LEVEL AND VOLATILITY FACTORS IN MACROECONOMIC DATA

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Abstract

The conventional wisdom in macroeconomic modeling is to attribute business cycle fluctuations to innovations in the level of the fundamentals. Though volatility shocks could be important too, their propagating mechanism is still not well understood partly because modeling the latent volatilities can be quite demanding. This paper suggests a simply methodology that can separate the level factors from the volatility factors and assess their relative importance without directly estimating the volatility processes. This is made possible by exploiting features in the second order approximation of equilibrium models and information in a large panel of data. Our largest volatility factor V_1 is strongly counter-cyclical, persistent, and loads heavily on housing sector variables. When augmented to a VAR in housing starts, industrial production, the fed-funds rate, and inflation, the innovations to V_1 can account for a non-negligible share of the variations at horizons of four to five years. However, V_1 is only weakly correlated with the volatility of our real activity factor and does not displace various measures of uncertainty. This suggests that there are second-moment shocks and non-linearities with cyclical implications beyond the ones we studied. More theorizing is needed to understand the interaction between the level and second-moment dynamics.

JEL: C3, C5, E3, E4.

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1 Introduction

A long tradition in macroeconomics is to summarize aggregate fluctuations with a handful of shocks. In a celebrated paper, King and Rebelo (1993) showed that a large fraction of macroeconomic variations at business cycle frequencies can be accounted for by a single shock to the level of technology. At lower frequencies, nearly all macroeconomic fluctuations are often attributed to technology shocks (e.g., King et al. (1991)). More elaborate macroeconomic models also incorporate shocks to policies, preferences, and other primitives. Although these newer models have richer features and theoretical foundations, it is fair to say that using a few "level" shocks to generate cyclical fluctuations and co-movements is at the heart of macroeconomic modeling.

More recently, there is a nascent theoretical literature suggesting that higher-order shocks, and more specifically, second-moment *volatility* shocks, can also be an important source of business cycles.[1] This alternative focus is motivated by the observation that realized volatility and expected future volatility (or uncertainty) tend to be high during recessions. This countercyclical feature of volatility is robust to whether the latent volatility variables are estimated or are replaced by proxy variables. Additional evidence that second-moment variations may have first-order effects is given in Fernandez-Villaderde and Rubio-Ramirez (2010), among others.

The need to model the dynamics of volatility has long been recognized. In a seminal paper, Engle (1982) presents evidence of autoregressive conditional volatility (also known as ARCH effects) in inflation data. Sims and Zha (2006) also conclude that time-varying volatility is an important feature that empirical macroeconomic models should incorporate. From estimation of structural models, Justiano and Primiceri (2008) find significant time-varying volatility in monetary policy and technology shocks, while Fernandez-Villaverde et al. (2015) find that a two-standard deviation shock to fiscal volatility can reduce output by up to 1.5 percentage points when the economy is at the zero lower bound. Work along this line tends to assume that volatility is exogenous and that its shocks are independent of the innovations to the level of the fundamentals.

Despite statistical and methodological progress made in modeling volatility, the source of volatility shocks as well as the interaction between the level and volatility dynamics remain open questions to a large extent. While exogenous time-varying volatility in productivity shocks is a natural starting point from a theoretical point of view, it may not necessarily be the most important source of volatility in the data. Furthermore, exogenous volatility precludes volatility-in-mean effects that allow for feedback between the first- and second-moment dynamics. But the stochastic volatility estimates are typically countercyclical, suggesting that volatility is likely related to and possibly

[1] See, for example, Schmitt-Grohe and Uribe (2004), Kim et al. (2008),Bloom (2009), Fernández-Villaverde et al. (2011), Fernandez-Villaverde et al. (2015), Jurado et al. (2015) and references there in.

predictable by observed cyclical variables[2], which is at odds with the assumed exogeneity of volatility.

While it may be tempting to criticize the limitations of the exogenous volatility assumption, relaxing the assumption is not at all easy for a number of reasons. To begin with, economic theory has focused on level shocks and does not provide much guidance about the source of volatility fluctuations and how the volatility process is supposed to evolve. It is quite common to adapt models designed for high-frequency financial data to macroeconomic data, even though the two data types have distinctive time series properties. Furthermore, with time-varying volatility, there can be many channels for generating equivalent first- and second-moment dynamics. Model identification and validation is difficult as the volatility is latent even ex-post.

Perhaps more important from a practical standpoint is that modeling non-linearity and volatility often requires computationally sophisticated methods. Non-linear VARs such as the one considered in Pesaran and Shin (1998) are already quite computationally demanding. Though conceptually simple, adding stochastic volatility to an otherwise standard VAR or dynamic stochastic general equilibrium (DSGE) model entails a significant change in the estimation methodology. It is relatively easy to assess the sensitivity of a homoskedastic model to alternative assumptions, but the flexibility disappears once the volatility process has to be explicitly modeled.

In this paper, we propose a simple and easy-to-implement framework for studying the interaction between the first- and second-moment dynamics. It preserves the traditional view that there are relatively few level shocks in macroeconomic data. However, it allows second-moment shocks to be a source of economic fluctuations and permits the second-moment factors to respond to the level shocks. Specifically, Benigno et al. (2013) shows that time-varying volatility has a second-order effect on the level of the endogenous variables. We demonstrate that if the data are generated according to a DSGE model and are observed without error, then under some additional assumptions, we can distinguish the "level" factors A from the "volatility" factors V. In practice, the V that we recover is likely a composite of second-moment factors whose interpretation we remain agnostic on. This limitation arises partly because there are likely shocks, some to second moments, that DSGE models fail to capture. Furthermore, the construction of V depends on the level factors estimated from a large panel of data, and these estimates are only consistent for the space spanned by the true factors. In other words, we only identify the true factors up to a rotation. The exercise is nonetheless of interest because it sheds light on the importance of the second-moment dynamics. After all, if the level shocks are the sole source of economic fluctuations, then the second-moment shocks should have no cyclical implications whatever their structural interpretation might be. We find that not only are the effects of the second-moment shocks significant, but their presence tends

[2]See, for example, Justiano and Primiceri (2008) and Carriero et al. (2016) among others.

to reduce the importance of the level factors previously used in FAVARs.

Our objective is to separate the level and the volatility factors in the data, and to quantify their individual contributions as well as the non-linear interactions. Previous macroeconomic analysis typically incorporates volatility processes into fully specified structural models estimated from a small number of variables (see Fernandez-Villaderde and Rubio-Ramirez (2010)). Estimation is rather complicated and the results rely on correct specification of both the economic model and the volatility processes that theory offers little guidance on. Our methodology requires the presence of pervasive volatilities but is not tied to any particular economic model. Instead, it relies on information contained in a monthly panel of 134 macroeconomic time series to recover the space spanned by the volatility processes.[3]

Several patterns in the level and volatility factors are noteworthy. First, even though there are eight factors, we suggest that only three are level factors. Our dominant volatility factor V_1 is estimated to be countercyclical and persistent. It rises during the Great Recession considerably and remains at an elevated level for many years, but it is weakly and negatively correlated with the stochastic volatility directly estimated from our real-activity level factor. Second, our estimated V_1 is only weakly correlated with measures of volatility/uncertainty constructed in previous studies such as Baker et al. (2015) and Jurado et al. (2015). Instead, V_1 is most correlated with the second factor estimated from X^2. There is also evidence of second moment shocks other than volatility in the data and which need to be further understood.

To study their dynamic effects, we augment estimates of A_1 and V_1 to a VAR in housing starts, federal funds rate, industrial production, and inflation. While the largest level factor in the data is unambiguously a real activity factor, its effects on inflation depend on whether we explicitly control for the presence of V_1. The responses to a positive shock to V_1 resemble those of a negative "demand" shock, while the responses to a negative shock to A orthogonalized to V_1 are reminiscent of responses to a negative supply shock. In short, we find that the level and second moment dynamics do interact. Volatility shocks appear only in higher-order approximations. Hence, their effect is expected to be smaller than those of the level shocks. An innovation to V_1 accounts for 5% to 20% of the variations in the data at the horizons of 4-5 years. Though this contribution may seem modest, volatility shocks are large in magnitude when they occur.

The rest of the paper is structured as follows. In the next section, we outline our framework. Specifically, we show how higher-order approximations of DSGE models can be used to separate "volatility" and "level" factors in theory. Section 3 then discusses how to recover the factors from

[3]Our emphasis is on the second-moment dynamics, hence distinct from the VAR proposed in Aruboa et al. (2013), whose focus is non-linearities. While Jurado et al. (2015) also exploits a data rich environment, their uncertainty measure concerns h step ahead volatility in the forecast errors. We evaluate the contemporaneous unconditional volatility; no forecasting model is involved in our approach.

the data. Section 4 presents the estimates and Section 5 uses impulse responses and variance decompositions to understand the business cycle implications of these factors. Section 6 concludes.

2 Level vs. Second-Moment Factors

This section consists of two parts. Subsection 1 uses the one-sector stochastic growth model to highlight the main issues involved in a simple setting. Subsection 2 uses the general second-order solution of dynamic stochastic equilibrium (DSGE) models as a guide to understand what are the common factors that can be expected from the level and square of the data.

2.1 A Simple Example

Consider the one-sector stochastic growth model. Let z_t be technology with homoskedastic innovations ψ_t, and let $\check{c}_t$ and $\check{k}_{t+1}$ be log-deviations of consumption and capital from the steady state, respectively. It is well known that the linearized solution is

$$\begin{aligned}
\check{c}_t &= b_k\check{k}_t + b_z z_t \\
\check{k}_{t+1} &= h_k\check{k}_t + h_z z_t \\
z_{t+1} &= \rho z_t + \psi_{t+1}.
\end{aligned}$$

Our point of departure is to modify the homoskedasticity assumption to allow for time-varying volatility in the technology shocks, i.e., $z_{t+1} = \rho z_t + u_t\epsilon_{t+1}$ where u_t (which has a mean of $\bar{u}$) governs the volatility of shocks to technology and ϵ_{t+1} is an i.i.d. zero mean, unit variance shock so that $\psi_{t+1} = u_t\epsilon_{t+1}$ is heteroskedastic. One can interpret changes in u_t^2 as volatility shocks.

