LEVEL AND VOLATILITY FACTORS IN MACROECONOMIC DATA

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Abstract

It is well documented that most representative panels of macroeconomic time series have a factor structure. The conventional wisdom is to associate the common shocks with innovations to the level of economic fundamentals. While second-moment shocks can be important drivers of economic fluctuations, there are few simple frameworks for understanding their dynamic effects. We suggest to first estimate a second-moment factor from the level and squared data, and then purge from it the non-linear variations in the level factors. Augmenting this orthogonalized second-moment factor $V_1$ to a FAVAR leads to a FAVARsq model that allows the dynamic effects of first and second-moment shocks to be studied without direct estimation of the latent volatility processes. Our $V_1$ is strongly counter-cyclical, persistent, but it not strongly correlated with the volatility of the real-activity factor. Its innovations explain a tangible share of the variations in housing permits, industrial production, the fed-funds rate, and inflation at horizons of four to five years. However, $V_1$ does not displace other second moment variations such as due to non-linearity or various measures of uncertainty. The overfall finding is that second moment variations have non-negligible macroeconomic effects and more theorizing is needed to understand the interaction between the level and second moment dynamics.


Keywords: volatility, business cycle fluctuations, common factors, robust principal components.

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1 Introduction

A long tradition in macroeconomics is to summarize aggregate fluctuations with a handful of shocks. In a celebrated paper, King and Rebelo (1993) showed that a large fraction of macroeconomic variations at business cycle frequencies can be accounted for by a single shock to the level of technology. At lower frequencies, nearly all macroeconomic fluctuations are often attributed to technology shocks (e.g., King et al. (1991)). More elaborate macroeconomic models also incorporate shocks to policies, oil prices, and preferences. Although these newer models have richer features and theoretical foundations, it is fair to say that using a few “level” shocks to generate cyclical fluctuations and co-movements is at the heart of macroeconomic modeling.

More recently, there is a nascent theoretical literature suggesting that higher-order shocks, and more specifically, second-moment volatility shocks, can also be an important source of business cycles. This alternative focus is motivated by the observation that realized volatility and expected future volatility (or uncertainty) tend to be high during recessions. This countercyclical feature of volatility is robust to whether the latent volatility variables are estimated or are replaced by proxy variables. Additional evidence that second-moment variations may have first-order effects is given in Fernandez-Villaderde and Rubio-Ramirez (2010).

The need to model the dynamics of volatility has long been recognized. In a seminal paper, Engle (1982) presents evidence of autoregressive conditional volatility (also known as ARCH effects) in inflation data. Sims and Zha (2006) also conclude that time-varying volatility is an important feature that empirical macroeconomic models should incorporate. From estimation of structural models, Justiano and Primiceri (2008) find significant time-varying volatility in monetary policy and technology shocks, while Fernandez-Villaverde et al. (2015) find that a two-standard deviation shock to fiscal volatility can reduce output by up to 1.5 percentage points when the economy is at the zero lower bound. Work along this line tends to assume that volatility is exogenous and that its shocks are independent of the innovations to the level of the fundamentals.

Despite statistical and methodological progress made in modeling volatility, the source of volatility shocks as well as the interaction between the level and volatility dynamics remain open questions to a large extent. While convenient, exogenous volatility precludes the possibility of volatility-in-mean effects that allow for feedback between the first- and second-moment dynamics. But the stochastic volatility estimates are typically countercyclical, suggesting that volatility is likely related to and possibly predictable by observed cyclical variables, which is at odds with the assumed exogeneity of volatility.

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1 See, for example, Schmitt-Grohe and Uribe (2004), Kim et al. (2008), Bloom (2009), Fernández-Villaverde et al. (2011), Fernandez-Villaverde et al. (2015), Jurado et al. (2015) and references there in.

2 See, for example, Justiano and Primiceri (2008) and Carriero et al. (2016) among others.
While it may be tempting to criticize the limitations of the exogenous volatility assumption, relaxing the assumption is not at all easy for a number of reasons. To begin with, economic theory treats level shocks as the dominant driver of economic fluctuations and does not provide much guidance as to what is the source of volatility fluctuations and how the volatility process is supposed to evolve. It is quite common to adapt models designed for high-frequency financial data to macroeconomic data, even though the two data types have distinctive time series properties. Furthermore, identifiability of linear dynamic models typically rely on the assumption of covariance stationarity. With time-varying volatility, covariance stationarity is no longer a valid maintained assumption. There can be many channels of how the conditional mean and volatility can generate observationally equivalent first- and second-moment dynamics. Model validation is difficult as the volatility is latent even ex-post.

Perhaps more important from a practical standpoint is that volatility modeling often requires computationally sophisticated methods. Though conceptually simple, adding stochastic volatility to an otherwise standard VAR or dynamic stochastic general equilibrium (DSGE) model entails a significant change in the estimation methodology. It is relatively easy to assess the sensitivity of a homoskedastic model to alternative assumptions, but the flexibility disappears once the volatility process has to be explicitly modeled.

In this paper, we propose a simple and easy-to-implement framework for studying the interaction between the first- and second-moment dynamics. It preserves the traditional view that only relatively few level shocks account for the bulk of variations in macroeconomic data but allows second-moment shocks to be drivers of economic fluctuations on the one hand and permits the second-moment factors to respond to the level shocks on the other. Specifically, Benigno et al. (2013) show that time-varying volatility has a second-order effect on the level of the endogenous variables, and their model solution suggests features in the data that can be used to distinguish “level” factors from “volatility” factors, at least in theory. However, in practice, we can only estimate the factors up to a rotation matrix. Furthermore, DSGE models only allow for time-varying volatility in the fundamentals and likely omit other second-moment shocks that may be empirically relevant. Hence, we can only recover a composite second-moment factor that we will call $V$. While our framework also omits many details that more sophisticated models would capture, the estimation exercise is nonetheless of interest because it provides a first glimpse of the importance of the second-moment dynamics. After all, if the level shocks are the sole source of economic fluctuations, then the second-moment shocks should have no cyclical implications whatever their structural interpretation might be. We find that not only are the effects of the second-moment shocks significant, their presence tends to reduce the importance of factors previously used in FAVARs.

Using a monthly panel of 134 macroeconomic time series, we estimate and find eight factors
in the level of the data $X$, which is consistent with previous studies.\(^3\) Given that the number of interesting primitive shocks in macro models is arguably smaller than eight, one might wonder why there are so many factors in the level of the data. According to our analysis, the factors in the data are a mixture of first- and second-moment common variations, implying that the number of genuine level shocks is likely less than eight.

Several patterns in the first- and second-moment factors are noteworthy. First, our dominant second-moment factor $V_1$ is estimated to be countercyclical and persistent. It rises during the Great Recession considerably and remains at an elevated level for many years. These basic properties indicate the potential of second-moment shocks as a source of fluctuations in aggregate series.

Second, our estimated $V_1$ is only weakly correlated with measures of volatility/uncertainty constructed in previous studies such as Baker et al. (2015) and Jurado et al. (2015). It is also only weakly correlated with the stochastic volatility series estimated directly from the real activity level factor. Our $V_1$ is most correlated with the first factor estimated from $X^2$, and which will be referred to as $S_1$ below.

Third, to study the dynamic effects of $V_1$ and $S_1$, we include $V_1$ in a small Factor-Augmented Vector Auto Regression (FAVAR) as in Bernanke et al. (2005). We refer to our framework as FAVARsq. Our emphasis is on the second-moment dynamics, hence distinct from the VAR proposed in Aruboa et al. (2013), whose focus is non-linearities. Our $V_1$ explains a modest-to-considerable share of variation in macroeconomic series such as housing permits, federal funds rate, industrial production, and inflation. The contribution of the second-moment shock at the horizons of 4-5 years ranges from 5% to 20%. The size of the estimated share is robust to alternative orderings of variables or adding alternative controls in FAVARsq. While the largest level factor in the data is unambiguously a real activity factor, its effects on inflation depend on whether we explicitly control for the presence of $V$. The responses to a positive shock to the second-moment factor $V$ are akin to the responses to a negative “demand” shock, while the responses to a negative shock in the orthogonalized level factor are reminiscent of responses to a negative supply shock.

