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Uncovering some subtleties of the uncovered set: Social choice theory and distributive politics

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Abstract. Although the uncovered set has occupied a prominent role in social choice theory, its exact shape has never been determined in a general setting. This paper calculates the uncovered set when actors have pork barrel, or purely distributive, preferences, and shows that in this setting nearly the entire Pareto set is uncovered. The result casts doubt on the usefulness of the uncovered set as a general solution concept and suggests that to predict the distribution of political benefits one must explicitly model the institutions that structure collective choice.

1. Introduction

Since its introduction, the uncovered set has played a prominent role in social choice theory. However, its exact shape has proven notoriously difficult to determine except in a few special cases. This essay calculates the uncovered set for a large class of games characteristic of distributive politics, such as the well-known divide-the-dollar, pork barrel allocation, and tax-and-spend games. We show that in these settings the uncovered set comprises nearly the entire Pareto set. Further, a closely related solution concept, the "strongly uncovered set" is exactly equal to the Pareto set. These findings have two important implications. First, the uncovered set may not be as useful a solution concept as previously assumed. And second, purely preference-based analysis has little to say about distributive politics, implying that the political institutions which structure the decision-making process will be all that more important in determining final policy outcomes.

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The remainder of the paper is organized as follows. The next section provides an overview of the uncovered set and its place in social choice theory. Section 3 presents the basic model and then defines the strongly uncovered set. Section 4 provides some basic descriptions of the strongly uncovered set, and then describes both the uncovered and strongly uncovered sets in a distributive politics setting. Section 5 summarizes the findings and concludes. An appendix provides proofs of all propositions.

2. The uncovered set in social choice theory

Two broad research traditions have arisen to meet the challenge of equilibrium identification in positive political theory: institutional analysis and social choice theory. While the former seeks equilibria in the formalized description of political processes, the latter seeks to restrict plausible outcomes on the basis of more abstract analysis of preferences and their aggregation through voting systems.

One of the earliest successes in the social choice literature was Black's [6] median voter theorem, which stated that in a unidimensional voting space, the ideal point of the median voter was an equilibrium. However, hopes that this result could be extended were dashed when Plott [17] proved that only if voter ideal points met stringent symmetry conditions would an equilibrium exist in more than one dimension. Even worse, McKelvey [12] proved that in a setting without a core[1], and assuming Euclidean preferences, sincere voting in a multidimensional setting could produce outcomes outside of the Pareto set. And Schofield [19] proved that the core was generically empty. Thus not only was there no equilibrium in more than one dimension under majority rule, but any alternative could result, leading to a complete inability to generate meaningful predictions.

However, one could argue that voters need not vote myopically at each opportunity, and Plott and McKelvey had assumed; rather, they may vote in a farsighted, or sophisticated, manner. In his classic exposition on the subject, Farquharson [8] showed that if the entire agenda is known before voting starts, the behavior of actors who look ahead and vote strategically may differ from those who vote sincerely, considering only their preferences over the alternatives offered at each stage. By wedding game theory and social choice theory, sophisticated voting offered the possibility of significantly narrowing the scope of outcomes that may result from majority-rule voting, thus approaching equilibrium predictions.

Miller [15] showed that, indeed, results under sophisticated voting are constrained to lie inside the "uncovered set," which he proved to be a subset of the Pareto set. Furthermore, in contrast to the lengthy agendas involved in the proof of McKelvey's chaos theorem, any alternative that can be reached from a given starting point under sophisticated voting can be reached in an agenda that contains no more than two steps. Miller's theorem considered only finite choice sets, but he speculated that in a spatial setting, the uncovered set would

[1] A point is in the core if no other point defeats it head-to-head.

be "a relatively small subset of [the Pareto optimal set], centrally located in the distribution of ideal points. (p. 84)"

Since Miller's original insight, Banks [1] has provided a method to identify all possible sophisticated winners from a finite choice set (Miller had shown that being uncovered was a necessary but not sufficient condition for an alternative to be an equilibrium outcome under sophisticated voting). Further, Shepsle and Weingast [23] extended Miller's work to a spatial setting with finite agendas. They proved that no policy could be a sophisticated winner in a finite amendment agenda if an alternative that covers it is also included in the agenda. They were, however, unable to confirm Miller's conjecture about the size of the uncovered set relative to the Pareto set.

Some progress was made on the size of the uncovered set when McKelvey [13] showed that it must lie in an area of radius no more than four times the size of the "yolk", the ball of minimum radius which intersects every median hyperplane. To date, however, the uncovered set has been calculated only for limited numbers of players with specific preference configurations (Feld [9]; Hartley and Kilgour [10]). The purpose of this paper is to calculate the size of the uncovered set in a general setting of particular interest to political scientists, distributive politics. Its main result is that in purely distributive games, the uncovered set is essentially equal to the Pareto set.