Applying the second-order approximate solution method of Benigno et al. (2013) and letting $v_t = u_t^2$ yields

$$\begin{aligned}
\check{c}_t &= b_k\check{k}_t + b_z z_t + \frac{1}{2}\left[b_{kk}\check{k}_t^2 + b_{zz}z_t^2 + b_{kz}\check{k}_t z_t + b_{uu}v_t\right] + \text{constant}, \\
\check{k}_{t+1} &= h_k\check{k}_t + h_z z_t + \frac{1}{2}\left[h_{kk}\check{k}_t^2 + h_{zz}z_t^2 + h_{kz}\check{k}_t z_t + h_{uu}v_t\right] + \text{constant}, \\
z_{t+1} &= \rho z_t + \psi_{t+1}.
\end{aligned}$$

There are now two independent but altogether three common sources of randomness: namely, z_t z_t^2, and v_t. As is evident even in the simple growth model, exogenous changes to the level and the second-moment factors do not have the same effects on $\check{c}$ and $\check{k}$. While the effects of z_t^2 are expected to be smaller than those of z_t, the quadratic effects omitted from the linear solution can still be important. Note the second-order approximation for z_t is identical to the first-order approximation because z_t is a conditionally linear process.[4]

Several observations are important to understand how we will disentangle the effects of the level shocks from the volatility shocks. First, the factors in $\check{c}$ and $\check{k}$ will, in general, be a combination of the level factors (e.g., z_t in the above example and the cross-product of z_t with the endogenous state variables) and volatility factors (e.g., v_t in the above example). Thus, when common components are extracted from a vector of macroeconomic variables, the extracted factors are likely a mix of level shocks, nonlinear terms, and volatility shocks.

Second, if we square both sides of the second-order approximate solution and omit the higher-order terms, we see that

$$\begin{aligned}
\check{c}_t^2 &= b_k^2 \check{k}_t^2 + b_z^2 z_t^2 + 2b_k b_z \check{k}_t z_t + \text{constant}, \\
\check{k}_{t+1}^2 &= h_k^2 \check{k}_t^2 + h_z^2 z_t^2 + 2h_k h_z \check{k}_t z_t + \text{constant}, \\
z_{t+1}^2 &= \rho^2 z_t^2 + 2\rho \bar{u} z_t \epsilon_{t+1} + \bar{u}^2 \epsilon_{t+1}^2 + v_t.
\end{aligned}$$

The last two terms in the last equation follow from $\psi_{t+1}^2 = u_t^2 \epsilon_{t+1}^2 \approx \bar{u}^2 \epsilon_{t+1}^2 + u_t^2 \bar{\epsilon}^2 = \bar{u}^2 \epsilon_{t+1}^2 + u_t^2$. The second term in the last equation is an approximation of $2\rho u_t z_t \epsilon_{t+1}$ because $E(z_t) \equiv \bar{z} = 0$ and $E(\epsilon_t) \equiv \bar{\epsilon} = 0$, $2\rho u_t \bar{z} \epsilon_{t+1} = 0$ and $2\rho u_t z_t \bar{\epsilon} = 0$. Note that the squared data $\check{c}_t^2$, $\check{k}_{t+1}^2$ and z_{t+1}^2 give information about z_t^2 and the cross-product terms but not about z_t itself, while volatility v_t appears in the level and squared data. z_{t+1}^2 give information about z_t^2 and the cross-product terms but not about z_t itself, while volatility v_t appears in the level and squared data.

Third, z_t^2 is common to both $\check{c}_t$ and $\check{c}_t^2$. This is also true of other endogenous variables. Thus, the factors in $\check{c}^2$ and $\check{k}^2$ can be a subset of the factors in $\check{c}$ and $\check{k}$. These three observations provide a basis to separate the level shocks and the volatility shocks v_t from the second-moment variations.

2.2 The Second-Order Solution

The stochastic growth model is useful for gaining intuition, but the presence of only one exogenous state variable is restrictive. Consider the following description of a generic DSGE model:

$$\begin{aligned}
0 &= E_t\{\boldsymbol{Q}(y_{t+1}, p_{t+1}, y_t, p_t)\} && \text{(1a)} \\
z_{t+1} &= \Lambda z_t + A\psi_{t+1} && \text{(1b)} \\
\psi_{t+1} &= U_t \epsilon_{t+1} && \text{(1c)} \\
u_{t+1}^2 &= \bar{u}^2 + \Xi u_t^2 + \Omega \eta_{t+1} && \text{(1d)}
\end{aligned}$$

where y_t is the vector of non-predetermined variables, $p_t = [k_t' \quad z_t']'$ is a vector of predetermined (state) variables, z_t is the vector of exogenous variables, k_t is the vector of endogenous predetermined

[4] We implicitly assume that z_t is observed by an econometrician because the series can be constructed from linear combinations of endogenous variables. For example, in the one-sector growth model, the production function implies $z_t = \check{y}_t - \alpha \check{k}_t$ where $\check{y}_t$ is the log deviation of output from its steady state.

(state) variables. We continue to use "checks" on the above variables to indicate that the variables are measuring deviations from steady-state. The shocks η_t and ϵ_t are mutually independent. $\psi_{t+1} = U_t \epsilon_{t+1}$ is the vector of shocks or rational expectation errors that is a product of i.i.d shocks $\epsilon_t \overset{iid}{\sim} (0, I)$ and volatility shocks collected into U_t, a diagonal matrix whose entries u_t follow a VAR(1) structure. The volatility innovations $\eta_t \overset{iid}{\sim} (0, I)$ have contemporaneous effects summarized by matrix Ω, see equation (1d). Equation (1a) summarizes the optimality conditions for economic agents, as given by vector function $\boldsymbol{Q}(\cdot)$. Equation (1b) describes dynamic properties of the forcing variables.

If we ignore the time-varying volatility, the model can be solved using the method of King and Watson (1998), Sims (2002), and Klein (2000), among others. The first-order solution is

$$\check{y}_t = \boldsymbol{M}(\check{p}_t, u_t) \tag{2a}$$

$$\check{p}_{t+1} = \boldsymbol{W}(\check{p}_t, u_t) + W_\psi \psi_{t+1} \tag{2b}$$

where $\boldsymbol{M}$ and $\boldsymbol{W}$ are vector functions. But z_{t+1} in equation (1b) is a conditionally linear process with heteroskedastic innovations. Applying the method of Benigno et al. (2013) yields the second-order approximate solution:

$$\check{y}_t = M_p \check{p}_t + \tfrac{1}{2}(I_y \otimes \check{p}_t') M_{pp} \check{p}_t + \tfrac{1}{2} M_{uu} u_t^2 + \text{constants} \tag{3a}$$

$$\check{p}_{t+1} = W_p \check{p}_t + \tfrac{1}{2}(I_p \otimes \check{p}_t') W_{pp} \check{p}_t + \tfrac{1}{2} W_{uu} u_t^2 + W_\psi \psi_{t+1} + \text{constants} \tag{3b}$$

where $\otimes$ denotes the Kronecker product.[5] The approximation given in equation (3b) is relevant only for endogenous state variables k_t because the exogenous process z_t is already conditionally linear.

The representation given by equations (3a)-(3b) has several important features. First, the dynamics have a factor structure. The common "factors" are the "level" of the state variables $\check{p}_t$, the shocks ψ_{t+1}, the "second moment" variables originating from the variances and covariances of the state variables, as well as volatility u_t^2. Hence, volatility shocks η_t have a direct effect on "level" variables. Given the conditional linearity of z_t and independence of the shocks, there is no interaction term between volatility u_t^2 and the state variables p_t in the second-order approximation. Note also that ψ_{t+1} directly affects only z_{t+1}, not k_{t+1}.

Second, the squared entries of $\check{y}_t$ or $\check{p}_t$ depend only on squares of "level" terms in equations

[5] The matrices $M_p, M_{pp}, M_{uu}, W_p, W_{pp}, W_{uu}$ correspond to the first- and second-order derivatives of functions $\boldsymbol{M}$ and $\boldsymbol{W}$ in equations (2a) and (2b). For example, $M_{pp} = \frac{\partial^2 \boldsymbol{M}(p,u)}{\partial p \partial p'}$. The CONSTANTS depend on the volatility of z_t and curvature in the optimality conditions that render $E(\check{y}_t) = 0$ and $E(\check{p}_t) = 0$.

(3a)-(3b). That is,

$$\begin{aligned}
\operatorname{diag}\{\breve{y}_t\breve{y}_t'\} &= \operatorname{diag}\{M_p\breve{p}_t\breve{p}_t'M_p'\} + \text{constant} &\quad (4a)\\
\operatorname{diag}\{\breve{p}_{t+1}\breve{p}_{t+1}'\} &= \operatorname{diag}\{W_p\breve{p}_t\breve{p}_t'W_p'\} + \operatorname{diag}\{W_p\breve{p}_t\psi_{t+1}'W_\psi'\} \\
&\quad +\operatorname{diag}\{W_\psi\psi_{t+1}\breve{p}_t'W_p'\} + \operatorname{diag}\{W_\psi\psi_{t+1}\psi_{t+1}'W_\psi'\} + \text{constant}. &\quad (4b)
\end{aligned}$$

Other terms when squared have orders greater than two and thus are ignored. The second and third terms in the squared solution are interactions of the "level" variables (i.e. state variables $\breve{p}_t$) and ψ_{t+1} (the innovations to a_t). Importantly, these interaction terms are not present in the second-order solution of $\breve{y}_t$ and $\breve{p}_{t+1}$ given in (3a)-(3b).[6] The last term in equation (4b) is a combination of volatility in u_t and ϵ_{t+1} as in the scalar case since $\psi_{t+1}^2 = u_t^2\epsilon_{t+1}^2 = \overline{u^2}\epsilon_{t+1}^2 + u_t^2\bar{\epsilon^2} +$ higher order terms $= \overline{u^2}\epsilon_{t+1}^2 + u_t^2 +$ higher order terms.

Third, the "level" factors appear linearly and quadratically in equations (3a)-(3b). Furthermore, the squared "level" terms appear in equations (3a)-(3b) and equations (4a)-(4b). Hence there is a "cross-equation" restriction not only in terms of factors but also in terms of loadings on the factors.

For subsequent analysis, it is useful to consider a representation in terms of mutually orthogonal factors instead of the system of equations (3a)-(4b). The reason is that the state-variables $\breve{p}_t$ are potentially correlated.[7] Hence we consider p_t's orthogonal components, denoted a_t. We then define g_t to be a vector of cross-products of entries in a_t (including its own product). Similarly, let v_t be the orthogonal components of u_t^2.[8] Note that a_t and v_t are contemporaneously uncorrelated because a cross-product of $\breve{p}_t$ and $\breve{u}_t^2$ is a negligible third-order term.