Our FAVARsq is simple to estimate, but simplicity comes at the expense of ignoring the possible dependence between the data $X$ and its square $X^2$. This is reminiscent of the trade-off between the simplicity of a static factor model and the richer theoretical structure of a dynamic factor model.\(^4\) But even with the simple structure of a FAVARsq, consideration of common second-moment shocks raises some methodological problems that need to be addressed. One issue is that the squared data

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\(^3\)See e.g. Sargent and Sims (1977); Quah and Sargent (1993); Forni et al. (2000); Stock and Watson (2002b); Bai and Ng (2002).

\(^4\)A static factor model omits the temporal dependence in the factors that a dynamic factor model would capture, but the static model is easier to implement. Extending a non-linear VAR such as the one considered in Pesaran and Shin (1998) to allow for common factors of different order is possible in principle, but it will be computationally more demanding. This defeats our goal of finding a computationally simple alternative to full-blown structural estimation.
can exaggerate the role of outliers, and principal components are known to be sensitive to influential observations (also known as “noise corruption”). A second issue is the need to avoid overfitting the data with factors that are of second-order importance, and theory suggests many such factors will be present. Thus in addition to estimating the factors by standard PCA which can be implemented as iterative least squares, we also estimate ‘robust’ factors by iterative generalized least squares and iterative ridge regressions.

The rest of the paper is structured as follows. In the next section, we outline our framework. Specifically, we show how higher-order approximations of DSGE models can introduce volatility shocks and how one can separate “volatility” and “level” factors empirically. In Section 3, we present our approach to recover factors when data are potentially corrupted with noise. Section 4 discusses basic properties of the extracted “level” and “volatility” factors. Section 5 presents impulse responses and variance decompositions. Section 6 concludes.

2 Level vs. Second-Moment Factors

This section consists of three parts. Subsection 1 uses the one-sector stochastic growth model to highlight the main issues involved in a simple setting. Subsection 2 uses the general second-order solution of dynamic stochastic equilibrium (DSGE) models as a guide to understand what are the common factors that can be expected from the level and square of the data. Subsection 3 then presents an empirical factor model to suggest other second-moment shocks that might arise.

2.1 A Simple Example

Consider the one-sector stochastic growth model. Let \( z_t \) be technology with homoskedastic innovations \( \psi_t \), and let \( \tilde{c}_t \) and \( \tilde{k}_{t+1} \) be log-deviations of consumption and capital from the steady state, respectively. It is well known that the linearized solution is

\[
\begin{align*}
\tilde{c}_t &= b_k \tilde{k}_t + b_z z_t \\
\tilde{k}_{t+1} &= h_k \tilde{k}_t + h_z z_t \\
z_{t+1} &= \rho z_t + \psi_{t+1}.
\end{align*}
\]

Our point of departure is to modify the homoskedasticity assumption to allow for time-varying volatility in the technology shocks, i.e., \( z_{t+1} = \rho z_t + u_t \epsilon_{t+1} \) where \( u_t \) (which has a mean of \( \bar{u} \)) governs the volatility of shocks to technology and \( \epsilon_{t+1} \) is an i.i.d. zero mean, unit variance shock so that \( \psi_{t+1} = u_t \epsilon_{t+1} \) is heteroskedastic. One can interpret changes in \( u_t^2 \) as volatility shocks.

Applying the second-order approximate solution method of Benigno et al. (2013) and letting
\(v_t = u_t^2\) yields
\[
\dot{c}_t = b_k \dot{k}_t + b_z z_t + \frac{1}{2} \left[ b_{kk} \dot{k}_t^2 + b_{zz} z_t^2 + b_{kz} \dot{k}_t z_t + b_{uu} v_t \right] + \text{constant},
\]
\[
\dot{k}_{t+1} = h_k \dot{k}_t + h_z z_t + \frac{1}{2} \left[ h_{kk} \dot{k}_t^2 + h_{zz} z_t^2 + h_{kz} \dot{k}_t z_t + h_{uu} v_t \right] + \text{constant},
\]
\[
z_{t+1} = \rho z_t + \psi_{t+1}.
\]
There are now two independent but altogether three common sources of randomness: namely, \(z_t\), \(z_t^2\), and \(v_t\). As is evident even in the simple growth model, exogenous changes to the level and the second-moment factors do not have the same effects on \(\dot{c}\) and \(\dot{k}\). While the effects of \(z_t^2\) are expected to be smaller than those of \(z_t\), the quadratic effects omitted from the linear solution can still be important. Note that the equation for technology is common to both \(\dot{c}\) and \(\dot{k}\). These three observations provide a basis to separate the level shocks and the volatility shocks \(v_t\) from the second-moment variations.

Our goal is to better understand and disentangle the effects of the level shocks from the volatility shocks. To this end, we make several observations. First, the factors in \(\dot{c}\) and \(\dot{k}\) will, in general, be a combination of the level factors (e.g., \(z_t\) in the above example and the cross-product of \(z_t\) with the endogenous state variables) and volatility factors (e.g., \(v_t\) in the above example). Thus, when common components are extracted from a vector of macroeconomic variables, the extracted factors are likely a mix of level shocks, nonlinear terms, and volatility shocks.

Second, if we square both sides of the second-order approximate solution and omit the higher-order terms, we see that
\[
\dot{c}_t^2 = b_k^2 \dot{k}_t^2 + b_z^2 z_t^2 + 2b_k b_z \dot{k}_t z_t + \text{constant},
\]
\[
\dot{k}_{t+1}^2 = h_k^2 \dot{k}_t^2 + h_z^2 z_t^2 + 2h_k h_z \dot{k}_t z_t + \text{constant},
\]
\[
z_{t+1}^2 = \rho^2 z_t^2 + 2\rho \ddot{z}_t \epsilon_{t+1} + \ddot{u}_t + v_t,
\]
where the last two terms in the last equation follow from \(\ddot{z}_{t+1} = u_t^2 \epsilon_{t+1}^2 + \ddot{u}_t^2 + \ddot{v}_t^2 \approx 2\rho \ddot{u}_t z_t \epsilon_{t+1} + \ddot{u}_t + v_t\) and the second term in the last equation is an approximation of \(2\rho u_t z_t \epsilon_{t+1}\). Note that the squared data give information about \(z_t^2\) and the cross-product terms but not about \(z_t\) itself. At the same time, volatility \(v_t\) appears in the level and squared data.

Third, \(z_t^2\) is common to both \(\dot{c}_t\) and \(\dot{c}_t^2\). This is also true of other endogenous variables. Thus, the factors in \(\dot{c}_t^2\) and \(\dot{k}_t^2\) can be a subset of the factors in \(\dot{c}\) and \(\dot{k}\). These three observations provide a basis to separate the level shocks and the volatility shocks \(v_t\) from the second-moment variations.

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5 We implicitly assume that \(z_t\) is a variable observed by an econometrician. We make this assumption because shock series can be constructed from linear combinations of endogenous variables. For example, in the one-sector growth model, the production function implies \(z_t = \bar{y}_t - \alpha \bar{k}_t\) where \(\bar{y}_t\) is the log deviation of output from its steady state.