3. Definitions and model

Consider a compact, convex subset $\mathbf{X}$ of $\mathbf{R}^k$, with elements $\mathbf{x} \in \mathbf{X}$ considered to be the feasible alternatives. There is a set $N = \{1, \ldots, n\}$ of individuals, with $n > 4$. Each of the individuals i has a preference correspondence $P_i : \mathbf{X} \to \mathbf{X}$, where for $\mathbf{x} \in \mathbf{X}$, $P_i(\mathbf{x})$ is an open, convex set of outcomes strictly preferred to $\mathbf{x}$. Assume that P_i can be represented by a strictly quasi-concave and continuous utility function $u_i : \mathbf{X} \to \mathbf{R}$ which attains its unique maximum at an ideal point $\mathbf{x}_i$. Individual utility functions summarize individual preference relations over pairs of alternatives, and these preference relations are aggregated to form social preference relations according to majority rule.

Definition 1. Individual i's preference ordering $\succ_i$ and social preferences $\succ$ over alternatives $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are given by:

$$\mathbf{x} \succ_i \mathbf{y} \Leftrightarrow u_i(\mathbf{x}) > u_i(\mathbf{y});$$

$$\mathbf{x} \succ \mathbf{y} \Leftrightarrow |\{i | \mathbf{x} \succ_i \mathbf{y}\}| \geq \frac{n+1}{2};$$

$$\mathbf{y} \succeq \mathbf{x} \Leftrightarrow \mathbf{x} \nsucc \mathbf{y}.$$

Thus social decisions are made by majority rule. Individual preferences are used to define the Pareto set:

Definition 2. Given two alternatives $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}$ is Pareto superior to $\mathbf{y}$ if $\mathbf{x} \succ_i \mathbf{y}$ for all $i \in N$. The set of all $\mathbf{x} \in \mathbf{X}$ for which there is no $\mathbf{y} \in \mathbf{X}$ such that $\mathbf{y}$ is Pareto superior to $\mathbf{x}$ is called the Pareto set of $\mathbf{X}$, $P(\mathbf{X})$.

One object is said to dominate another if it is at least as good in every respect and better in at least one respect. Pareto superiority is an example of a dominance relation, and the Pareto set is defined as the set of undominated elements. The requirement that alternatives be undominated according to some relevant criterion is often used in economics to place limits on plausible outcomes without specifying a selection process. In the present context, to require that final allocations be in the Pareto set means that no utility is wasted in the sense that someone could be made better off without making anyone else worse off.

Social preferences are used to define the covering relation and the uncovered set. Two equivalent definitions are offered; the first emphasizes that covering is another example of a dominance relation, and the second defines covering in a manner that makes the upcoming definition of the strongly uncovered set more easily understood.

Definition 3

1. Given two alternatives $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{y}$ covers $\mathbf{x}$ ($\mathbf{y}C\mathbf{x}$) iff (a) $\mathbf{y} \succ \mathbf{x}$ and (b) $\mathbf{z} \succ \mathbf{y} \Rightarrow \mathbf{z} \succ \mathbf{x}$. The set of all $\mathbf{x} \in \mathbf{X}$ for which there is no $\mathbf{y} \in \mathbf{X}$ such that $\mathbf{y}$ covers $\mathbf{x}$ is called the uncovered set of $\mathbf{X}$, $U(\mathbf{X})$.
2. Equivalently, an alternative $\mathbf{x}$ is uncovered iff for every $\mathbf{y} \succ \mathbf{x}$ there exists an alternative $\mathbf{z} \in \mathbf{X}$ such that $\mathbf{z} \succ \mathbf{y}$ and $\mathbf{x} \succeq \mathbf{z}$.

Thus alternative **y** covers **x** if it not only defeats **x** heat-to-head, but also any alternative that is majority preferred to **y** is also preferred to **x**. Part 2 of Definition 3 states that for any uncovered alternative **x** and any alternative **y** that beats it, there must exist a third alternative **z** such that $\mathbf{x}, \mathbf{y}$, and **z** constitute a three-element majority-rule cycle. This observation forms the basis of Miller's "two-step" theorem mentioned above that from any starting point, an uncovered alternative can be reached in an amendment agenda of no longer than two stages.

Part 1 of Definition 3 is phrased so as to make obvious the parallels between the uncovered set and the Pareto set as undominated alternatives. Whereas Pareto superiority is based on individual preferences in the direct comparison of two alternatives, covering is based on social preferences and the indirect comparison of two alternatives against various third alternatives. Consider, for instance, a two-candidate election where each candidate simultaneously adopts a platform in **X**. Then a platform **x** would never be chosen if there exists another platform **y** which covers it, for any other platform which defeats **y** would also defeat **x**. Thus the uncovered set comprises the set of undominated candidate platforms.

Figures 1 and 2 illustrate the covering relation. If **y** covers **x**, then the set of points preferred to it is a subset of those preferred to **x**. So if **y** is preferred to **x** but does not cover it, there must be some point $\mathbf{z}'$ inside the set of $\mathbf{z} \succ \mathbf{y}$ but outside the set of $\mathbf{z} \succ \mathbf{x}$. This implies, as shown in Fig. 3, that **x**, **y**, and $\mathbf{z}'$ form a classic Condorcet majority rule voting cycle. From these preferences we can construct a two-stage amendment agenda that begins at **y** and has **x** as its sophisticated winner (the arrows in the voting tree indicate the direction sophisticated voters will go at each node). Thus any uncovered point can be reached from any other point in no more than two steps of sophisticated voting.