If we define $\breve{x}_t \equiv [\breve{y}_t' \quad \breve{p}_t']'$ and $\breve{x}_t^2 \equiv \operatorname{diag}\{\breve{x}_t\breve{x}_t'\}$, we can re-write equations (3a)-(4b) as

$$\begin{bmatrix}\breve{x}_t^2\\ \breve{x}_t\end{bmatrix} = \begin{bmatrix}\lambda_{2g} & \lambda_{2v} & 0\\ \lambda_{1g} & \lambda_{1v} & \lambda_{1a}\end{bmatrix}\begin{bmatrix}g_t\\ v_t\\ a_t\end{bmatrix} \qquad (5)$$

where the λ parameters in the two equations can be interrelated. The point to note is that the common factors in $\breve{x}_t$ are a mixture of level factors a_t and second-moment factors (g_t, v_t). However, the level factors a_t do not appear in the equation for $\breve{x}_t^2$. That is, $\breve{x}_t^2$ is contemporaneously uncorrelated with a_t conditional on v_t and g_t. The next step is to exploit this structure to separate the level and the second-moment factors.

Proposition 1 *Let h be a vector of unique cross-products of the components of $f = (g, v, a)$ Define $\tilde{x} = M_h\breve{x}$ to be the residuals from projecting $\breve{x}$ on h where M_h is the orthogonal projection matrix.*

[6] Given the structure of W_ψ, the second and third terms in equation (4b) are actually zero for the endogenous state variables k_t.

[7] This is not an issue for other "level" variables ψ_{t+1} (shocks to exogenous variables) because it is conventional to assume that these shocks are independent.

[8] v_t actually includes u_t^2 and ϵ_{t+1}^2. One may be able to separate these shocks: ϵ_{t+1}^2 does not enter the "level" (i.e. (3a)-(3b)) while u_t^2 does.

Similarly, let $\tilde{x}^2 = M_h \check{x}^2$ be the residuals from projecting $\check{x}^2$ on h. Then

$$\begin{pmatrix} \tilde{x}_t^2 \\ \tilde{x}_t \end{pmatrix} = \begin{pmatrix} \lambda_{2v} & 0 \\ \lambda_{1v} & \lambda_{1a} \end{pmatrix} \begin{pmatrix} v_t \\ a_t \end{pmatrix}.$$

The idea behind Proposition 1 is that the factors f in $\check{x}$ are a mixture of level and second moment variations. Its quadratic components are g, $a \cdot g$, $a \cdot v$, $g \cdot g$, $g \cdot v$, and $v \cdot v$ which we collect into h. Importantly, h does not depend on a or v. A projection of $\check{x}^2$ on h provides residuals $\tilde{x}^2$ that are purged of the higher order variations of a and v, but preserve the variations of a and v themselves, along with terms of higher order that are presumed negligible. Though the residuals $\tilde{x}$ from projecting $\check{x}$ on h are still a linear combination of a and v, the volatility factor v is already recovered from $\tilde{x}^2$. Hence $\tilde{x}$ allows recovery of the level factor a.

3 Econometric framework

The model variables denoted with a 'check' are not observed. In this section we discuss several adjustments that are needed before we can make use of Proposition 1 in practice. We also present details on how one can estimate factors.

3.1 From the Model Solution to the Data

To begin, we let uppercase denote empirical analog of the model variables. For $t = 1, \dots, T$, let $X_t = (X_{1t}, \dots, X_{Nt})'$ and $X_t^2 = (X_{1t}^2, \dots, X_{Nt}^2)'$ be $N \times 1$ vectors of observables standardized to be mean zero with unit variance. Let A_t and V_t be the latent common level and volatility factors that we seek to recover, and G_t be a vector of cross-products of the components of A_t.

The theoretical setup provides a useful point of departure for isolating V_t, but it is limited in several ways. First, there is no role for series-specific shocks in equation (5). Or, from the viewpoint of Boivin and Giannoni (2006), the model variables $\check{x}_t$ and $\check{x}_t^2$ are assumed to have exact empirical counterparts. To accommodate this, we introduce non-pervasive errors e_{1t} to X_t and e_{2t} to X_t^2. These can be errors omitted from the theoretical model, or may simply be measurement errors. We leave the source of these errors agnostic.

Second, the theoretical model only allows volatility to the innovations of the fundamentals A_t, omitting other sources of second-moment variations that could be empirically relevant. For example, the idiosyncratic errors ϵ_{it} can have common time varying volatility σ_{ct} so that $e_{it} = \sigma_{ct}\epsilon_{it}$, $\epsilon_{it} \sim (0,1)$. If there were no level factors and $X_{it} = e_{it}$, then for large N, $\frac{1}{N}\sum_{i=1}^N X_{it}^2 = \frac{1}{N}\sum_{i=1}^N e_{it}^2 \xrightarrow{p} \sigma_{ct}^2$. Such a common volatility factor will appear in X_t^2. The theoretical model also ignores time variation in its loadings. For example, if $X_{it} = (\bar{\lambda}_{fi} + \lambda_{At}^c)A_t + e_{1t}$, then $\lambda_{At}^c A_t$ is another source of common second order variations. Hence it is reasonable to expect that the V that we recover will not be limited to volatility in the fundamentals as in economic models, and we leave their interpretations agnostic. Our premise is that if second-moment variations have no business cycle implications, they should not be in the data regardless of the name these variations are given. With these considerations in mind, and under the maintained assumption that the level factors A_t have no contemporaneous effect on X^2 conditional on G_t and V_t, the data can be represented by

$$\begin{pmatrix} X_{it}^2 \\ X_{it} \end{pmatrix} = \begin{pmatrix} \Lambda_{i2,G} & \Lambda_{i2,V} & 0 \\ \Lambda_{i1,G} & \Lambda_{i1,V} & \Lambda_{i1,A} \end{pmatrix} \begin{pmatrix} G_t \\ V_t \\ A_t \end{pmatrix} + \begin{pmatrix} e_{i2t} \\ e_{i1t} \end{pmatrix}. \tag{6}$$

Notably, the factors in X_t are linear combinations, or a mixture, of the level factor (A_t) and second-moment factors (G_t, V_t). The factors in X_t^2 are a mixture of volatility factor V_t and interaction of the level factors G_t).

Let r_A and r_V be the number of level and volatility factors, respectively. Guided by Proposition 1, we propose to estimate A and V as follows:

Algorithm-AV

i Estimate r_F factors $F = (G, V, A)$ from X and let H be the $\frac{r_F(r_F+1)}{2}$ unique cross products of F.

ii For each $i = 1, \dots N$, project X_i on H to obtain residual $\tilde{X}_i$. Also project X_i^2 on H to obtain residual $\tilde{X}_i^2$.

iii Estimate V from $\tilde{X}^2$.

iv Estimate r_A factors from $\tilde{X}$ given an estimate of V from (iii); that is, project $\tilde{X}$ on V and extract factors from the residual.

A one-dimensional V can be obtained from Step (iii) by averaging over $\tilde{X}^2$, or r_V factors can be constructed from $\tilde{X}^2$. Yalcin and Amemiya (2001) suggests that one might find a large number of factors in the data if higher order factors are omitted from the linear factor model. We provide a constructive result. We use the second order solution of a DSGE model to show that the omitted higher other terms can allow recovery of V. One path forward is to estimate a non-linear factor model. Chen et al. (2009) provide the theory for nonlinear principal components and suggests using sieves estimation in implementation, but the empirical properties of the procedure are unknown. Yalcin and Amemiya (2001) suggests a maximum-likelihood estimator for a large T small N setting that incorporates the cross-restrictions in the first and second moment on the Λ matrices. We do not impose such restrictions, but we exploit information in a data rich environment. Estimation of factors is an important part of the algorithm, which is discussed in the next section.

3.2 Estimation of Factors

For $i = 1, \ldots, N$, the scaled data is $Z_i = \frac{X_i}{\sqrt{NT}}$ where X_i is a $T \times 1$ series standardized to have mean-zero with unit variance. The scaling yields $\sum_i \sum_t Z_{it}^2 = 1$, which is useful in what follows. The factor analytic model for the scaled data $Z = (Z_1, Z_2, \ldots, Z_N)$

$$Z = F^* \Lambda^{*\prime} + e^*$$

We assume that Z has singular-value decomposition UDV^T, U is a $T \times T$ unitary matrix of left eigenvectors, D is a diagonal $T \times N$ matrix of eigenvalues in descending order, and V is a $N \times N$ unitary matrix of right eigenvectors. If there are r factors, the best low rank approximation of Z is given by $U_r D_r V_r^T$ where U_r collects the first r left eigenvectors corresponding to the top $r \times r$ submatrix of D_r. For any specified number of factors k, the principal components (PCA) minimizes the unweighted sum of squared residuals $\text{SSR}_k(F, \Lambda) = \sum_{i=1}^N \sum_{t=1}^T (Z_{it} - \Lambda_i' F_t)^2$ to obtain $\hat{F} = U_k D_k^{1/2}$.[9] Bai and Ng (2017) shows that $\sqrt{N}(\hat{F}_t - M' F_t^*) \approx N(0, \Sigma_F)$. In general, M is not an identity matrix and $\hat{F}$ is only an estimate of the space spanned by F. Nonetheless, it can be used in many applications as though they were observed in empirical work. Bai and Ng (2002) suggests to determine r by a criterion that can be written as

$$\hat{r} = \min_k \log \left(1 - \sum_{j=1}^k D_{jj}^2 \right) + k g(N, T)$$

where $g(N,T) = \frac{N+T}{NT} \log(\frac{NT}{T+T})$, D_{jj}^2 is the variance in Z explained by factor j, and $\text{SSR}_k = 1 - \sum_{j=1}^k D_{jj}^2$.

Our analysis requires estimating factors in Z and Z^2. Two issues arise. First, the squared data can exaggerate the role of outliers, and principal components are known to be sensitive to influential observations (also known as "noise corruption"). As robustness check, we also weigh the observations as in Boivin and Ng (2006), leading to a GLS type estimator of the factors. For isolating A and V, the more important problem is that $\hat{r}$ tends to accept too many 'weak' factors. This matters since we need to project X on H, and overfitting is likely if H is high dimension. The machine-learning literature suggests to handle both outliers and 'weak' factors by regularization. Minimizing $\text{SSR}_k + \lambda(||F||_F^2 + ||\Lambda||_F^2)$ yields regularized principal components (RPCA) defined as $\bar{F} = U_r (D_r^+)^{1/2}$ and $\bar{\Lambda} = V_r (D_r^+)^{1/2}$ where $D_r^+ = (D_r - \lambda I_r)_+$. The regularization tilts the objective towards finding a low rank component of Z that has the smallest rank possible. It does so

[9] The literature on factor models is large. See, for example,Sargent and Sims (1977); Quah and Sargent (1993); Forni et al. (2000); Stock and Watson (2002a,b); Bai and Ng (2002, 2006). More commonly used is the method of asymptotic principal components (APCA) due to Connor and Korajczyk (1993) which gives $\hat{F} = \sqrt{T} U_r$ with $\frac{F^T F}{T} = I_r$. The PCA estimator uses the normalization $F^T F = D_r$ instead of $\frac{F' F}{T} = I_r$. But the PCA factors are perfectly correlated with the APCA factors. See Bai and Ng (2017).

by shrinking the PCA estimates towards zero, and discards those with eigenvalues below a threshold λ. Regularization suggests a new criterion for determining r (see Bai and Ng (2017)):

$$\bar{r} = \min_k \log \left(1 - \sum_{j=1}^{k} (D_{jj} - \lambda)^2\right) + kg(N,T).$$

Adding a data-dependent term to the penalty gives a more conservative estimate of r. It should be noted that the RPCA estimates that survive thresholding are perfectly correlated with the PCA factors (since they are both spanned by $U_{\bar{r}}$).