6 Because \(E(z_t) \equiv \ddot{z} = 0\) and \(E(\epsilon_t) \equiv \ddot{\epsilon} = 0\), \(2\rho u_t z_t \epsilon_{t+1} = 0\) and \(2\rho u_t z_t \ddot{\epsilon} = 0\).
2.2 The Second-Order Solution

The stochastic growth model is useful for gaining intuition, but the presence of only one exogenous state variable is restrictive. Consider the following description of a generic DSGE model:

\[
\begin{align*}
0 &= E_t\{Q(y_{t+1}, p_{t+1}, y_t, p_t)\} \\
z_{t+1} &= \Lambda z_t + A\psi_{t+1} \\
\psi_{t+1} &= U_t\epsilon_{t+1} \\
u_{t+1}^2 &= \bar{u}^2 + \Xi u_t^2 + \Omega\eta_{t+1}
\end{align*}
\]

where \(y_t\) is the vector of non-predermined variables, \(p_t = [k_t' \ z_t']'\) is a vector of predetermined (state) variables, \(z_t\) is the vector of exogenous variables, \(k_t\) is the vector of endogenous predetermined (state) variables. We continue to use “checks” on the above variables to indicate that the variables are measuring deviations from steady-state. The shocks \(\eta_t\) and \(\epsilon_t\) are mutually independent. \(\psi_{t+1} = U_t\epsilon_{t+1}\) is the vector of shocks or rational expectation errors that is a product of i.i.d shocks \(\epsilon_t \sim i.i.d(0, I)\) and volatility shocks collected into \(U_t\), a diagonal matrix whose entries \(u_t\) follow a VAR(1) structure. The volatility innovations \(\eta_t \sim i.i.d(0, I)\) have contemporaneous effects summarized by matrix \(\Omega\), see equation (1d). Equation (1a) summarizes the optimality conditions for economic agents, as given by vector function \(Q(\cdot)\). Equation (1b) describes dynamic properties of the forcing variables.

If we ignore the time-varying volatility, the model can be solved using the method of King and Watson (1998), Sims (2002), and Klein (2000), among others. The first-order solution is of the form

\[
\begin{align*}
\tilde{y}_t &= M(p_t, u_t) \\
\tilde{p}_{t+1} &= W(p_t, u_t) + W\psi_{t+1}
\end{align*}
\]

where \(M\) and \(W\) are vector functions. But \(z_{t+1}\) in equation (1b) is a conditionally linear process with heteroscedastic innovations. Applying the method of Benigno et al. (2013) yields the second-order approximate solution:

\[
\begin{align*}
\tilde{y}_t &= M_p\tilde{p}_t + \frac{1}{2}(I_y \otimes \tilde{p}_t')M_{pp}\tilde{p}_t + \frac{1}{2}M_{uu}u_t^2 + \text{constants} \\
\tilde{p}_{t+1} &= W_p\tilde{p}_t + \frac{1}{2}(I_p \otimes \tilde{p}_t')W_{pp}\tilde{p}_t + \frac{1}{2}W_{uu}u_t^2 + W\psi_{t+1} + \text{constants}
\end{align*}
\]

where \(\otimes\) denotes the Kronecker product.\(^7\) The approximation given in equation (3b) is relevant only for endogenous state variables \(k_t\) because the exogenous process \(z_t\) is already conditionally linear.

\(^7\)The matrices \(M_p, M_{pp}, M_{uu}, W_p, W_{pp}, W_{uu}\) correspond to the first- and second-order derivatives of functions \(M\) and \(W\) in equations (2a) and (2b). For example, \(M_{pp} = \frac{\partial^2 M(p, u)}{\partial p \partial p'}\). The constants depend on the volatility of \(z_t\) and curvature in the optimality conditions that render \(E(\tilde{y}_t) = 0\) and \(E(\tilde{p}_t) = 0\).
The representation given by equations (3a)-(3b) has several important features. First, the dynamics have a factor structure. The common “factors” are the “level” of the state variables \( \tilde{p}_t \), the shocks \( \psi_{t+1} \), the “second moment” variables originating from the variances and covariances of the state variables, as well as volatility \( u_t^2 \). Hence, volatility shocks \( \eta_t \) have a direct effect on “level” variables. Given the conditional linearity of \( z_t \) and independence of the shocks, there is no interaction term between volatility \( u_t^2 \) and the state variables \( p_t \) in the second-order approximation. Note also that \( \psi_{t+1} \) directly affects only \( z_{t+1} \), not \( k_{t+1} \).

Second, the squared entries of \( \tilde{y}_t \) or \( \tilde{p}_t \) depend only on squares of “level” terms in equations (3a)-(3b). That is,

\[
\text{diag}\{\tilde{y}_t \tilde{y}_t'\} = \text{diag}\{M_p \tilde{p}_t \tilde{p}_t' M_p'\} + \text{constant} \tag{4a}
\]

\[
\text{diag}\{\tilde{p}_t \tilde{p}_t'_{t+1}\} = \text{diag}\{W_p \tilde{p}_t \tilde{p}_t'_{t+1} W_p'\} + \text{diag}\{W_p \tilde{p}_t \psi_{t+1} W_p'\}
+ \text{diag}\{W_p \psi_{t+1} \psi_{t+1}' W_p'\} + \text{constant}. \tag{4b}
\]

Other terms when squared have orders greater than two and thus are ignored. The second and third terms in the squared solution are interactions of the “level” variables (i.e. state variables \( \tilde{p}_t \)) and \( \psi_{t+1} \) (the innovations to \( \tilde{a}_t \)). Importantly, these interaction terms are not present in the second-order solution of \( \tilde{y}_t \) and \( \tilde{p}_t \) given in (3a)-(3b). The last term in equation (4b) is a combination of volatility in \( u_t \) and \( \epsilon_{t+1} \).\(^9\)

Third, the “level” factors appear linearly and quadratically in equations (3a)-(3b). Furthermore, the squared “level” terms appear in equations (3a)-(3b) and equations (4a)-(4b). Hence there is a “cross-equation” restriction not only in terms of factors but also in terms of loadings on the factors.

For subsequent analysis, it is useful to consider a representation in terms of mutually orthogonal factors instead of the system of equations (3a)-(4b). The reason is that the state-variables \( \tilde{p}_t \) are potentially correlated.\(^10\) Hence we consider \( \tilde{p}_t \)’s orthogonal components, denoted \( \tilde{a}_t \). We then define \( g_t \) to be a vector of cross-products of entries in \( \tilde{a}_t \) (other than its own product). Similarly, let \( v_t \) be the orthogonal components of \( u_t^2 \).\(^11\) Note that \( a_t \) and \( v_t \) are contemporaneously uncorrelated because a cross-product of \( \tilde{p}_t \) and \( u_t^2 \) is a negligible third-order term.

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\(^8\) Given the structure of \( W_p \), the second and third terms in equation (4b) are actually zero for the endogenous state variables \( k_t \).

\(^9\) Recall that, in the case of scalar \( u_t \) and \( \epsilon_{t+1} \), we have \( \psi_{t+1}^2 = u_t^2 \epsilon_{t+1}^2 = u_t^2 \epsilon_{t+1}^2 \) + higher order terms = \( u_t^2 \epsilon_{t+1}^2 + u_t^2 \) + higher order terms. Hence the last term depends on the volatility \( u_t^2 \) and \( \epsilon_{t+1}^2 \).

\(^10\) This is not an issue for other “level” variables \( \psi_{t+1} \) (shocks to exogenous variables) because it is conventional to assume that these shocks are independent.