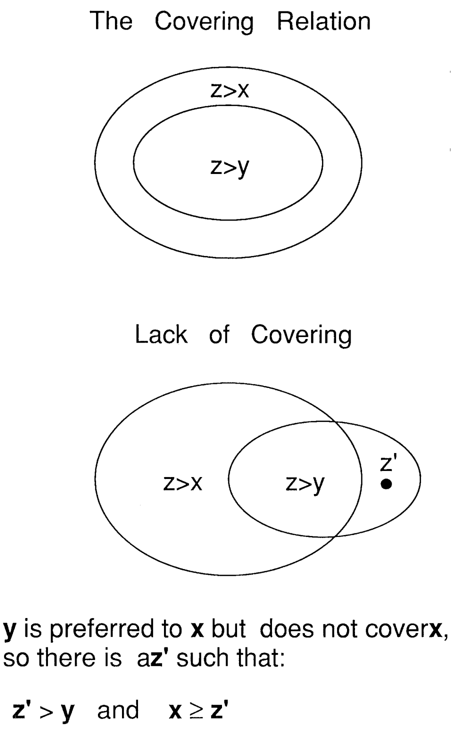
The Covering Relation
z>x
z>y
Lack of Covering
z>x
z>y
z'
y is preferred to x but does not coverx,
so there is az' such that:
z' > y and x ≥ z'

Fig. 1.

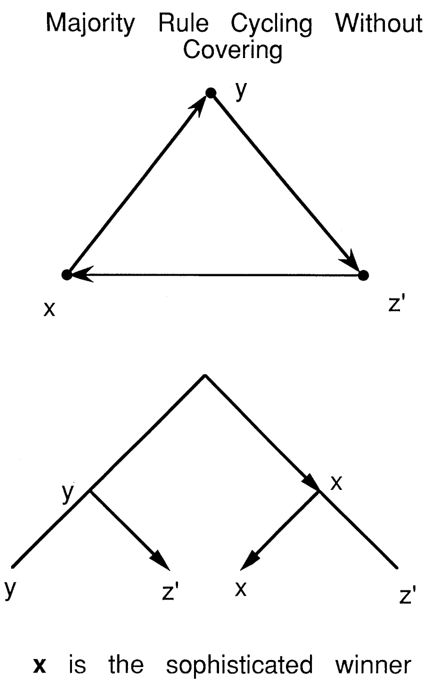
Majority Rule Cycling Without Covering
y
x
z'
y
x
y
z'
x
z'
x is the sophisticated winner

Fig. 2.

We next introduce the *strongly uncovered set*, which slightly strengthens the necessary conditions for an alternative $\mathbf{y}$ to cover another alternative $\mathbf{x}$. In contrast to Part 1 of Definition 3, the new definition requires that not only must every point $\mathbf{z}$ that is majority-preferred to $\mathbf{y}$ also beat $\mathbf{x}$, but it must also be possible to specify some arbitrarily small region surrounding $\mathbf{x}$ such that $\mathbf{z}$ beats all points in this region. In terms of sophisticated voting, the strongly uncovered set will contain all points which can be approximated arbitrarily closely by sophisticated voting over an amendment agenda.

The new requirement that any alternative $\mathbf{z}$ preferred to $\mathbf{y}$ also be preferred to all alternatives in some small neighborhood of $\mathbf{x}$ makes covering more difficult. Thus the new relation will be referred to as strong covering. To formalize its definition, an ε-neighborhood (denoted U_ε) of any point $\mathbf{x}$ for a small positive value ε is defined as the set of points $\{\mathbf{x}^\varepsilon \in \mathbf{X}$ for which $d(\mathbf{x}, \mathbf{x}^\varepsilon) < \varepsilon\}$, where $d(\cdot, \cdot)$ is the usual Euclidean distance metric.

Definition 4

1. Given two points $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, we say that $\mathbf{y}$ *strongly covers* $\mathbf{x}$ ($\mathbf{y}SC\mathbf{x}$) iff $\mathbf{y} \succ \mathbf{x}$ and there exists some ε-neighborhood U_ε of $\mathbf{x}$ such that $\mathbf{z} \succ \mathbf{y} \Rightarrow \mathbf{z} \succ \mathbf{x}'$ for all $\mathbf{x}' \in U_\varepsilon$. The set of all $\mathbf{x} \in \mathbf{X}$ for which there is no $\mathbf{y} \in \mathbf{X}$ such that $\mathbf{y}$ strongly covers $\mathbf{x}$ is called the strongly uncovered set of $\mathbf{X}$, $SU(\mathbf{X})$.
2. Equivalently, a point $\mathbf{x}$ is strongly uncovered iff for every $\mathbf{y} \succ \mathbf{x}$ there exists a point $\mathbf{z}$ such that $\mathbf{z} \succ \mathbf{y}$ and for every ε-neighborhood U_ε of $\mathbf{x}$ there exists $\mathbf{x}' \in U_\varepsilon$ such that $\mathbf{x}' \succeq \mathbf{z}$.