Algorithm-AV is simple, but comes at the expense of ignoring the non-linear dependence between the A and G, for example.[10] It is thus useful to verify the adequacy of the approximation. We conduct a monte-carlo experiment to check the effectiveness of Algorithm-AV. Using the second-order approximate solution as guide, a panel of $T = 400$ and $N = 100$ observations is simulated as

$$\begin{aligned} X_{it} &= \Lambda'_{1,A} A_t + \Lambda'_{1,G} G_t + \Lambda_{1,V} V_t + \epsilon_{1it} = \Lambda'_1 F_t + \epsilon_{1it} \\ X^2_{it} &= \Lambda'_{2,G} G_t + \Lambda_{2,V} V_t + \epsilon_{2it} = \Lambda'_2 S_t + \epsilon_{2it} \end{aligned}$$

We assume $r = 2$ level factors: $A_{1t} = \rho A_{1t-1} + \sqrt{v_t} e_{1t}$ and $A_{2t} = e_{2t}$. Here, $G_t = \{A^2_{1t}, A^2_{2t}, A_{1t}A_{2t}\}$. Two volatility processes are considered: a chi-square process with one degree of freedom, and a beta-distributed process with parameters 5 and 3. The errors $e_{1t}, e_{2t}, \epsilon_{1it}, \epsilon_{2it}$ are independent normally distributed with mean zero and unit variance. The Λ parameters are calibrated to mimic the data. For example, in the chi-square distribution case, the level factors A explain about 66%, the G factors explain 0.2%, the volatility factor V explains 5% of the variation of X. In the beta-distribution case, the shares are 0.5, 0.02, and 0.09, respectively. The u series are simulated only once, but all other errors are re-generated in the 1000 monte-carlo experiments.

The dimension of F is estimated by $\hat{r}_F$ to be 4 and by $\bar{r}_F$ to be 2.5 with the maximum number of factors set to eight, both higher than the true value of two. On average, $\hat{r}_S$ suggests five factors in X^2 while $\bar{r}_S$ suggests three. In each replication, we use $\bar{r}_F$ as r_F to in Algorithm-AV. In the chi-square distribution case, the dimensions of A and V are estimated to be two and one on average. The average dimension of V is estimated to be 2.4 with a median of 2. In the beta-distribution case, r_A and r_V are 1.3 and 1.5. Though we over-estimate the number of Vs, the correlation between the average $\hat{V}_1$ and V is 0.95 in the simulations.

[10]This is analogous to the drawback that A_t and A_{t-1} are treated as separate factors in a factor model.

4 Factors in X and X^2

We estimate the factors using data from FRED-MD McCracken and Ng (2016), a macroeconomic database consisting of a panel of 134 series over the sample 1960M1-2015M12. Consistent with previous studies, the data are transformed by taking logs and first differencing before the factors are estimated. Algorithm-AV uses regressions of X and X^2 on H to obtain $\tilde{X}$ and $\tilde{X}^2$. The goodness of fit of these regressions are indicative of the importance H. The average R^2 in the X regressions is 0.119; the median is 0.033, and the maximum is 0.836. The three series with the highest R^2 are housing starts, total housing starts, and housing starts in the northeast. Thus, the interaction of the level factors have strongest effects on the housing variables. The average R^2 in the X^2 regressions is 0.201; the median is 0.146, and the maximum of 0.794. The highest R^2s are recorded for the price of commodities, followed by the price of non-durables and CPI-ex shelter. Interestingly, the correlation between X^2 and $\tilde{X}^2$ is strongest for non-borrowed reserves (0.998) and weakest for commodity prices (0.204). The correlation between X and $\tilde{X}$ is strongest for non-borrowed reserves and weakest for housing starts.

Data Z	Factors	Components	$\hat{r}$	$\bar{r}$	4σ Spikes
X	F	(G, V, A)	8	3	(4): 74M12, 80M5, 08M12-09M1
X^2	S	(G, V)	8	3	(8): 74M12, 80M4-M5, 08M9-09M1
$\tilde{X}$	A	A	8	3	(1): 80M5
$\tilde{X}^2$	V	V	6	2	(1): 80M5

The common factors in $\tilde{X}$ are A and V, and the common factors in $\tilde{X}^2$ are V, respectively. The Bai and Ng (2002) criterion finds $\hat{r}_F = 8$ PCA factors, which is also the maximum number of factors considered. Consistent with previous work using this data, the RPCA estimate of F_1 loads heavily on production and employment variables, F_2 on term spreads, and F_3 on prices. We set λ to 0.05. In the data, D_{11} is 0.40 and explains about 0.16 of the variations in the data. After thresholding, $D_{11}^+ = 0.35$ and the explanatory power falls to 0.125. The smallest eigenvalue, D_{88} is 0.154 and still survives thresholding. But according to the heavier penalty, $\bar{r}$ is only three. Our analysis attributes the five weak factors to interactions of the level factors and to second-moment factors. There are also eight GLS factors. Though the first GLS factor still loads heavily on production and employment variables, it also loads heavily on several housing sector variables. Furthermore, both F_2 and and F_3 load heavily on interest rate variables. Recall that the PCA and RPCA factors are both spanned by U_r, hence perfectly correlated. Without loss of generality, we only plot the RPCA estimates. The top panel of Figure 1 plots the RPCA and GLS estimates of F_1, standardized to be mean zero with unit variance.[11] The bivariate correlations between RPCA and GLS is 0.94. The

[11] Because the factor estimates are only identified up to sign, the normalization is chosen so that the series is negative in the 1980 recession.

estimate of F_1 is strongly procyclical. During the 1973, 1978, and 2008 recessions, the estimates are more than three standard deviations below the mean.

The factors in X^2 are denoted S. The second panel of Figure 1 shows that S_1 is cyclical and highly volatile. During major recessions, it is as many as six standard deviations above the mean of zero. The RPCA estimate (black line) has larger spikes than the GLS estimate. The correlation between RPCA and GLS estimate of S_1 is 0.77. While the RPCA estimate indicates that S_1 is strongly correlation with production variables, the GLS estimate of S_1 loads more heavily on the housing sector variables. Evidently, GLS weighting changes the loadings but not the estimated number of factors. In what follows, we focus on the results for the RPCA estimates.

The level factors A and volatility factors V are new to this line of work. They are estimated from $\tilde{X}$ and $\tilde{X}^2$, which are purged of the quadratic effects of F. The PCA and RPCA estimates are no longer perfectly correlated because $\hat{r}_F > \bar{r}_F$. Nonetheless, the two estimates of A_1 have a correlation of 0.72. Both load heavily on production and employment variables. The third panel of Figure 1 shows the estimates of A_1. They are low during the 1973, 1982, and 2001 recessions but slightly below mean during 1990 and 2008 recessions, both thought to be of financial origins. While PCA gives an A_2 that loads on prices, the RPCA factor that loads on prices is A_3. For RPCA, A_2 loads on term spreads and housing sector variables.

The bottom panel of Figure 1 plots our V_1 normalized to be positive during the 1980 recession. The V_1 series is clearly cyclical and has a spike in each of the recessions. Both the PCA and RPCA estimate of V_1 load heavily on the housing sector variables. The two series have a correlation of 0.76. The estimates of V_2 load heavily on term spreads. In the rest of the section we attempt to relate the estimated V_1 to alternative measures of volatility/uncertainty.

The logic of Section 2.1 suggests that if there is a single factor in the first moment of the data, then V_1 should measure stochastic volatility in the innovations of that factor. To verify this, we directly estimate the stochastic volatility in A_1 instead of backing it out from X^2 as suggested in Algorithm-AV. Precisely, we first estimate an AR(4) for A_1 and then fit a first-order autoregressive stochastic volatility model to the residuals of the AR(4).[12] As seen from the top panel of Figure 2, the "directly estimated" volatility series is negatively correlated with our estimate of V_1 ($\rho = -0.34$). This result can be attributed to the fact that we have not one, but multiple factors in the data, and we can only estimate a linear combination of them. Our estimate of V_1 potentially includes the common volatility to the idiosyncratic shocks which the theoretical framework cannot accommodate. Hence, we think of V_1 as a composite volatility factor.

With the above caveat in mind, how does our estimate of V_1 relate to volatility/uncertainty measures available in the literature? The first measure for comparison is the economic policy

[12] We use the package STOCHVOL in R.

uncertainty (EPU) index constructed by Baker et al. (2015). While economic policy uncertainty is only a part of what can qualify as a second-order shock, EPU is an example of a higher-order shock identified with a narrative approach and a number of other sources. We find that V_1 and EPU are moderately correlated ($\rho = 0.40$). The behavior of the series also differs across recessions (the top panel of Figure 2). For example, both series soared during the Great Recession, but EPU increased during the 2001 recession while V_1 did not. Furthermore, V_1 rose during the Volcker recession while EPU move up only by a tad. In short, EPU and V_1 have independent variations.

The second comparison is with the uncertainty series of Jurado et al. (2015) (henceforth JLN). Their approach estimates stochastic volatility in the h step-ahead idiosyncratic forecast error for a panel of macroeconomic series, then averaged to form an aggregate uncertainty series. While both the present paper and JLN analyze stochastic volatility, the key difference is that JLN considers the common variations in the expected volatility of a panel of macroeconomic series, while the present paper focuses on the current volatility in the factors common to the level of N series. These are conceptually distinct objects. Figure 2 shows that the two series are weakly correlated ($\rho = 0.22$). Analysis of specific episodes also exposes important differences. For example, during the Great Recession both series increased considerably. Yet, the JLN series increased a lot more and then declined to zero by 2011 while V_1 stayed elevated until mid 2015.