\(^11\) \( V_t \) actually includes \( u_t^2 \) and \( \epsilon_{t+1}^2 \). One may be able to separate these shocks: \( \epsilon_{t+1}^2 \) does not enter the “level” (i.e. (3a)-(3b)) while \( u_t^2 \) does.
If we define \( \tilde{x}_t \equiv [y_t' \quad \tilde{p}_t']' \) and \( \tilde{x}_t^2 \equiv \text{diag}\{\tilde{x}_t \tilde{x}_t'\} \), we can re-write equations (3a)-(4b) as

\[
\begin{bmatrix}
\tilde{x}_t^2 \\
\tilde{x}_t
\end{bmatrix} = \begin{bmatrix}
\Lambda_{2a^2} & \Lambda_{2g} & \Lambda_{2v} & 0 \\
\Lambda_{1a^2} & \Lambda_{1v} & \Lambda_{1v} & \Lambda_{1a}
\end{bmatrix} \begin{bmatrix}
a_t^2 \\
g_t \\
v_t \\
a_t
\end{bmatrix}
\] (5)

where the \( \Lambda \) parameters in the two equations can be interrelated. The point to note is that the common factors in \( \tilde{x}_t \) are a mixture of level factors \( a_t \) and second-moment factors \( (a_t^2, g_t, v_t) \). However, the level factors \( a_t \) do not appear in the equation for \( \tilde{x}_t^2 \). The next step is to exploit this structure to separate the level and the second-moment factors.

**Proposition 1** Let \( h = (a^2, g, v, a) \) and \( \bar{h} \) be the image from a projection of \( h \) on the span of \( h^2 \). Define \( \bar{x} = M_{\bar{h}} \bar{x} \) and \( \bar{x}^2 = M_{\bar{h}} \bar{x}^2 \). Then

\[
\begin{bmatrix}
\bar{x}_t^2 \\
\bar{x}_t
\end{bmatrix} = \begin{bmatrix}
A_{2v} & 0 \\
A_{1v} & A_{1a}
\end{bmatrix} \begin{bmatrix}
v_t \\
a_t
\end{bmatrix}.
\]

The idea behind Proposition 1 is that \( h^2 \) is spanned by \( a^2, g, g^2, v^2, \) and \( a^4 \). It does not, however, depend on \( a \) and \( v \). Hence, to isolate the variations \( a \) and \( v \) from \( h \), we construct \( \bar{h} \) as the projection on the span of \( h^2 \). Projecting \( \bar{x} \) and \( \bar{x}^2 \) on the orthogonal space of \( \bar{h} \) then purges variations due to \( \bar{h} \). The residuals from the projection, denoted \( \bar{x} \) and \( \bar{x}^2 \), retain information in \( a_t \) and \( v_t \). Specifically, \( \bar{x}^2 \) is \( v \) plus negligible higher order terms. In a similar spirit, \( \bar{x} \) is a linear combination of volatility shocks \( v \) and level shocks \( a \). If we project \( \bar{x} \) on the \( v \) that is recovered from \( \bar{x}^2 \), the residual from the projection is \( a \).

### 2.3 From the Model Solution to the Data

A limitation of the above theoretical setup is that there is no role for series-specific shocks in equation (5). Or, from the viewpoint of Boivin and Giannoni (2006), the model variables \( \tilde{x}_t \) and \( \tilde{x}_t^2 \) are assumed to have exact empirical counterparts. We now relax this assumption.

For \( t = 1, \ldots, T \), let \( X_t = (x_{1t}, \ldots, x_{Nt})' \) and \( X_t^2 = (x_{1t}^2, \ldots, x_{Nt}^2)' \) be \( N \times 1 \) vectors of observables. Also let \( e_{1t} \) and \( e_{2t} \) be non-pervasive errors associated with the data. These can be errors omitted from the theoretical model, or simply measurement errors. We leave the source of these errors agnostic. Guided by theoretical considerations, the data are now represented by

\[
\begin{bmatrix}
X_t^2 \\
X_t
\end{bmatrix} = \begin{bmatrix}
\Lambda_{2,A^2} & \Lambda_{2,G} & \Lambda_{2,V} & 0 \\
\Lambda_{1,A^2} & \Lambda_{1,G} & \Lambda_{1,V} & \Lambda_{1,A}
\end{bmatrix} \begin{bmatrix}
A_t^2 \\
G_t \\
V_t \\
A_t
\end{bmatrix} + \begin{bmatrix}
e_{2t} \\
e_{1t}
\end{bmatrix}
\] (6)

where \( A_t \) and \( V_t \) are the level and volatility factors in the data, \( G_t \) is a vector of cross-products of in the components of \( A_t \) other than itself. The data representation inherits the theoretical structure.
that the level factors $A_t$ have no contemporaneous effect on $X^2$ conditional on $A_t^2, G_t$ and $V_t$. Motivated by Proposition 1, we consider the following:

**Algorithm**

i Estimate factors $H = (A^2, G, V, A)$ from $(X \ X^2)$ and form $H^2$, i.e. squares and interaction of the estimated factors $H$.

ii Project $X$ and $X^2$ on $H^2$ to obtain residuals $\tilde{X}$ and $\tilde{X}^2$.

iii Extract $V$ from $\tilde{X}^2$. Given an estimate of $V$, we project $\tilde{X}$ on $V$ and extract $A$ from the residual of the projection.

To be clear about step (i), we can only estimate the space spanned by the factors but not the factors themselves. That is, the estimate of $H$ is a rotation of $(A^2, G, V, A)$ and unless we impose additional restriction, we cannot recover $(A^2, G, V, A)$ from the estimated $H$ alone. As a consequence, we cannot make use of the cross-equation constraints between the $\Lambda$ parameters in the two equations.

The solution of DSGE models is useful in guiding us how to interpret the factors that will be recovered from $X_t$. The theoretical model only allows for one type of second-moment variation, namely, volatility to the innovations of the fundamentals. However, other sources of second-moment variations could be empirically relevant. For example, we have so far ignored the idiosyncratic errors in equation (6) that map the factor model implied by theory to the data. But these errors could be another source of common volatility shocks. Consider the simple example $x_{it} = e_{it}$ that has no factor structure in the level of the data. Suppose that there is a common volatility shock that takes the form $e_{it} = \sigma_{ct} \epsilon_{it}$ and $\epsilon_{it} \sim (0, 1)$ for all $i$. Then for large $N$,

$$\frac{1}{N} \sum_{i=1}^{N} x_{it}^2 = \frac{1}{N} \sum_{i=1}^{N} \epsilon_{it}^2 \overset{p}{\rightarrow} \sigma_{ct}^2.$$  

Such a common volatility factor will appear in $X_t^2$, but it will not appear be in $X_t$.

The theoretical model also ignores time variation in its parameters which could induce instability in $\Lambda_F$ and $\Lambda_S$. To illustrate the issue, consider the case of a single factor with loadings $\lambda_t$ that exhibit time variations around the mean $\bar{\lambda}$. Suppose that

$$X_t = \lambda_t F_t + e_t = \bar{\lambda} F_t + (\lambda_t - \bar{\lambda}) F_t + e_t.$$  

Suppose also that the variation in $\lambda_t$ is idiosyncratic and independent of $F_t$. This can be reformulated as a factor model with constant loadings $\bar{\lambda}$ on $F_t$ and an additional level factor $\lambda_t F_t$ with
unit loadings. Even if $\lambda_t F_t$ has variations too small to be detected as a factor, a new volatility factor $\sigma^2_{\lambda,t} F_t^2$ may still be possible. Similarly, if parameter variation over time is common, say $\lambda^c F_t$, we have $X_{it} = (\lambda_{fi} + \lambda^c F_t) F_t + \epsilon_{1t}$. Such parameter variation may appear either as an additional level factor, a second-moment factor, or not at all if the variations are small. The outcome is data dependent and ultimately an empirical matter.

