
    g1 = A*g0*A' + Q - k0*s0*k0';
    b1 = A*g1*C' + P;
    s1 = C*g1*C' + R;
    k1 = b1/s1;
    dd=max(max(abs(k1-k0)));
    g0=g1;
end

K=k1;
S=s1;
```

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