Given that V_1 is a component of common variation in X^2, and S are the common factors in X^2, might S be a good proxy for V_1? We see from the bottom panel of Figure 2 that the estimated V_1 and S_1 also have independent variations. The divergence during the Great Recession is particularly notable: V_1 rises two standard deviations, while S_1 increases by more than four standard deviations. The correlation between V_1 and S_1 is 0.29. What explains the low correlation between the two series? In our approach, V is formed by purging G from S. Since the "level" factors do not have a direct "level" effect on the second moments according to equation (6), the low correlation between V_1 and S_1 suggests that a considerable fraction of the common variations in X^2 comes from G, i.e. non-linear functions of the level factors. Consistent with this logic, we find that when we regress S_1 on squares and interactions of $\{F_1, F_2, F_3\}$, we obtain R^2 of 0.75. The bottom panel of Figure 2 also reveals that V_1 closely tracks S_2, the second factor extracted from X^2, with a correlation of 0.83. Since by construction S_1 and S_2 are uncorrelated, we conclude that our particular rotation of factors S roughly separates the "volatility" factor into S_2 and the higher-order component due to non-linear effects of "level" factors into S_1.

5 FAVARsq Analysis

To understand the dynamic responses of macroeconomic variables to the level an second-moment shocks, we augment a FAVAR with second-moment factors. This leads to a reduced-form FAVARsq

$$\begin{pmatrix} \text{Factors}_t \\ Y_t \end{pmatrix} = \sum_{k=1}^{p} A_k \begin{pmatrix} \text{Factors}_{t-k} \\ Y_{t-k} \end{pmatrix} + \begin{pmatrix} \eta_{\text{Factors}t} \\ \eta_{Yt} \end{pmatrix}.$$

where FACTORS are common components in the data (e.g. F_1, S_1, A_1 and V_1). In this paper, the reduced-form errors η are orthogonalized by Cholesky decomposition to obtain structural shocks. We consider

$$Y_t = \begin{pmatrix} \text{HOU}_t & \text{IP}_t & \text{INFL}_t & \text{FFR}_t \end{pmatrix}'.$$

where HOU is housing starts of total new privately owned (series 50), FFR is the Federal Feds Rate, and IP is industrial production (series 6). Annual inflation is INFL=log(CPIAUSL$_t$) − log(CPIAUSL$_{t-12}$) where CPIAUCSL (series 113) is the consumer price index. Although the original FAVAR in Bernanke et al. (2005) did not include housing as an "outcome" variable in vector Y_t, we add this variable for two reasons. First, housing is key for understanding the Great Recession. Second, some of the estimated factors load heavily on housing, but we do not know if if the housing sector variables respond to volatility shocks.

Without a fully specified model, we will not able to provide a complete structural interpretation of F. But by letting the data speak, we can still shed more light on how the factors at different levels interact. In what follows, we consider a series of models with different choices of FACTORS to understand the role of V in economic fluctuations and its relation with the level factors. To preserve space, we consider only factors F_1, A_1, S_1, V_1 estimated with RPCA and report results for selected orderings as alternative orderings yield similar results.

Baseline Model: FACTORS=(F_1)

The largest factor F in the data is well documented to be a real activity factor. Our point of reference is therefore the simplest FAVAR with FACTORS set equal to F_1, the largest factor in X.[13] The decomposition of variance is reported in Table 1. The effect of an F_1 shock has large short-run effects on IP and HOU but the effects decline after the initial peak. In contrast, the effects on FFR and INFL grow over time. After 60 months, an F_1 shock explains approximately 50% of variation in FFR and approximately 30% of variation in INFL.

Model I: FACTORS=$(A_1 \quad V_1 \quad F_1)$

[13] Results are similar when we consider a larger set of factors which includes F_2, F_3, etc.

To isolate the effect of volatility shocks, we introduce the largest level factor A_1 and the largest volatility factor V_1 to the set of FACTORS. By ordering A_1 and V_1 before F_1, we examine the importance of conventionally used F_1 after conditioning on a volatility factor and a level factor. That is, we can isolate variation in F_1 due to A_1, V_1 and other factors such $A_2, A_3, ...$ and $V_2, V_3, ...$ as well as terms quadratic in $A_1, A_2, ...$ Note that if we order F_1 *before* A_1 and V_1, we will not be able to do this decomposition.

Figure 3 plots the impulse responses (along with 68% bootstrap confidence intervals) of macroeconomic variables in the model to V_1. We find that after a standard deviation increase in V_1, housing starts (HOU), fed funds rate (FFR), industrial production (IP), and inflation (INFL) all decline. This pattern is qualitatively similar to what one may expect after a negative demand shock. Although qualitatively the effects of a V_1 shock on the model variables are all negative, there are differences across variables. Housing starts (HOU) and industrial production (IP) decline on impact after a shock to V_1. In contrast, we observe little short-run effects on inflation. The fed funds rate declines on impact and then continuously falls.

Figure 4 contrasts the responses to a one-standard deviation shock to F_1 (black line) to a one-standard deviation shock to A_1 (blue line). After a shock to F_1, housing starts, industrial production, inflation and fed funds rate all decline. These dynamics are similar to what one may expect after a negative demand shock. On the other hand, a level shock to A_1 decreases housing starts, industrial production, and the fed funds rate, but increases inflation. This pattern is consistent with the dynamics one may observe after a negative supply shock. The differences in the responses signal that the presence of the volatility factors can influence the "level" shocks that are being identified. Indeed, given that F_1 is positively correlated with A_1 and negatively with V_1, our results support the premise of this paper that F_1 is a mixture of level and volatility factors. Hence the responses to a shock in F_1 is likely a mixture of responses to "level" (supply-like) shocks A_1 and "volatility" (demand-like) shocks V_1.

To assess the quantitative significance of the effects, Table 2 reports the decomposition of variances. We see that a V_1 shock generally has weaker effects on real activity, inflation, and the fed funds rate in the short run relative to the responses at longer horizons. For example, V_1 accounts for less than 1% of variation in INFL at the one-quarter horizon but 20% at the 4-5 year horizons. V_1 accounts for a modest fraction (4-9%) of variation for housing (HOU) and industrial production (IP) at the same horizons. Compared to the baseline model without V_1 in Table 1, the effects of F_1 are reduced. For example, F_1 in the baseline model accounts for nearly 30% of variation in inflation at the 5-year horizon but the contribution is closer to 15% in the model that includes V_1. For horizons longer than one year, the total contribution of shocks to A_1, V_1 and F_1 is generally close to the total contribution of shocks to F_1 in Table 1. Thus, we reach our first important conclusion that factors F is a mix of a variety of shocks some of which may capture innovations to volatility.

Bernanke et al. (2005) shows that using the factors F enhances identification of monetary policy shocks in VARs as the factors can provide a better summary of the information set available to central bankers. We explore if decomposing F into A and V can further enhance identification and allow potentially differential responses of monetary policy to level and volatility shocks. The responses of recursively identified monetary policy shocks from our FAVARsq appear to be only slightly different from the baseline (FAVAR) model. Monetary policy shocks account for similar fractions of fluctuations in housing, inflation and industrial production whether the central bank's information set is proxied with (F) or (A, V, F). These findings are consistent with the view that the central bank reacts to changes in inflation and output regardless of where these changes come from. In other words, the reaction of the Federal Reserve System to a one percentage point increase in inflation due to a volatility shock is similar to the reaction of the Fed to a one percentage point increase in inflation due to e.g. a supply-side shock.

Model II: FACTORS$=\begin{pmatrix} A_1 & V_1 & S_1 \end{pmatrix}$:

Our analysis of the second-order approximate solution of a generic DSGE model suggests that dynamics of macroeconomic aggregates can be influenced by non-linear effects and second-moment shocks. In Model I, we have attempted to assess the contribution of volatility shocks. In the next exercise, we try to quantify the contribution of non-linear effects. Specifically, we note that, according to the second-order approximate solution, variation in X^2 comes from V (volatility factors) and G (squares and interactions of level factors). Thus when we extract S from X^2, we should recover a rotation of V and G. Since we have an estimate of V, we can obtain a rotation of G. More precisely, orthogonalizing the shocks in S with respect to those in V should isolate second-moment variation that is not directly due to stochastic volatility in the fundamentals. Therefore, the orthogonalized shocks to S can be broadly interpreted as a composite of non-linear effects on the level of macroeconomic variables.

We consider a FAVARsq by adding S_1 to Model I. By putting S_1 last in the list of factors, we orthogonalize innovations in S_1 to innovations in A_1 and V_1. The impulse response functions are shown in Figure 5. Shocks to S_1 have effects qualitatively similar to V_1: real activity contracts, and the shocks trigger modest short-run responses in the Fed-Funds rate rate and inflation. The decomposition of variances are reported in Table 3. A shock to S_1 has a non-trivial short-run impact on housing starts, and its effect on the Fed-funds rate and inflation are comparable to that of the F_1 shock at long horizons. Compared to the results in Table 2, we see that adding S_1 to the FAVARsq does not change the significance of V_1 much. Our simple framework suggests that some of the variations in X are potentially due to non-linear interaction effects (i.e., G in our notation). They are also consistent with the finding in Bai and Ng (2008) that the estimates of S and of F^2 both have predictive power for inflation and some measures of real economic activity.

Model III: FACTORS=$(A_1 \quad V_1 \quad \text{JLN} \quad \text{EPU})$

As we have discussed above, the volatility factor identified in our framework is only weakly related to alternative measures of volatility such as the economic policy uncertainty (EPU) index constructed by Baker et al. (2015) or the volatility (JLN) index constructed by Jurado et al. (2015). However, the *innovations* to these indices could be more interrelated. It is conceivable that various measures of second-moment shocks influence the same set of macroeconomic variables in a similar way so that little is lost by focusing on one of the measures of second-moment shocks. To investigate these hypotheses, we include both EPU and JLN in FACTORS of our FAVARsq. Because V_1 can be contemporaneously correlated with JLN and EPU, we consider two orderings: i) V_1 before JLN and EPU; and ii) V_1 after JLN and EPU. We find that the estimated responses in this model are close to those for Model I, and the alternative ordering makes little difference for the responses. In a similar spirit, the variance decompositions for V_1 (Table 4) are also close to the results for Model I and are insensitive to the ordering. In addition, the contribution of the level factor A_1 is unaffected by inclusion of JLN and EPU.

This exercise demonstrates that the contributions of V_1, JLN and EPU vary across shocks, variables and horizons, thus underscoring that different measures of volatility capture different channels of macroeconomic fluctuations. For example, JLN accounts for a large share of variation in housing starts (HOU) at long horizons while V_1 and EPU contribute about 1.4% and 0.7% respectively. On the other hand, V_1 is more important for explaining variation in inflation and the fed funds rate at longer horizons than either JLN or EPU. Generally, we find that EPU tends to contribute less than either V_1 or JLN. The collective contribution of volatility shocks V_1, JLN, and EPU to variation of the macroeconomic variables ranges from approximately 30% for HOU and FFR to about 10% for IP.