The foregoing discussion suggests the following representation of the data

$$X_t = \Lambda_{1,A} A_t + \Lambda_{1,V} V_t + \Lambda_{1,A^2} A_t^2 + \Lambda_{1,G} G_t + \epsilon_{1t}$$

$$X_t^2 = \Lambda_{2,V} V_t + \Lambda_{2,A^2} A_t^2 + \Lambda_{2,G} G_t + \epsilon_{2t}$$

The factors $F_t$ in the level of $X_t$ is a mixture of level and second-moment factors. While the factors $V_t$ in the squared data $X_t^2$ do not contain quadratic functions of $A_t$, they will be a mixture of volatility and other second-moment variations. We cannot interpret $V_t$ more precisely without further assumptions. Because theory provides little guidance for these assumptions, we leave the interpretation of $V$ agnostic. In spite of this limitation, the estimation of $V$ is still interesting because if second-moment variations have no business cycle implications, this should be reflected in the data regardless of the name these variations are given.

3 Estimation of Factors

Consider the generic factor analytic model for data $Z = (Z_1, Z_2, \ldots, Z_N)$

$$Z = F\Lambda' + e$$

where for $i = 1, \ldots, N$, $Z_i = (Z_{i1}, \ldots, Z_{iT})'$ is a $T \times 1$ vector, $F$ is a $T \times r$ matrix of common factors, $\Lambda$ is a $N \times r$ matrix of factor loadings, and $e$ is a $T \times N$ matrix of idiosyncratic errors. The method of asymptotic principal components (PCA), due to Connor and Korajczyk (1993), estimates $F$ as the $r$ eigenvectors corresponding to the $r$ largest eigenvalues of $ZZ^T/T$. Stock and Watson (2002a), Bai and Ng (2002) show that under some assumptions, $\min(N,T) \frac{1}{T} \sum_{t=1}^{T} \|\hat{F}_t - MF_t\| = O_p(1)$ as $N, T \to \infty$ when $N$ and $T$ are large. That is, $\hat{F}_t$ consistently estimates a rotation of $F_t$, where $M$ is the rotation matrix. The estimated factors can be used as though they were observed in empirical work.

The PCA solution can be computed by alternating least squares (ALS) as follows:

• Given the iterate at step $k$ denoted $F^k$, solve $\Lambda^{k+1} = (F^{kT} F^k)^{-1} F^{kT} Z$
• Given $\Lambda^{k+1}$, solve $F^{k+1} = (\Lambda^{(k+1)T} \Lambda^{k+1})^{-1} \Lambda^{(k+1)T} Z$.

Then the PCA estimate is the converged solution, namely, $\hat{F} = F^k$ when $||F^k - F^{k-1}||$ is smaller than some pre-specified tolerance. This view of PCA is useful because of two estimation issues that need to be considered.

The first issue relates to the fact that PCA weighs each observation equally, which is inefficient if the idiosyncratic error variances are not roughly the same. Generalized least squares theory suggests that efficiency gains can be obtained by weighing a series according to how much information it contributes. This can be particularly relevant in the present context because the squared data can be highly variable. Outliers (also known as “noise corruption” in the machine learning literature) can make the principal component estimates imprecise. A fully efficient GLS correction is, however, not possible because the estimated idiosyncratic error-covariance matrix is reduced rank. As a compromise, Boivin and Ng (2006) suggests a two-step approach that uses a diagonal weighting matrix. Specifically, let $W$ be a $N \times N$ diagonal matrix where $W_{ii}$ is the estimated idiosyncratic error variance of the $i$-th series in a first-step estimation of $F$ and $\Lambda$ using the data $Z$ that have been demeaned and standardized. For a given $i$, let the $T \times 1$ vector $Z_i^* = Z_i / \sqrt{W_{ii}}$. The $T \times N$ matrix of weighted data $Z^* = (Z_1^* Z_2^* \ldots Z_N^*)$ is used instead of $X$ in the ALS algorithm above. This is tantamount to a iterative GLS.

The second problem pertains to the possibility of overfitting the data with factors that contribute little to the common component of $Z$. This issue is relevant here because the second-order approximation suggests a quadratic increase in the number of factors over what is implied by the first-order solution, and many of these factors will be ‘weak’. Put differently, the low rank component of the data will have many small eigenvalues. To alleviate this problem, we shrink the eigenvalues of the low rank component towards zero. There are many ways to perform shrinkage. We use $L_2$ shrinkage which amounts to iterative ridge regressions instead of iterative least squares to estimate the factors. Specifically, given iterate $F^k$:

1. Perform ridge regressions of $Z_i$ of $F^k$ to obtain $\hat{\Lambda}^k_i = (F^k F^k + \tau I)^{-1} F^k Z_i$.

2. Perform a ridge regression of $Z$ on $\Lambda^{k+1}$ gives $\hat{F} = Z \Lambda^{k+1} (\Lambda^{k+1T} \Lambda^{k+1} + \tau I)^{-1}$.

This method, which we refer to as RPCA, is further studied in Bai and Ng (2016). While GLS downweighs observations with large idiosyncratic noise, RPCA reweighs to shrink the eigenvalues of the common component.\footnote{Yamamoto (2016) provides an upper bound of jump magnitudes for asymptotic inference to be accurate. In machine learning, methods that make PCA robust to outliers are generically referred to as robust principal components. See, for example, Lin et al. (2013); Zhou et al. (2010); Wright et al. (2009).\footnote{The Rank-Restricted Soft SVD algorithm of Hastie et al. (2015) delivers robust RPCA by iterative ridge estimation as we do. But to achieve a low rank component of rank $r^*$, the $r^* + 1$ and smaller eigenvalues are explicitly threshold to zero.}}
4 Factors in $X$ and $X^2$

We estimate the factors using data from FRED-MD (McCracken and Ng (2016)), a macroeconomic database consisting of a panel of 134 series over the sample 1960M1-2015M12. Consistent with previous studies, the data are transformed by taking logs and first differencing before the factors are estimated. As noted above, regularization shrinks the length of the factor estimates to zero. The common component is therefore smaller the larger is the shrinkage factor $\tau$. PCA obtains as a special case of RPCA when $\tau = 0$. The following factors will be considered.

<table>
<thead>
<tr>
<th>Data Z</th>
<th>Factors</th>
<th>Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$F$</td>
<td>$(A^2, G, V, A)$</td>
</tr>
<tr>
<td>$X^2$</td>
<td>$S$</td>
<td>$(A^2, G, V)$</td>
</tr>
<tr>
<td>$(X, X^2)$</td>
<td>$H$</td>
<td>$(A^2, G, V, A)$</td>
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</table>

The Bai and Ng (2002) criterion finds that the eight dimensional $F$ (estimated with PCA) explains 0.43 of the variation in the data. The eight GLS factors explain 0.42 of the variation in the data. In contrast, the eight RPCA factors $F$ explain 0.40, 0.37 and 0.32 of the variation when $\tau$ is set to 2, 5, and 10, respectively. Hereafter, we focus on results with $\tau = 2$ which yields the smallest shrinkage. The goal is to study the properties of the level factors $A_t$ and the volatility factors $V_t$.

The top panel of Figure 1 plots the largest factor $F_1$ in $X$ estimated using PCA, GLS, and RPCA respectively, all estimates are standardized to be mean zero with unit variance. Because the factor estimates are only identified up to sign, the normalization is chosen so that the series is negative in the 1980 recession. The three sets of estimates are highly correlated. The bivariate correlations between PCA and GLS is 0.94, between PCA and RPCA is 0.99, and between GLS and RPCA is 0.95. The estimate of $F_1$ is strongly procyclical. During the 1973, 1978, and 2008 recessions, the estimates are more than three standard deviations below the mean. Given that the properties of $F_1$ are rather insensitive to the choice of estimator, it is not surprising to find that irrespective of estimation methodology, $F_1$ loads heavily on real activity variables, consistent with what is reported in the literature. Nonetheless, the real activity factor can fluctuate because of shocks from different sources. FAVAR models that only incorporate the composite factor $F$ cannot disentangle these variations.