Miller showed that the uncovered set is nonempty for any well-defined set of preferences and is equal to the core of a game when one exists. It follows from the definitions of covering and strong covering that any point that is uncovered is also strongly uncovered (just take $\varepsilon = 0$), and therefore $U(\mathbf{X}) \subseteq SU(\mathbf{X})$. Combined with the result that the uncovered set is always nonempty, this implies that the strongly uncovered set is also nonempty. Furthermore, the existence of a core implies that there exists a point which is socially preferred to all the other points in the space. If we denote the core by $\mathbf{x}_C$, then it is clear that since there is no point $\mathbf{z}$ such that $\mathbf{z} \succ \mathbf{x}_C$, the definitions of covering and strong covering are equivalent in games with a core. So the strongly uncovered set also converges to the core if one exists[2].

4. Results

The foundation has now been laid to investigate the shapes of the sets $U(\mathbf{X})$ and $SU(\mathbf{X})$ in a distributive politics setting. The strategy will be the following: first, all points not in the Pareto set will be shown to be strongly covered, which implies that they are also covered. Then we shall see that if a point is strongly covered, it is strongly covered by a point in the Pareto set, which will allow us to concentrate our analysis solely on the Pareto set. Finally, it will be shown that for a certain class of games, including pork-barrel politics, the entire Pareto set is strongly uncovered. We begin with some basic properties of the strongly uncovered set.

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[2] One could use the logic presented in Cox [7] to prove that this convergence to the core is continuous.

Theorem 1. 1. *If* $\mathbf{y}SC\mathbf{x}$ *and* $\mathbf{z}SC\mathbf{y}$, *then* $\mathbf{z}SC\mathbf{x}$.
2. $SU(\mathbf{X}) \subseteq P(\mathbf{X})$.
3. *If* $\mathbf{y}SC\mathbf{x}$ *then there exists an* ε*-neighborhood* U_ε *of* $\mathbf{x}$ *such that* $\mathbf{y}SC\mathbf{x}'$ *for all* $\mathbf{x}' \in U_\varepsilon$.
4. *The set* $SU(\mathbf{X})$ *is closed*[3].

Proof. All proofs are provided in the appendix.

These results show that the strong covering relation, like the covering relation, is transitive. And the strongly uncovered set, like the uncovered set, lies within the Pareto set. Finally, the last two results establish the important proposition that the strongly uncovered set is closed, unlike the covered set, which in general is neither open nor closed (see the discussion below).

Note that, in combination with results in the previous section, we have established that the strongly uncovered set contains the uncovered set, yet lies within the Pareto set (see Fig. 3). The possibility still remains, of course, that the strongly uncovered set is in fact equal to the uncovered set, in which case adding the ε perturbations makes no difference. On the other hand, $SU(\mathbf{X})$ might be as large as the Pareto set, in which case any Pareto optimal point can be approximated arbitrarily closely via sophisticated voting.

The proof of Theorem 1 gives rise to the observation that if all individuals unanimously prefer an alternative $\mathbf{y}$ to another alternative $\mathbf{x}$, then the points that are majority-preferred to $\mathbf{y}$ are also majority-preferred to $\mathbf{x}$. This in turn provides the intuition for the following corollary:

Corollary 1. *If* $\mathbf{y}SC\mathbf{x}$ *and there exists a point* $\mathbf{z}$ *such that for all individuals,* $\mathbf{z} \succ_i \mathbf{y}$, *then* $\mathbf{z}SC\mathbf{x}$.

Corollary 1 states that given a point $\mathbf{y}$ which strongly covers $\mathbf{x}$ and another point $\mathbf{z}$ which is closer to the Pareto frontier in the sense of being unanimously preferred to $\mathbf{y}$, then $\mathbf{z}$ also strongly covers $\mathbf{x}$. The following lemmas states that this process can be continued all the way to the Pareto frontier, so that a point which is strongly covered must be strongly covered by a point in the Pareto set.

Lemma 1. *If any point* $\mathbf{x} \in \mathbf{X}$ *is strongly covered by some point* $\mathbf{y}$ *not in* $P(\mathbf{X})$, *then* $\mathbf{x}$ *is also strongly covered by a point in* $P(\mathbf{X})$.

It has thus been shown that any point which is not on the Pareto surface is strongly covered by a point on the Pareto surface. Furthermore, if a point on the Pareto surface is strongly covered, then it is strongly covered by another point on the Pareto surface. So future candidates for strongly uncovered points as well as alternatives that strongly cover a given point can now be restricted to only those alternatives in the Pareto set.

So far, the results obtained in this paper have been extensions to strong covering of results known previously (Miller [15]; Shepsle and Weingast [23]) in the context of the covering relation. Whereas these previous authors were unable to give a general description of the uncovered set in finite agenda

[3] These results may be extended past majority rule to any q-rule, requiring a fraction q ($\geq n/2$) of voters to pass a policy. See McKelvey and Schofield [14], Banks [2] and Saari [18] for treatments of q-cores.