In summary, what we have shown is the following. First, from Model 1, the importance of the level factors is reduced once V_1 is controlled for. A V_1 shock has features of a negative demand shock, similar to an F_1 shock. An A_1 shock has features of a negative supply shock. Second, Model II demonstrates that the second-moment variations are uncorrelated with V_1, which could be understood as variations due to non-linear effects. Third, Model III shows that our volatility factor is distinct from measures of uncertainty with V_1 being more important in explaining inflation and the fed-funds rate than other measures of uncertainty/volatility. Together, the results show that V_1 has significant dynamic effects on the economic activity, but unlike the main channel postulated by theory, we only find a weak correlation between V_1 and the stochastic volatility of A_1. Furthermore, there is evidence of potentially important second-moment variations other than V_1.

6 Concluding remarks

Identifying the sources of fluctuations in aggregate data has been a long-standing agenda in macroeconomics. Although the conventional approach in this area is to use only shocks to levels of fundamentals, recent advances in theoretical and empirical macroeconomics suggest that higher-order shocks (volatility, uncertainty, etc.) could be an important determinant too. Structural attempts to separate these two distinct forces have to overcome identification and computational challenges and, even if successful, the results are conditional on correct specification of the model.

This paper contributes to the effort by developing a novel FAVARsq framework that allows macroeconomic fluctuations to be determined by these two distinct types of factors. Apart from being computationally attractive, this framework can be suitable in other applications provided that we have a large panel of data that have common level and second moment variations, and our identifying restriction is satisfied. Examples are asset prices, sectoral and international data.

Our analysis of three FAVARsq leads us to three important conclusions. First, the largest "volatility" factor V_1 is countercyclical, persistent, and accounts for a modest but tangible share of variation in macroeconomic variables especially at longer horizons. There is evidence that some of its variations have indeed been attributed to F_1, the first factor in X. Second, while the common variations in X^2 are influenced by V_1, the interaction of the level factors can have non-trivial cyclical implications. In the VAR considered, the effects on inflation and interest rate are the strongest. Third, our V_1 has variations that are independent of measures of uncertainty as well as stochastic volatility directly estimated from the largest level factor. The data thus suggest multiple sources of second-moment variations.

While we make progress in isolating the "volatility" factors from "level" factors, further restrictions are needed to give more precise interpretations to the volatility factor. For example, our V_1 is likely a composite of volatility from different sources, some of which may have no role in theoretical macroeconomic models. The interaction between the first- and second-order dynamics is worthy of more theorizing in light of the evidence for non-trivial second-moment variations.

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Table 1: Decomposition of Variances: Baseline Model; shock to F_1.

h	F_1	HOU	IP	INFL	FFR
1	1.000	0.104	0.701	0.004	0.085
6	0.802	0.089	0.668	0.065	0.368
12	0.732	0.056	0.656	0.161	0.504
24	0.674	0.043	0.650	0.269	0.597
60	0.651	0.056	0.646	0.283	0.506

Note: The table reports variance decomposition for FAVAR with $\text{Factors}=\big(F_1\big)'$. The factor is estimated with robust principal components. HOU is housing pertmits, FFR is the Federal Funds Rate, IP is industrial production, INFL is the CPI inflation rate. See Section 5 for more details.

Table 2: Decomposition of Variances: Model I

Ordering	A_1	V_1	F_1	HOU	IP	INFL	FFR
h	Shock A_1						
1	1.000	0.015	0.517	0.003	0.543	0.006	0.006
6	0.906	0.020	0.354	0.006	0.447	0.005	0.010
12	0.905	0.031	0.369	0.005	0.448	0.004	0.064
24	0.886	0.028	0.339	0.012	0.439	0.025	0.135
60	0.817	0.056	0.323	0.047	0.430	0.037	0.116
h	Shock V_1						
1	0.000	0.985	0.057	0.083	0.010	0.000	0.054
6	0.015	0.830	0.074	0.048	0.032	0.028	0.068
12	0.017	0.706	0.069	0.029	0.032	0.097	0.120
24	0.016	0.559	0.071	0.040	0.037	0.187	0.168
60	0.015	0.472	0.109	0.087	0.049	0.194	0.142
h	Shock F_1						
1	0.000	0.000	0.427	0.108	0.198	0.030	0.161
6	0.036	0.025	0.398	0.092	0.251	0.122	0.476
12	0.033	0.035	0.331	0.067	0.245	0.173	0.462
24	0.039	0.029	0.298	0.052	0.243	0.159	0.411
60	0.055	0.022	0.264	0.049	0.237	0.131	0.322

Note: The table reports variance decomposition for FAVARsq with $\text{Factors}=\big(A_1 \quad V_1 \quad F_1\big)'$. F_1 is the first factor in extracted from X. V_1 is the first volatility factor net of quadratic variations in the level factors. A_1 is the first level factor. V_1 and A_1 are extracted as described in Section 2. The factors are estimated with robust principal component analysis. HOU is housing starts, FFR is the Federal Funds Rate, IP is industrial production, INFL is the CPI inflation rate. See Section 5 for more details.

Table 3: Decomposition of Variances: Model II

Ordering	A_1	V_1	S_1	HOU	IP	INFL	FFR
h	Shock to A_1						
1	1.000	0.016	0.025	0.005	0.529	0.006	0.004
6	0.946	0.016	0.056	0.003	0.464	0.003	0.014
12	0.943	0.025	0.090	0.002	0.462	0.008	0.076
24	0.922	0.022	0.086	0.009	0.453	0.044	0.149
60	0.850	0.051	0.099	0.039	0.444	0.058	0.124
h	Shock to V_1						
1	0.000	0.984	0.078	0.086	0.009	0.000	0.054
6	0.019	0.834	0.048	0.046	0.035	0.021	0.069
12	0.021	0.715	0.042	0.027	0.035	0.086	0.122
24	0.020	0.558	0.057	0.038	0.039	0.173	0.164
60	0.019	0.471	0.097	0.088	0.051	0.176	0.131
h	Shock to S_1						
1	0.000	0.000	0.898	0.097	0.094	0.002	0.015
6	0.005	0.004	0.785	0.146	0.138	0.122	0.161
12	0.005	0.004	0.688	0.129	0.134	0.180	0.177
24	0.011	0.009	0.621	0.117	0.132	0.184	0.209
60	0.032	0.010	0.550	0.101	0.132	0.190	0.243

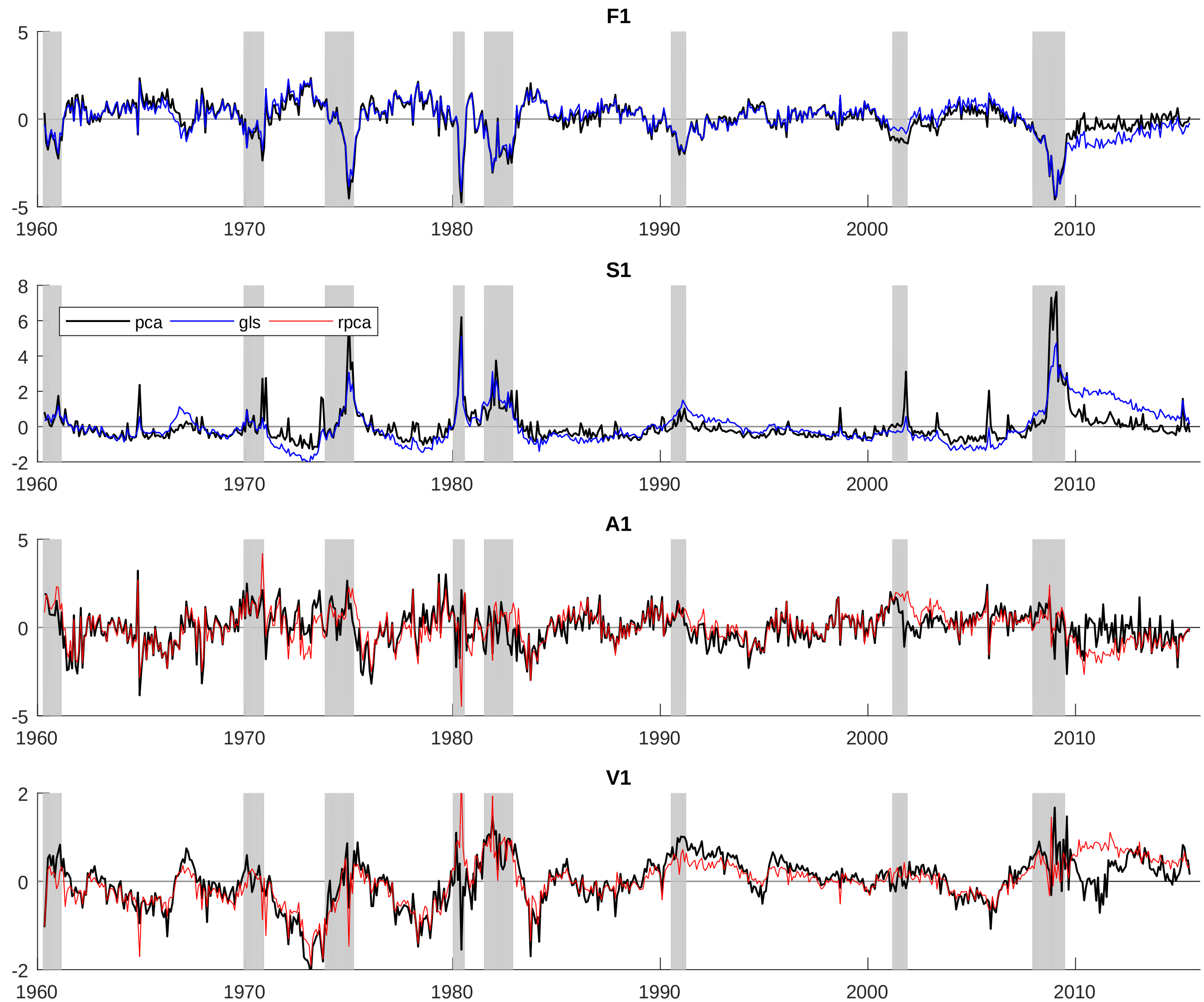
Note: The table reports variance decomposition for FAVARsq with FACTORS=$\begin{pmatrix}A_1 & V_1 & S_1\end{pmatrix}'$. V_1 is the first "volatility" factor net of quadratic variations in the level factors. A_1 is the first level factor. V_1 and A_1 are extracted as described in Section 2. S_1 is the first factor extracted from squared data X^2. The factors are estimated with robust principal component analysis. HOU is housing pertmits, FFR is the Federal Funds Rate, IP is industrial production, INFL is the CPI inflation rate. See Section 5 for more details.