To highlight the importance of the second moment factors, the analysis to follow only uses the estimates of $F_1$, but properties of the remaining factor estimates are worth mentioning. Just like the estimates of $F_1$, all estimates of $F_2$ are strongly below mean during the 1982 recession. But unlike the estimates for $F_1$, the estimates of $F_2$ are all above the mean in the 2008 recession. All three estimates of $F_2$ load heavily on interest rate spreads. But while the PCA estimate for $F_3$
load heavily on price variables, the GLS and RPCA estimates load more heavily on the housing variables. In general, factors $F_3$ to $F_8$ are noticeably more sensitive to the estimator. The PCA estimates of these smaller factors are generally more volatile.

The second panel of Figure 1 shows the time series for the first factor in $X^2$, which we have denoted by $S$. Bai and Ng (2008) estimates $S$ by extracting the principal components in the squared data and referred to them as quadratic principal components.\(^{14}\) We estimate $S$ by robust principal components and find $S_1$ to be countercyclical and highly volatile. During major recessions, the estimates of $S_1$ are as many as six standard deviations above the mean of zero. Unlike $F_1$ which is robust to the estimation methodology, the estimation method makes a tangible difference for $S_1$. One can see that the PCA estimate (black line) has larger spikes than either the GLS estimate or the RPCA estimate and, in turn, the RPCA estimate is smoother than than the GLS estimate.

The third panel of Figure 1 plots the three series of estimates for the level factor $A_1$. The series are highly correlated, but not as strongly as what is found for $F_1$. The correlation between PCA and RPCA is 0.95, between PCA and GLS is 0.78, and between RPCA and GLS is 0.64. This level factor is also low during the 1973, 1982, and 2001 recessions. Interestingly, $A_1$ is not below mean during 1990 and 2008 recessions thought to be of financial origins.

The bottom panel of Figure 1 plots our $V_1$ normalized to be positive during the 1980 recession. The $V_1$ series is clearly counter-cyclical as it is high during all recessions. The PCA estimate of $V_1$ has a correlation of 0.99 with the RPCA estimate, and 0.86 with the GLS estimate. The correlation between the GLS and RPCA estimates of $V_1$ is 0.85. In the rest of the section we attempt to relate estimated $V_1$ to alternative measures of volatility/uncertainty.

The logic of Section 2.1 suggests that if there is a single factor in the first moment of the data, then $V_1$ should measure stochastic volatility in the innovations of that factor. To verify this, we directly estimate stochastic volatility in the estimated innovations of $A_1$ instead of backing it out from $X^2$ as suggested in Proposition 1. Since the three estimated $V_1$ series are highly correlated, we use RPCA estimates as the baseline without loss of generality. As seen from Figure 2, the “directly estimated” volatility series is only weakly correlated with our estimate of $V_1$; the correlation between the two series is 0.15. This can be attributed to the fact that we have not one, but multiple factors in the data, and we can only estimate a linear combination of them. Furthermore, our estimate of $V_1$ potentially includes the common volatility to the idiosyncratic shocks which the theoretical framework cannot accommodate. Hence, we think of $V_1$ as a composite second moment factor.

With the above caveat in mind, how does our estimate of $V_1$ relate to volatility/uncertainty measures available in the literature? The first measure for comparison is the economic policy

\(^{14}\)The correlation between the PCA and RPCA estimate of $S_1$ is 0.66. The correlation between PCA and GLS estimate is 0.77, while the correlation between the GLS and RPCA estimate is 0.95.
uncertainty (EPU) index constructed by Baker et al. (2015). While economic policy uncertainty is only a part of what can qualify as a second-order shock, EPU is an example of a higher-order shock identified with a narrative approach and a number of other sources. We find that $V_1$ and EPU are weakly correlated ($\rho = 0.28$). The behavior of the series also differs across recessions (see Figure 2). For example, both series soared during the Great Recession, but EPU increased during the 2001 recession while $V_1$ did not. Furthermore, $V_1$ rose during the Volcker recession while EPU move up only by a tad. In short, EPU and $V_1$ have independent variations.

The second comparison is with the uncertainty series of Jurado et al. (2015) (henceforth JLN). Their approach estimates stochastic volatility in the $h$ step-ahead idiosyncratic forecast error for a panel of macroeconomic series, then averaged to form an aggregate uncertainty series. While both the present paper and JLN analyze stochastic volatility, the key difference is that JLN consider the common variations in the expected volatility of a panel of macroeconomic series, while the present paper focuses on the current volatility in the factors common to the level of $N$ series. These are conceptually distinct objects. Figure 2 shows that the two series are weakly correlated ($\rho = 0.12$). Analysis of specific episodes also exposes important differences. For example, during the Great Recession both series increased by nearly three standard deviations. Yet, the JLN series declined to zero by 2011 while $V_1$ stayed elevated until mid 2015.

Given that $V_1$ is a component of common variation in $X^2$, and $S$ are the common factors in $X^2$, might $S$ be a good proxy for $V_1$? We see from Figure 2 that the estimated $V_1$ and $S_1$ also have independent variations. The divergence during the Great Recession is particularly notable: $V_1$ rises two standard deviations, while $S_1$ increases by one standard deviation. In spite of this difference during the Great Recession, the two series are quite strongly correlated in the rest of the sample. The correlation between $V_1$ and $S_1$ is 0.67, and the correlation increases to 0.79 if we exclude the Great Recession. What explains the high correlation between the two series? In our approach, $V$ is formed by purging $A^2$ and $G$ from $S$. Since the “level” factors do not have a direct “level” effect on the second moments according to equation (6), the high correlation between $V_1$ and $S_1$ suggests that a considerable fraction of the common variations in $X^2$ comes not from non-linear functions of the level factors $A^2$ and $G$, but from the second-moment factor that we call $V_1$. We now proceed to analyze its dynamic effects.

5 **FAVARsq Analysis**

The goal of our analysis is to understand the dynamic responses of macroeconomic variables to level and second-moment shocks. Our point of departure is to augment a FAVAR with second-moment
factors. This leads to a reduced-form FAVARsq

\[
\begin{pmatrix}
\text{Factors}_t \\
Y_t
\end{pmatrix} = \sum_{k=1}^{p} A_k \begin{pmatrix}
\text{Factors}_{t-k} \\
Y_{t-k}
\end{pmatrix} + \begin{pmatrix}
\eta_{\text{Factors}_t} \\
\eta_{Y_t}
\end{pmatrix}.
\]

where FACTORS are common components in the data (e.g. \(F_1, S_1, A_1\) and \(V_1\)). In this paper, the reduced-form errors \(\eta\) are orthogonalized by Cholesky decomposition to obtain structural shocks. We consider

\[Y_t = (\text{HOU}_t \quad \text{IP}_t \quad \text{INFL}_t \quad \text{FFR}_t)^\prime.\]

where HOU is housing starts of total new privately owned (series 50), FFR is the Federal Feds Rate, IP is industrial production (series 6). Annual inflation is \(\text{INFL} = \log(\text{CPIAUSL}_t) - \log(\text{CPIAUSL}_{t-12})\) where CPIAUCSL (series 113) is the consumer price index. To preserve space, we consider only factors estimated with RPCA and report results for selected orderings as alternative orderings yield similar results.

**Baseline Model:** \text{FACTORS}=(\text{F}_1)

The largest factor \(F\) in the data is well documented to be a real activity factor. Our point of reference is therefore the simplest FAVAR with \text{FACTORS} set equal to \(F_1\), the largest factor in \(X\).\footnote{Results are similar when we consider a larger set of factors which includes \(F_2, F_3\), etc.}

The decomposition of variance is reported in Table 1. The effect of an \(F_1\) shock has large short-run effects on IP and HOU but the effects decline after the initial peak. In contrast, the effects on FFR and INFL grow over time. After 60 months, an \(F_1\) shock explains approximately 50% of variation in FFR and approximately 30% of variation in INFL.