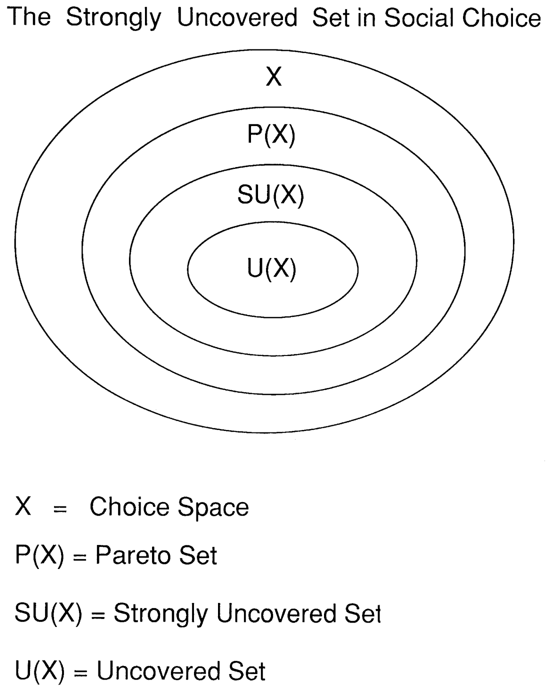

Fig. 3.

(Miller) or spatial (Shepsle and Weingast) settings, we are now in a position to define a class of alternative spaces in which both the uncovered and strongly uncovered sets can be precisely described.

Definition 5. A pair $(\mathbf{X}, \{u_i\})$ of alternatives and utility functions is *Pareto constant* if and only if for every point $\mathbf{x} \in P(\mathbf{X}), \sum_{i=1}^{n} u_i(\mathbf{x}) = L$ for some constant L. Given Pareto constant $(\mathbf{X}, \{u_i\})$, let u_i^0 be the *minimum utility* individual i receives at any point in $P(\mathbf{X})$.

Social choice theory has various examples of Pareto constant settings. For instance, in the divide-the-dollar game, $\mathbf{X} = \{\mathbf{x} \in \mathbf{R}^n \ \sum_{i=1}^{n} x_i \leq 1$ and $x_i \geq 0, i = 1 \ldots n\}$, and $u_i(\mathbf{x}) = x_i$. Here, the Pareto set includes all allocations in which the entire dollar is divided, and each player's minimum utility is 0[4]. The tax-and-spend game uses the same alternative set $\mathbf{X}$, but has $u_i(\mathbf{x}) = x_i - (1/n)\sum_{j=1}^{n} x_j$, in which case $u_i^0 = -1/n$ for all $i \in N$. This is a zero-sum game, and the Pareto set in this case equals the entire feasible set $\mathbf{X}$. We are now in a position to state our main result.

Theorem 2. *If* $(\mathbf{X}, \{u_i\})$ *is Pareto constant, then* $P(\mathbf{X}) \subseteq SU(\mathbf{X})$. *Combined with the result of Theorem 1 above, this implies that in Pareto constant settings,* $SU(\mathbf{X}) = P(\mathbf{X})$. *In addition, in Pareto constant settings the uncovered set* $U(\mathbf{X})$

[4] Note that in general, since $P(\mathbf{X})$ is closed, each player's minimum utility is obtained at some point within the set.

is the set of all Pareto allocations that give over half the voters more than their minimum utility.

Theorem 2 describes the uncovered set in Pareto constant games; that is, those games characterized by distributive politics. It states that the strongly uncovered set is equal to the Pareto set ($SU(\mathbf{X}) = P(\mathbf{X})$), and the uncovered set is equal to those allocations that leave at least half the voters with greater than their minimum utility. For instance, in a nine-player tax-and-spend game, the uncovered set is all policies that give at least five districts positive net benefits. Thus the uncovered set is equal to the entire Pareto set, minus some of the boundary points; to put it another way, the difference between the Pareto set and the uncovered set is a set of measure zero.

This suggests an interesting relation between the uncovered and strongly uncovered sets. For any set A, the closure of A, denoted $\overline{A}$, is defined as A plus its boundary points. In a three-player divide-the-dollar game, the uncovered set consists of the Pareto triangle minus the three corner points, a set which is neither open nor closed, while the strongly uncovered set is the entire Pareto triangle, which is closed. In general, for distributive political games, the strongly uncovered set is the closure of the uncovered set; $SU(\mathbf{X}) = \overline{U(\mathbf{X})}$.

5. Discussion

Theorem 2 implies that any alternative in the Pareto set can be approximated arbitrarily closed in an amendment agenda, even with sophisticated voting. Equivalently, the only positive prediction made by social choice theory in a distributive politics setting is that no funds will be wasted. Though not quite so bad as the standard chaos results, this state of affairs is still far from an equilibrium that significantly circumscribes the set of possible outcomes.

This result has important implications for both the study of the uncovered set and for the study of distributive politics. First, ever since its introduction by Miller, the uncovered set has been viewed as a useful generalization of the core. McKelvey [13], for instance, examines several institutional settings (two-candidate competition, cooperative behavior in small committees, and sophisticated voting over endogenously-determined agendas) and proves that in each case outcomes fall within the uncovered set. He then suggests that the uncovered set may serve as a general restriction on voting outcomes which is "institution-free" in the sense of being based solely on the geometry of the choice set and the logic of sophisticated voting.