Table 4: Decomposition of Variances: Model III

h	A_1	V_1	JLN	EPU	HOU	IP	INFL	FFR
ordering (A_1, V_1, JLN , EPU , HOU, IP, INFL, FFR)								
Shock to V_1								
1	0.000	0.986	0.027	0.003	0.094	0.010	0.000	0.050
6	0.012	0.790	0.015	0.011	0.060	0.037	0.029	0.084
12	0.012	0.658	0.013	0.024	0.033	0.036	0.111	0.149
24	0.012	0.492	0.012	0.036	0.018	0.037	0.205	0.196
60	0.013	0.369	0.045	0.036	0.014	0.043	0.233	0.184
Shock JLN								
1	0.000	0.000	0.943	0.010	0.012	0.004	0.013	0.008
6	0.008	0.011	0.858	0.038	0.154	0.054	0.021	0.010
12	0.008	0.010	0.807	0.052	0.265	0.066	0.011	0.011
24	0.013	0.071	0.738	0.070	0.397	0.072	0.007	0.014
60	0.054	0.202	0.597	0.103	0.502	0.072	0.056	0.126
Shock EPU								
1	0.000	0.000	0.000	0.985	0.000	0.000	0.000	0.002
6	0.012	0.030	0.019	0.922	0.011	0.014	0.001	0.004
12	0.018	0.056	0.020	0.883	0.013	0.020	0.001	0.007
24	0.021	0.066	0.021	0.833	0.010	0.022	0.005	0.017
60	0.019	0.051	0.019	0.780	0.007	0.022	0.023	0.029
ordering (A_1, JLN , EPU, V_1 , HOU, IP, INFL, FFR)								
Shock V_1								
1	0.000	0.957	0.000	0.000	0.104	0.012	0.000	0.041
6	0.011	0.763	0.070	0.020	0.087	0.050	0.023	0.084
12	0.011	0.639	0.068	0.042	0.058	0.050	0.103	0.155
24	0.011	0.484	0.052	0.060	0.035	0.049	0.200	0.208
60	0.016	0.355	0.080	0.061	0.023	0.055	0.254	0.226
Shock JLN								
1	0.000	0.028	0.970	0.011	0.003	0.002	0.013	0.015
6	0.010	0.044	0.806	0.033	0.129	0.042	0.027	0.011
12	0.010	0.036	0.756	0.042	0.241	0.054	0.019	0.007
24	0.015	0.086	0.700	0.055	0.381	0.061	0.014	0.005
60	0.051	0.221	0.565	0.086	0.494	0.062	0.040	0.091
Shock EPU								
1	0.000	0.001	0.000	0.987	0.000	0.000	0.000	0.003
6	0.011	0.025	0.017	0.918	0.009	0.013	0.001	0.003
12	0.017	0.049	0.017	0.875	0.011	0.019	0.001	0.005
24	0.020	0.060	0.019	0.824	0.009	0.021	0.003	0.013
60	0.019	0.047	0.017	0.771	0.006	0.020	0.018	0.024

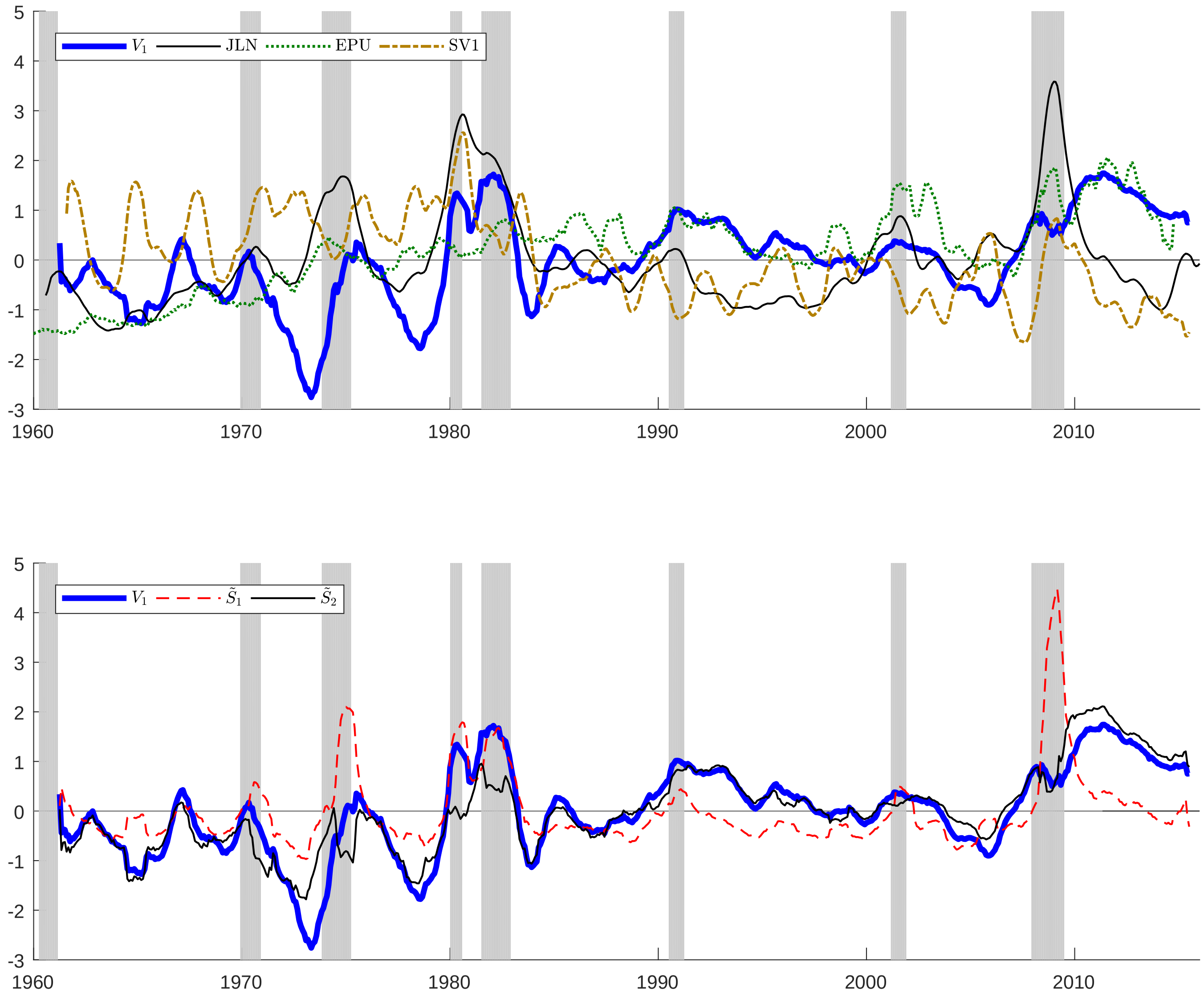
Note: The table reports variance decomposition for FAVARsq with Factors=$\begin{pmatrix}A_1 & V_1 & JLN & EPU\end{pmatrix}'$ and Factors=$\begin{pmatrix}A_1 & JLN & EPU & V_1\end{pmatrix}'$. V_1 is a "volatility" factor. A_1 is a "level" factor. The factors are estimated with robust principal components following the procedure described in Section 2. JLN is the volatility index constructed by Jurado et al. (2015). EPU is the economic policy uncertainty index constructed by Baker et al. (2015). HOU is housing starts, FFR is the Federal Funds Rate, IP is industrial production, INFL is the CPI inflation rate. See Section 5 for more details.

Figure 1: Factor Estimates



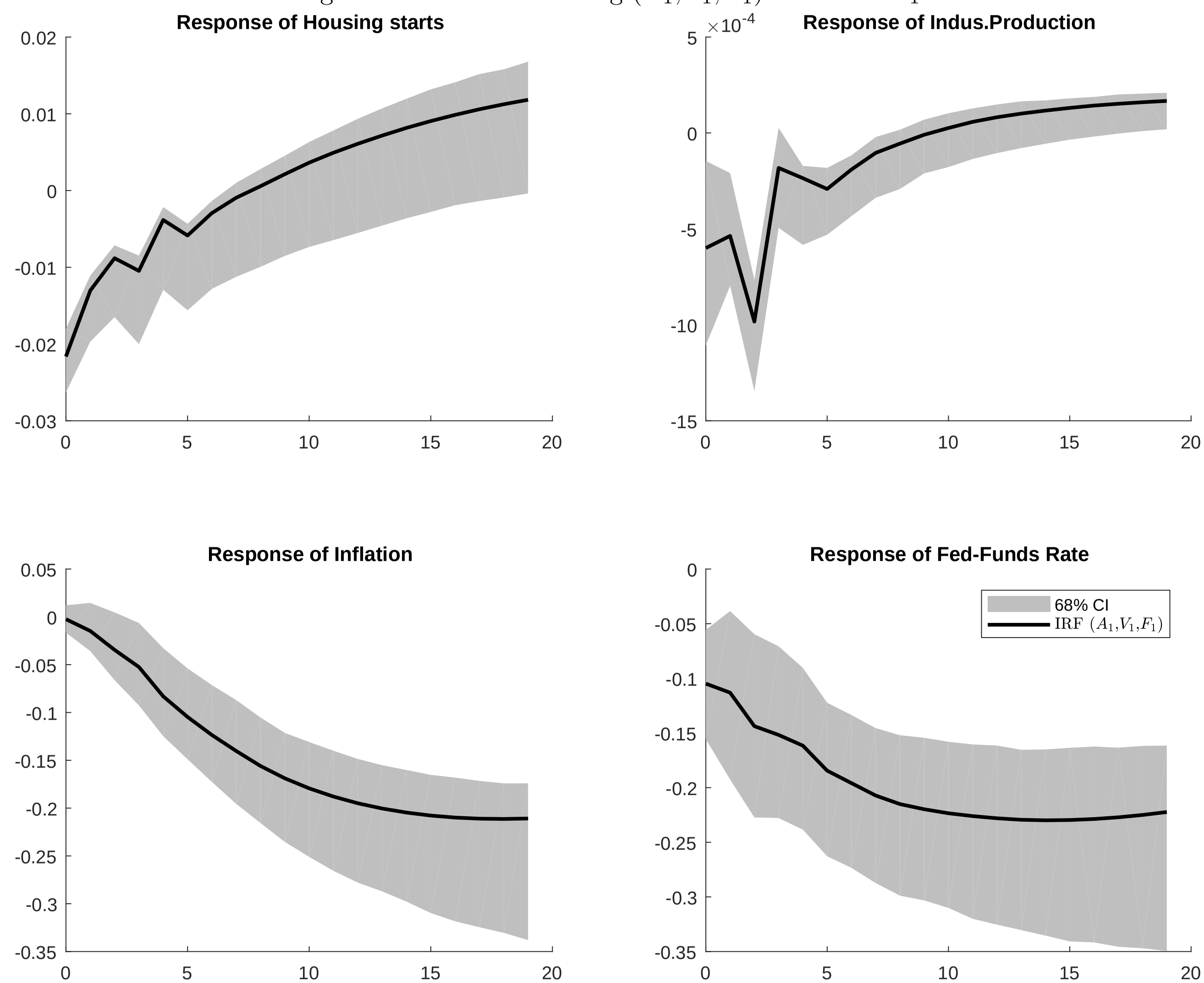
Note: The figure reports the time series of estimated factors. F_1 is the first factor extracted from the levels of data series X. S_1 is the first factor extracted from the squares of data series X^2. V_1 and A_1 are factors extracted as described in Proposition 1. V_1 is the first volatility factor. A_1 is the first level factor. Factors are extracted using principal component analysis (PCA; black line), robust principal component analysis (RPCA; red line), and GLS principal component analysis (GLS; blue line). See Section 4 for more details.