In what follows, we consider a series of models with different choices of \text{FACTORS}. Model I shows the significance of shocks to the second-moment factor \(V_1\) especially over longer horizons and the reduced importance of the level factors once \(V_1\) is controlled for. A \(V_1\) shock has features of a negative demand shock, similar to an \(F_1\) shock. An \(A_1\) shock has features of a negative supply shock. Model II shows that second moment shocks uncorrelated with \(V_1\) have effects similar to \(V_1\) and can be quantitatively important. Model II also provides a sense of how important non-linear effects are in the data. Model III shows that our volatility factor is distinct from measures of uncertainty with \(V_1\) being more important in explaining inflation and the fed-funds rate that other measures of uncertainty/volatility.

**Model I:** \text{FACTORS}=(\text{A}_1 \quad \text{V}_1 \quad \text{F}_1)

According to Section 2, the factors \(F\) in \(X\) can include a portion of the “volatility” factors \(V\). Proposition 1 suggests a way to distinguish between the “level” factors \(A\) from \(V\). To isolate the
effect of volatility shocks, we introduce the largest level factor $A_1$ and the largest volatility factor $V_1$ to the set of FACTORS. By ordering $A_1$ and $V_1$ before $F_1$, we also examine the importance of conventionally used $F_1$ after conditioning on a volatility factor and a level factor.

Figure 3 plots the impulse responses (along with 68% bootstrap confidence intervals) of macroeconomic variables in the model to $V_1$. We find that after a standard deviation increase in $V_1$, housing permits (HOU), fed funds rate (FFR), industrial production (IP), and inflation (INFL) all decline. This pattern is qualitatively similar to what one may expect after a negative demand shock. Even though qualitatively, the effects of a $V_1$ shock on the model variables are all negative, there are differences across variables. Housing permits (HOU) decline on impact after a shock to $V_1$. Industrial production (IP) has the largest decline shortly after the shock hits. In contrast, we observe little short-run effects on inflation as well as the fed funds rate.

Figure 4 contrasts the responses to a one-standard deviation shock to $F_1$ (black line) to a one-standard deviation shock to $A_1$ (blue line). After a shock to $F_1$, housing starts, industrial production, inflation and fed funds rate all decline. These dynamics are similar to what one may expect after a negative demand shock. On the other hand, a level shock to $A_1$ decreases housing permits, industrial production, and the fed funds rate, but increases inflation. This pattern is consistent with the dynamics one may observe after a negative supply shock. Obviously, the interpretation of the differences is rather tentative because $F_1$ and $A_1$ are not structural, in spite of being accepted to be real activity factors. However, the differences in the responses signal that the presence of the volatility factors can influence the “level” shocks that are being identified. Indeed, given that $F_1$ is positively correlated with $A_1$ and negatively with $V_1$, our results support the premise of this paper that $F_1$ is a mixture of level and volatility factors. Hence the responses to a shock in $F_1$ is likely a mixture of responses to “level” (supply-like) shocks $A_1$ and “volatility” (demand-like) shocks $V_1$.

To assess the quantitative significance of the effects, Table 2 reports the decomposition of variances. We see that a $V_1$ shock generally has weaker effects on real activity, inflation, and the fed funds rate in the short run relative to the responses at longer horizons. For example, $V_1$ accounts for less than 1% of variation in FFR or INFL at the one-quarter horizon but 15%-20% at the 4-5 year horizons. $V_1$ accounts for a modest fraction (4-8%) of variation for housing (HOU) and industrial production (IP). Compared to the baseline model without $V_1$ in Table 1, the effects of $F_1$ are reduced. For example, $F_1$ in the baseline model accounts for nearly 30% of variation in inflation at the 5-year horizon but the contribution is closer to 20% in the model that includes $V_1$. For horizons longer than one year, the total contribution of shocks to $A_1$, $V_1$ and $F_1$ is generally close to the total contribution of shocks to $F_1$ in Table 1. However, the $V_1$ and $A_1$ shocks do not fully capture the short-run variations in real activity. In the baseline model reported in Table 1,
accounts for more variations in the four variables within one year. Bernanke et al. (2005) show that using factors $F$ enhances identification of monetary policy shocks in VARs as the factors can provide a better summary of the information set available to central bankers. We explore if decomposing $F$ into $A$ and $V$ can further enhance identification. The responses of recursively identified monetary policy shocks from our FAVARsq appear to be only slightly different from the baseline (FAVAR) model. Monetary policy shocks account for similar fractions of fluctuations in housing, inflation and industrial production whether the central bank’s information set is proxied with $(F)$ or $(A, V, F)$. These findings are consistent with the view that the central bank reacts to changes in inflation and output regardless of where these changes come from. In other words, the reaction of the Federal Reserve System to a one percentage point increase in inflation due to a volatility shock is similar to the reaction of the Fed to a one percentage point increase in inflation due to e.g. a supply-side shock.

**Model II:** Factors $= (A_1 \ V_1 \ S_1)$:

In our framework, $V$ are the common factors in $X^2$ after the quadratic variations of $A$ are purged. If $V$ is indeed the stochastic volatility in the fundamentals, orthogonalizing the shocks in $S$ with respect to those in $V$ should isolate second-moment shocks that are not directly due to stochastic volatility in the fundamentals. As $X^2$ may include the squared “level” factors, the orthogonalized shocks to $S$ can be broadly interpreted as a collection of forces with non-linear effects on the level of macroeconomic variables, which is consistent with our analysis of the second-order approximation of DSGE models. To explore further the importance of volatility shocks and shocks with non-linear effects, we consider a FAVARsq by adding $S_1$ to Model I. By putting $S_1$ last in the list of factors, we attribute to $S_1$ only what is not explained by $A_1$ and $V_1$.

The impulse response functions are shown in Figure 5. Shocks to $S_1$ have effects qualitatively similar to $V_1$: real activity contracts, and the shocks trigger modest short-run responses in the Fed-Funds rate rate and inflation. The decomposition of variances are reported in Table 3. A shock to $S_1$ has non-trivial short-run impact on housing starts, and its effect on the Fed-funds rate and inflation are comparable to that of the $F_1$ shock at long horizons. Compared to the results in Table 2, we see that adding $S_1$ to the FAVARsq does not change the significance of $V_1$ much. Thus, our simple framework suggests that second-moment shocks not exclusively due to $V$ can have important effects, especially on inflation and interest rate. These results are also consistent with potentially large variations in the data due to non-linear effects (i.e., $A^2$ and $G$ in our notation). They are also consistent with the finding in Bai and Ng (2008) that the estimates of $S$ and of $F^2$ both have predictive power for inflation and some measures of real economic activity.
Model III: \[ \text{FACTORS} = (A_1 \ V_1 \ \text{JLN} \ \text{EPU}) \]

As we discussed above, the volatility factor identified in our framework is only weakly related to alternative measures of volatility such as the economic policy uncertainty (EPU) index constructed by Baker et al. (2015) or the volatility (JLN) index constructed by Jurado et al. (2015). However, it is possible that innovations to these indices could be more interrelated. To investigate this possibility, we include both EPU and JLN in FACTORS of our FAVARsq. Because \( V_1 \) can be contemporaneously correlated with JLN and EPU, we consider two orderings: i) \( V_1 \) before JLN and EPU; and ii) \( V_1 \) after JLN and EPU. We find that the estimated responses in this model are close to those for Model I, and the alternative ordering makes little difference for the responses. In a similar spirit, the variance decompositions for \( V_1 \) (Table 4) are also close to the results for Model I and are insensitive to the ordering. In addition, the contribution of the level factor \( A_1 \) is unaffected by inclusion of JLN and EPU.