However, McKelvey and others have been unable to show exactly how large the uncovered set is relative to the Pareto set. McKelvey does show that the uncovered set converges smoothly to the core when one exists, so it is "small" whenever preferences come "close" to satisfying the Plott conditions (see also Cox [7]). But this says little about the size of the uncovered set when preferences are not so neatly arranged; it may stay relatively small, or it may quickly expand to fill the Pareto set. The one general statement about the size of the uncovered set to date is provided in Theorem 2, which shows that in distributive politics, restricting outcomes to the uncovered set is not very powerful. Thus restrictions on the size of the uncovered set have yet to be demonstrated when preferences are not close to having a core[5]. These results suggest that social choice theory could benefit from a more restrictive, easily-calculated substitute for the uncovered set, such as the "heart" (Schofield [20, 21]).

As for the study of distributive politics, Pareto constant games have the common property that preferences are maximally opposed, in that one player's gain is necessarily another player's loss. The result in Theorem 2 implies that in these settings, social choice theory is unable to place meaningful restrictions on equilibrium outcomes. Consequently, the political institutions which shape policy, such as committees, bicameralism, and the rules governing floor debate, play a correspondingly more significant role in these issue areas. That is, the rules which dictate the play of the game are especially important in distributive politics. This essay thus argues for the importance of institution-based studies such as Baron and Ferejohn [5] and Baron [3, 4], as crucial to understanding the broad division of political benefits. That is, as politics becomes more distributive in nature, institutional features become more important in determining final policy outcomes.

Appendix

Proof of Theorem 1. (1) Assume that $\mathbf{y}SC\mathbf{x}$ and $\mathbf{z}SC\mathbf{y}$. First, by the transitivity of the covering relation (as shown in Miller [15]), $\mathbf{z}$ covers $\mathbf{x}$. So any alternative preferred to $\mathbf{z}$ is also strictly preferred to $\mathbf{x}$. Then by the continuity of the functions u_i, these alternatives are also preferred to any point in an ε-neighborhood of $\mathbf{x}$. So $\mathbf{z}$ strongly covers $\mathbf{x}$.

(2) Given a point $\mathbf{x}$ not in $P(\mathbf{X})$, by the definition of Pareto superiority there exists a point $\mathbf{x}'$ such that $\mathbf{x}i' \succ_i \mathbf{x}$ for some individual i. We now prove that $\mathbf{x}'$ strongly covers $\mathbf{x}$. Since for all i, $\mathbf{x}' \succ_i \mathbf{x}$, it is certainly true that $\mathbf{x}' \succ \mathbf{x}$. What remains to be shown is that for any point $\mathbf{x}''$ socially preferred to $\mathbf{x}'$ there exists an ε such that $\mathbf{x}''$ is also preferred to all points within an ε neighborhood of $\mathbf{x}$. Let $\delta = \min_{i \in N} u_i(\mathbf{x}') - u_i(\mathbf{x})$, so that all voters have a utility differential of at least δ when comparing $\mathbf{x}'$ and $\mathbf{x}$. Now select an ε such that for all $\mathbf{x}^\varepsilon$ in an ε-neighborhood of $\mathbf{x}$ and all i, $u_i(\mathbf{x}^\varepsilon) - u_i(\mathbf{x}) < \delta$.

Any point $\mathbf{x}'' \succ \mathbf{x}'$ must satisfy $u_i(\mathbf{x}'') > u_i(\mathbf{x}')$ for at least $(n+1)/2$ individuals; denote these individuals $C_{x''} \subseteq N$. The ε-neighborhood was constructed to ensure that $u_i(\mathbf{x}') > u_i(\mathbf{x}^\varepsilon)$ for all $i \in N$, so, in particular, this inequality holds for all $i \in C_{x''}$. By transitivity, we have shown that $u_i(\mathbf{x}'') > u_i(\mathbf{x}^\varepsilon)$ for all $i \in C_{x''}$. Thus a strict majority of individuals prefer $\mathbf{x}''$ to any $\mathbf{x}^\varepsilon$ in the ε-neighborhood of $\mathbf{x}$ defined above, and so $\mathbf{x}'$ strongly covers $\mathbf{x}$.

[5] Note that other results also call into question the usefulness of the uncovered set in analyzing voting outcomes. Kramer and McKelvey [11] show that the minmax set is not necessarily included in the uncovered set. Ordeshook and Schwartz [16] prove that sophisticated voting with non-amendment agendas, such as those actually used in Congress, can yield outcomes outside the uncovered set. And procedures which produce Pareto-dominated outcomes, such as Shepsle's [22] institution-induced equilibrium, also fall outside the ambit of the uncovered set. Thus there are other reasons to suspect that political outcomes may not coincide with the uncovered set.