Figure 2: Comparison of factors



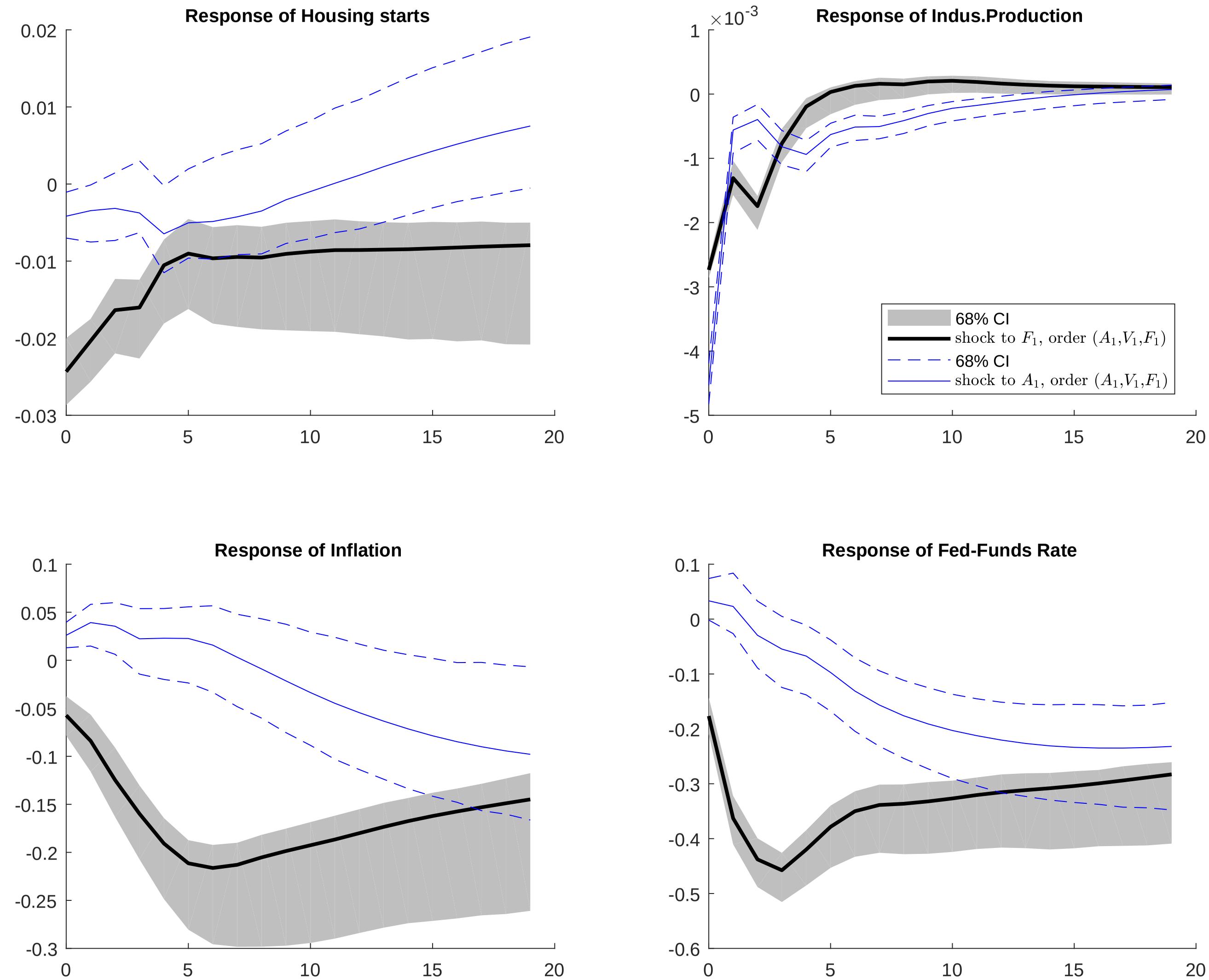
Note: The figure reports estimates of the first factor S_1 and the second factor S_2 extracted from the squares of data series X^2, and the volatility factor (V_1) estimated according to the framework outlined in Section 2. Also plotted is SV_1, the stochastic volatility estimated on A_1, the JLN uncertainty index constructed by Jurado et al. (2015), and the EPU economic policy uncertainty index constructed by Baker et al. (2015). Estimated factors are based on robust principal component analysis. See Section 4 for more details.

Figure 3: Model I: Ordering (A_1, V_1, F_1): Shock to V_1



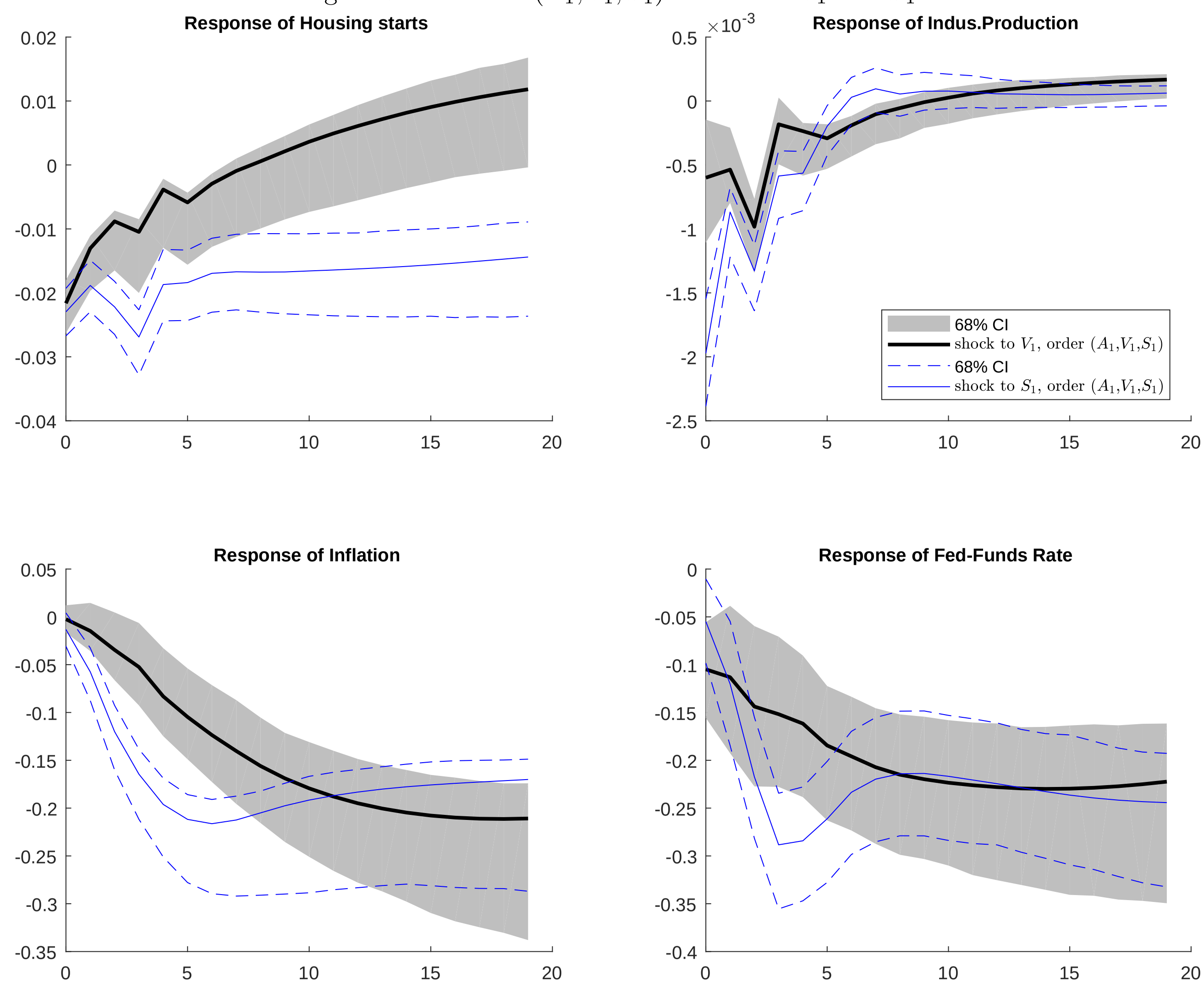
Note: The figure reports impulse responses to a one-standard deviation shock to volatility the factor V_1 in FAVARsq (Model I). The level factors (A) and volatility factors (V) are as outlined in Section 2. F_1 is the first factor extracted from the levels of data series X. Estimated factors are based on robust principal component analysis. See Section 5 for more details.

Figure 4: Model I: Ordering (A_1, V_1, F_1): Shocks to A_1 and F_1



Note: The figure reports impulse responses to a one-standard deviation shock to volatility the factor V_1 in FAVARsq (Model I). The "level" factors (A) and "volatility" factors (V) are constructed as outlined in Section 2. V_1 is the first volatility factor. A_1 is the first level factor. F_1 is the first factor extracted from the levels of data series X. All factors are extracted using robust principal component analysis. See Section 5 for more details.

Figure 5: Model II: (A_1, V_1, S_1): shocks to V_1 and S_1



Note: The figure reports impulse responses to a one-standard deviation shock to volatility the factor V_1 and S_1 in FAVARsq (Model II). The “level” factors (A) and “volatility” factors (V) are constructed as outlined in Section 3. Factor S_1 is the first factor extracted from squares of the data X^2. All factors are extracted using robust principal component analysis. See Section 5 for more details.