The point to highlight from this exercise is that the contributions of \( V_1 \), JLN and EPU vary across shocks, variables and horizons, thus underscoring that different measures of volatility capture different channels of macroeconomic fluctuations. For example, JLN accounts for a large share of variation in housing permits (HOU) at long horizons while \( V_1 \) and EPU contribute about 4% and 0.7% respectively. On the other hand, \( V_1 \) is much more important for explaining variation in inflation and the fed funds rate at longer horizons than either JLN or EPU. Generally, we find that EPU tends to contribute less than either \( V_1 \) or JLN. The collective contribution of volatility shocks \( V_1 \), JLN, and EPU to variation of the macroeconomic variables ranges from approximately 30% for HOU and FFR to about 10% for IP.

6 Concluding remarks

Identifying the sources of fluctuations in aggregate data has been a long-standing agenda in macroeconomics. Although the conventional approach in this area is to use only first moments of the data, recent advances in theoretical and empirical macroeconomics suggest that higher-order shocks (volatility, uncertainty, etc.) could be an important determinant too. This paper contributes to the effort by first showing that the factors estimated from the data will, in general, be a mixture of “level” and “volatility” factors. We then develop a novel FAVARsq framework that allows researchers to model macroeconomic fluctuations as determined by these two distinct types of factors. While we analyze macroeconomic data in this paper, the framework may be suitable for other applications (e.g., financial data) provided that our identifying restriction is satisfied.

Our analysis of three FAVARsq leads to the following findings. First, the largest “volatility” factor \( V_1 \) is countercyclical, persistent, and accounts for a tangible share of variation in macroeconomic
variables especially at longer horizons. There is evidence that some of its variations have indeed been attributed to the first factor \( X \). Second, while the common variations in \( X^2 \) are dominated by \( V_1 \), non-linear effects such as due to squared level factors can have non-trivial contribution to cyclical variation in \( X \), especially on inflation and interest rate. Third, our \( V_1 \) has variations that are independent of measures of uncertainty as well as stochastic volatility directly estimated from the largest level factor. The data thus suggest multiple sources of second-moment variations.

A caveat of the present analysis is that our \( V_1 \) is likely a composite of volatility from different sources, some of which have no role in theoretical macroeconomic models. While we make progress in isolating the “volatility” factors from “level” factors, further restrictions are needed to give more precise interpretations to the volatility factor. The interaction between the first- and second-order dynamics is worthy of more theorizing in light of the evidence for non-trivial second moment variations.
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Table 1: Decomposition of Variances: Baseline Model; shock to $F_1$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$F_1$</th>
<th>HOU</th>
<th>IP</th>
<th>INFL</th>
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Note: The table reports variance decomposition for FAVAR with $\text{FACTORS} = (F_1)'. F_1$ is the factor estimated with robust principal components. HOU is housing permits, FFR is the Federal Funds Rate, IP is industrial production, INFL is CPI inflation rate. See Section 5 for more details.

Table 2: Decomposition of Variances: Model I

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Note: The table reports variance decomposition for FAVARsq with $\text{FACTORS} = (A_1 \ V_1 \ F_1)'. F_1$ is the first factor in extracted from $X$. $V_1$ is the first volatility factor net of quadratic variations in the level factors. $A_1$ is the first level factor. $V_1$ and $A_1$ are extracted as described in Section 2. The factors are estimated with robust principal component analysis. HOU is housing permits, FFR is the Federal Funds Rate, IP is industrial production, INFL is CPI inflation rate. See Section 5 for more details.
Table 3: Decomposition of Variances: Model II

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Note: The table reports variance decomposition for FAVARsq with $\text{FACTORS}=(A_1 \ V_1 \ S_1)'$. $V_1$ is the first “volatility” factor net of quadratic variations in the level factors. $A_1$ is the first level factor. $V_1$ and $A_1$ are extracted as described in Section 2. $S_1$ is the first factor extracted from squared data $X^2$. The factors are estimated with robust principal component analysis. HOU is housing permits, FFR is the Federal Funds Rate, IP is industrial production, INFL is CPI inflation rate. See Section 5 for more details.
Table 4: Decomposition of Variances: Model III

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<th>HOU</th>
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<td>0.847</td>
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<td>0.020</td>
<td>0.033</td>
</tr>
<tr>
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<td>0.039</td>
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<td>0.799</td>
<td>0.007</td>
<td>0.025</td>
<td>0.042</td>
<td>0.043</td>
</tr>
</tbody>
</table>

Note: The table reports variance decomposition for FAVARsq with FACTORS=$\{A_1, V_1, JLN, EPU\}$ and FACTORS=$\{A_1, JLN, EPU, V_1\}$. $V_1$ is a “volatility” factor. $A_1$ is a “level” factor. The factors are estimated with robust principal components following the procedure described in Section 2. $JLN$ is the volatility index constructed by Jurado et al. (2015). $EPU$ is the economic policy uncertainty index constructed by Baker et al. (2015). HOU is housing permits, FFR is the Federal Funds Rate, IP is industrial production, INFL is CPI inflation rate. See Section 5 for more details.
Figure 1: Factor Estimates

Note: The figure reports the time series of estimated factors. $F_1$ is the first factor extracted from the levels of data series $X$. $S_1$ is the first factor extracted from the squares of data series $X^2$. $V_1$ and $A_1$ are factors extracted as described in Proposition 1. $V_1$ is the first volatility factor. $A_1$ is the first level factor. Factors are extracted using principal component analysis (PCA; black line), robust principal component analysis (RPCA; red line), and GLS principal component analysis (GLS; blue line). See Section 4 for more details.
Figure 2: Comparison of factors

Note: The figure reports estimates of the first factor $S_1$ extracted from the squares of data series $X^2$, and the volatility factor ($V_1$) estimated according to the framework outlined in Section 2. Also plotted is $SV_1$, the stochastic volatility estimated on $A_1$, the $JLN$ uncertainty index constructed by Jurado et al. (2015), and the $EPU$ economic policy uncertainty index constructed by Baker et al. (2015). Estimated factors are based on robust principal component analysis. See Section 4 for more details.
Figure 3: Model I: Ordering \((A_1, V_1, F_1)\): Shock to \(V_1\)

Response of Housing permits

Response of Indus. Production

Response of Inflation

Response of Fed-Funds Rate

Note: The figure reports impulse responses to a one-standard deviation shock to volatility the factor \(V_1\) in FAVARsq (Model I). The level factors \((A)\) and volatility factors \((V)\) are as outlined in Section 2. \(F_1\) is the first factor extracted from the levels of data series \(X\). Estimated factors are based on robust principal component analysis. See Section 5 for more details.
Figure 4: Model I: Ordering \((A_1, V_1, F_1)\): Shocks to \(A_1\) and \(F_1\)

Note: The figure reports impulse responses to a one-standard deviation shock to volatility the factor \(V_1\) in FAVARsq (Model I). The “level” factors \((A)\) and “volatility” factors \((V)\) are constructed as outlined in Section 2. \(V_1\) is the first volatility factor. \(A_1\) is the first level factor. \(F_1\) is the first factor extracted from the levels of data series \(X\). All factors are extracted using robust principal component analysis. See Section 5 for more details.
Figure 5: Model II: \((A_1, V_1, S_1)\): shocks to \(V_1\) and \(S_1\)

Note: The figure reports impulse responses to a one-standard deviation shock to volatility the factor \(V_1\) and \(S_1\) in FAVARsq (Model II). The “level” factors \((A)\) and “volatility” factors \((V)\) are constructed as outlined in Section 3. Factor \(S_1\) is the first factor extracted from squares of the data \(X^2\). All factors are extracted using robust principal component analysis. See Section 5 for more details.