(3) Since $\mathbf{y} \succ \mathbf{x}$, and given that $P_i(\mathbf{x})$ is open, there is some neighborhood U_y for which $\mathbf{x}' \in U_y \Rightarrow \mathbf{y} \succ \mathbf{x}'$. Similarly, we know from the definition of strong covering that there is a neighborhood U_ε such that $\mathbf{z} \succ \mathbf{y} \Rightarrow \mathbf{z} \succ \mathbf{x}'$ for all $\mathbf{x}' \in U_y$. Since these neighborhoods are open sets, for all $\mathbf{x}'' \in U_y \cap U_\varepsilon$, $\mathbf{y}SC\mathbf{x}''$.

(4) This follows directly from point (3). Take any sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converging to the point $\mathbf{x}^*$. Assume that $\mathbf{x}_i \in SU(\mathbf{X})$ for all i, but $\mathbf{x}^* \notin SU(\mathbf{X})$. Then $\mathbf{x}^*$ is strongly covered by some point $\mathbf{y}$, which implies that some point $\mathbf{x}_N$ is also strongly covered by $\mathbf{y}$ for large enough N, contradicting our original assumption.

Proof of Corollary 1. Suppose $\mathbf{y}SC\mathbf{x}$ and $\mathbf{z} \succ_i \mathbf{y}$ for all $i \in N$. Then $\mathbf{z} \succ_i \mathbf{y} \succ_i \mathbf{x}$ for all i in some winning coalition A; hence $\mathbf{z} \succ \mathbf{x}$. Suppose that $\mathbf{w} \succ \mathbf{z}$. Then $\mathbf{w} \succ_i \mathbf{z} \succ_i \mathbf{y}$ for all i in some winning coalition B; hence $\mathbf{w} \succ \mathbf{y}$. By the definition of strong covering, there exists a neighborhood U_ε of $\mathbf{x}$ such that $\mathbf{w} \succ \mathbf{x}'$ for all $\mathbf{x}' \in U_\varepsilon$. Thus $\mathbf{z}SC\mathbf{x}$.

Proof of Lemma 1. Given the transitivity of the strong covering relation, the lemma follows immediately from Corollary 1 by letting $\mathbf{z}$ by any point which is Pareto superior to $\mathbf{y}$. Since $\mathbf{y}$ is not in the Pareto set, we know that such a point $\mathbf{z}$ exists. Now $\mathbf{z}$ strongly covers $\mathbf{y}$. Then the fact that $\mathbf{y}$ strongly covers $\mathbf{x}$ implies that $\mathbf{z}$ also strongly covers $\mathbf{x}$.

Proof of Theorem 2. The proof is by construction; that is, given any point in the Pareto set $\mathbf{x}$, any point $\mathbf{y}$ in the Pareto set that defeats it, and any $\varepsilon > 0$, a method is given to construct points $\mathbf{z}$ and $\mathbf{x}^\varepsilon$ such that $\mathbf{z} \succ \mathbf{y}$, $\mathbf{x}^\varepsilon \succeq \mathbf{z}$, and $\mathbf{x}^\varepsilon$, is within an ε-neighborhood of $\mathbf{x}$. Thus no point in the Pareto set is strongly covered by another point in the Pareto set, which, given Lemma 2 above, implies that no point in the Pareto set is strongly covered at all. The size of the uncovered set $U(\mathbf{X})$ will also be calculated during the course of the proof.

Given the convexity of $\mathbf{X}$ and the continuity of the u_i's, we know that given two feasible alternatives and their associated utilities, there is a point in which any convex combination of those utilities is attained. So in the remainder of the proof when we speak in terms of assigning individuals certain utility level, this should be understood as shorthand for finding the point within the Pareto set at which those utility levels are achieved.

For all points in the Pareto set, the sum of the players' utilities is a constant. Let $L = sum_{i=1}^{N} u_i(\mathbf{x})$ be this total level of utility of all $\mathbf{x} \in P(\mathbf{X})$. Also, let u_i^0 be the minimum utility individual i receives at any point in $P(\mathbf{X})$. For convenience, recalibrate utilities so that $u_i(\mathbf{x}) \geq 0$ for all $i \in N$ and $\mathbf{x} \in P(\mathbf{X})$.

It is useful to order the players such that if K individuals receive $u_i(\mathbf{x}) > ui^0$ in the initial allocation, they are positioned as individuals 1 through K. We claim that for at least one of these K individuals, $u_i(\mathbf{y}) < u_i(\mathbf{x})$. To see this, note first that $\mathbf{y} \succ \mathbf{x} \Rightarrow u_i(\mathbf{y}) > u_i(\mathbf{x})$ for at least $(n+1)/2$ voters. Since the sum of the utilities is constant within the Pareto set, this means that for at least one voter $u_i(\mathbf{y}) < u_i(\mathbf{x})$. But, by assumption, voters $K+1$ through N are already at their minimum utility. Thus at least one voter in $\{1, \ldots, K\}$ receives lower utility from $\mathbf{y}$ than from $\mathbf{x}$.

Further note that for at least one of the individuals[6] $i \in \{2, \ldots, (n+1)/2\}$, $u_i(\mathbf{y}) > u_i^0$. This can be seen by observing that for at least $(n+1)/2$

[6] The proof as presented assumes that N is odd. For the case of N even, substitute $n/2+1$ for $(n+1)/2$ in the following formulas.

individuals $\mathbf{y} \succ_i \mathbf{x}$, and by definition individual 1 receives less from $\mathbf{y}$ than from $\mathbf{x}$, so there must be some individual in the range $(2, \ldots, (n+1)/2)$ for whom $u_i(\mathbf{y}) > u_i(\mathbf{x}) \geq u_i^0$. For easier reference, let $A = \sum_{i=2}^{(n+1)/2} [u_i(\mathbf{y}) - u_i^0]$, so $0 < A \leq L$.

The next step is to construct a point $\mathbf{z}$ such that $\mathbf{z} \succ \mathbf{y}$ and for any $\varepsilon > 0$ there exists some $\mathbf{x}^\varepsilon$ in U_ε such that $\mathbf{x}^\varepsilon \succeq \mathbf{z}$. Let $u_i(\mathbf{z}) = u_i^0$ for $i \in (2, \ldots, (n+1)/2)$. This frees up utility A to distribute to the other individuals. By assumption, $u_1(\mathbf{y}) < u_1(\mathbf{x})$. Set $u_1(\mathbf{z})$ so that $u_1(\mathbf{y}) < u_1(\mathbf{z}) < u_1(\mathbf{x})$ and also $u_1(\mathbf{y}) < u_1(\mathbf{z}) < u_1(\mathbf{y}) + A$. This leaves $A - [u_1(\mathbf{z}) - u_1(\mathbf{y})]$ units of utility left; distribute this equally to voters $((n+3)/2, \ldots, n)$. Then the $(n+1)/2$ individuals in the set $(1, (n+3)/2, (n+5)/2, \ldots, n)$ all prefer $\mathbf{z}$ to $\mathbf{y}$, thus assuring that $\mathbf{z} \succ \mathbf{y}$.

Finally, two cases must be considered depending on the size of K, the number of voters for whom $u_i(\mathbf{x}) > u_i^0$. First, assume that $K \geq (n+1)/2$. Then we claim that the point $\mathbf{z}$ constructed above satisfies $\mathbf{z} \succ \mathbf{x}$. In particular, voter 1 has $u_1(\mathbf{z}) < u_1(\mathbf{x})$ by construction. And for all $i \in (2, \ldots, (n+1)/2)$, $u_i(\mathbf{z}) = u_i^0 < u_i(\mathbf{x})$. So for the cases in which $K \geq (n+1)/2$, $\mathbf{x} \succ \mathbf{z}$, completing a majority-rule cycle. Thus for any given $\mathbf{y} \succ \mathbf{x}$ we have constructed a point $\mathbf{z}$ such that $\mathbf{z} \succ \mathbf{y}$ and $\mathbf{x} \succeq \mathbf{z}$, proving that $\mathbf{x}$ is uncovered. And since $\mathbf{x}$ can be any point where more than half the players receive a non-minimal allocation, we conclude that these are exactly the points belonging to $U(\mathbf{X})$.

For the case where $K < (n+1)/2$, the task is now to construct a point $\mathbf{x}^\varepsilon$ within an ε-neighborhood of $\mathbf{x}$ for any given ε satisfying $\mathbf{x}^\varepsilon \succ \mathbf{z}$. Recall that for voters 2 through K, $u_i(\mathbf{z}) = u_i^0 < u_i(\mathbf{x})$, so they will strictly prefer $\mathbf{x}$ to $\mathbf{z}$. For these voters, let $u_i(\mathbf{x}^\varepsilon) = u_i(\mathbf{x})$, so that they will strictly prefer $\mathbf{x}^\varepsilon$ to $\mathbf{z}$.

By construction, $u_1(\mathbf{z}) < u_1(\mathbf{x})$, so voter 1 strictly prefers $\mathbf{x}$ to $\mathbf{z}$. Also, $u_i(\mathbf{z}) = u_i^0 = u_i(\mathbf{x})$ for all $i \in (K+1, \ldots, (n+1)/2)$, so these voters are indifferent between $\mathbf{x}$ and $\mathbf{z}$. We now construct $\mathbf{x}^\varepsilon$ so that voter 1 still prefers $\mathbf{x}^\varepsilon$ to $\mathbf{z}$, and voters $K+1$ through $(n+1)/2$ also strictly prefer $\mathbf{x}^\varepsilon$ to $\mathbf{z}$. Set $u_1(\mathbf{x}^\varepsilon) = u_1(\mathbf{x}) - v > u_1(\mathbf{z})$ and divide the remaining v of utility equally among voters $(K+1, \ldots, (n+1)/2)$. Now v can be decreased until the resulting allocation is within the required ε-neighborhood of $\mathbf{x}$. Then, as required, all individuals $(1, \ldots, (n+1)/2)$ strictly prefer $\mathbf{x}^\varepsilon$ to $\mathbf{z}$, and by constructing $\mathbf{x}^\varepsilon$ is within an ε-neighborhood of $\mathbf{x}$. Thus $\mathbf{x}^\varepsilon \succeq \mathbf{z}$, so $\mathbf{x}$ is not strongly covered by $\mathbf{y}$. Since the original point $\mathbf{x}$ was chosen arbitrarily, the entire Pareto set is strongly uncovered.

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