"Budget Management Strategies in Repeated Auctions"*

Technical Report with Supplementary Materials on the Uniqueness and Stability Properties in the Independent Setting

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TR.1 Introduction

In this technical report, we consider the dynamical systems corresponding to natural implementations of budget management mechanisms in Table 1 of the main paper and characterize the uniqueness and stability properties of system equilibria of these mechanisms in the independent setting. Often in practice, budget management strategies are implemented dynamically by adjusting parameters over time. One simple approach involves adjusting the parameter of each buyer independently until the system converges to an equilibrium.

Recall, for all budget management mechanisms under consideration, that a system equilibrium can be alternatively characterized as a solution to the following nonlinear complementarity problem:

$$G_i(s) \leq B_i \quad \perp \quad s_i \geq \underline{s}_i, \quad \forall i \in [n]\,, \tag{TR-1}$$

and that the expenditure function $G_i(s)$ of each buyer is strictly decreasing in his parameter s_i. Note, by Theorem 3.2 of the main paper, the budget management mechanisms always admit a system equilibrium in any independent problem instance. Let $\mathcal{S}$ be the parameter space as determined in Theorem 3.2 of the main paper. Then, when the expenditure $G_i(s)$ is above the budget B_i, the seller could increase the parameter s_i to reduce the expenditure. Conversely, when the expenditure $G_i(s)$ is below the budget B_i, the seller could decrease the parameter s_i to increase the expenditure. A simple update rule involves increasing the parameter of each buyer proportionally to the difference between the actual expenditure and the budget. Similar additive update rules have been extensively studied in the literature (see, e.g., Rosen 1965; Borgs et al. 2007; Jiang et al. 2014). Given initial system parameters $s \in \mathcal{S}$, the update rule is given by:

$$s^{(k+1)} = \mathrm{proj}_{\mathcal{S}}\left(s^{(k)} + \epsilon \cdot (G(s^{(k)}) - B)\right)\,, \tag{TR-2}$$

for iteration $k \geq 0$ with $s^{(0)} = s$. Here, $\epsilon > 0$ is the step size and $\mathrm{proj}_{\mathcal{S}} : \mathbb{R}^n \to \mathcal{S}$ is the projection map defined as $\mathrm{proj}_{\mathcal{S}}(s') = \arg\min_{s \in \mathcal{S}} \|s' - s\|$. Note that, in general, this procedure is not guaranteed to

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†Part of this work was done when the author was an intern at Google, Inc.

converge because the expenditure of each buyer is impacted by the parameters of all buyers in the auctions.

While adjustments are typically discrete in nature, we consider the limit in which parameters are continuously updated and steps are infinitesimal in size. Thus motivated, given initial system parameters s, we consider the following projected dynamical system:

$$\dot{s}(t) = \Pi_{\mathcal{S}}\left(s(t), \epsilon \cdot (G(s(t)) - B)\right) , \tag{TR-3}$$

with $s(0) = s$. Note $\epsilon > 0$ is the step size of the continuous updates and $\Pi_{\mathcal{S}}$ is the projection operator that keeps the vector s of system parameters in $\mathcal{S}$ at all times. More precisely, given $s \in \mathcal{S}$ and $\Delta \in \mathbb{R}^n$, $\Pi_{\mathcal{S}}$ is the projection map of vector Δ with respect to s and is defined as $\Pi_{\mathcal{S}}(s, \Delta) = \lim_{\epsilon' \to 0} \frac{\text{proj}_{\mathcal{S}}(s+\epsilon' \cdot \Delta) - s}{\epsilon'}$. This is given by $\Pi_{\mathcal{S}}(s, \Delta)_i = \Delta_i$ if $s_i \in (\underline{s}_i, \bar{s}_i)$, $\Pi_{\mathcal{S}}(s, \Delta)_i = \max\{0, \Delta_i\}$ if $s_i = \underline{s}_i$, and $\Pi_{\mathcal{S}}(s, \Delta)_i = \min\{0, \Delta_i\}$ if $s_i = \bar{s}_i$. By Lemma 2.2 in Zhang and Nagurney (1995), a system equilibrium can be then characterized as a stationary point to (TR-3).

Given the characterizations of system equilibria as solutions to the nonlinear complementarity problem (TR-1) and the projected dynamical system (TR-3), we can analyze the uniqueness and stability properties of system equilibria of the budget management mechanisms in Table 1 of the main paper. We first provide preliminaries including stability concepts in Section TR.2 then present the characterizations of the system equilibria of the budget management mechanisms in Section TR.3. When there is one buyer (i.e., $n = 1$), the buyer's expected expenditure is determined in terms of his system parameter only and it is straightforward to see the system equilibrium is unique and stable for all budget management mechanisms. Henceforth, we focus on the independent problem instances with the number of buyers $n \geq 2$. For notations and general setup in the independent setting (both asymmetric and symmetric), we follow Sections 2 and 4 of the main paper.

TR.2 Preliminaries

Recall $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ is the product space of the parameter spaces for the buyers where $\mathcal{S}_i = [\underline{s}_i, \bar{s}_i] \subseteq \mathbb{R}_+$ is the buyer i's parameter space as determined in Theorem 3.2 of the main paper. Let $\text{int}\, \mathcal{S}$ and $\partial \mathcal{S}$ denote the interior and boundary of the product space, respectively.

Let $\bar{G}_i(s) = B_i - G_i(s)$ be the *residual expenditure function* and $\bar{J}(s) = \nabla \bar{G}(s)$ be the *residual Jacobian matrix*, or simply, residual Jacobian. Note that the Jacobian matrix J of the projected dynamical system (TR-3) is $J = \epsilon \cdot \left(\frac{\partial G_i}{\partial s_j}\right)_{ij}$. For ease of exposition, we drop the multiplicative factor $\epsilon > 0$ and let the Jacobian matrix be $J = \left(\frac{\partial G_i}{\partial s_j}\right)_{ij}$. Then, $J = -\bar{J}$ and a condition or property in terms of one can be equivalently written in terms of the other. We use both Jacobian and residual Jacobian as convenient.

Given initial system parameters $s \in \mathcal{S}$, with some abuse of notation, we let $s(t)$ be the trajectory of the parameters as a function of time t produced by the projected dynamical system (TR-3) with initialization $s(0) = s$. If $s(t)$ stays fixed, i.e., $s = s(t)$ for all $t \geq 0$, then s is a *stationary point* (or equivalently, an equilibrium). For $s \in \mathcal{S}$ and $r > 0$, let $\mathcal{B}(s, r)$ denote the ball with radius r centered at s intersected with $\mathcal{S}$. An equilibrium point s^* is *stable* if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $s \in \mathcal{B}(s^*, \delta)$ and $t \geq 0$, $s(t) \in \mathcal{B}(s^*, \epsilon)$. On the other hand, we say that an equilibrium point s^* is *unstable* if it is not stable, that is, there exists some $\epsilon > 0$ such that for any $\delta > 0$, there exists $s \in \mathcal{B}(s^*, \delta)$ for which $s(t)$ goes outside of $\mathcal{B}(s^*, \epsilon)$. In addition, we use the following stronger notions of stability in theorem statements and proofs: *asymptotic stability*, *exponential stability* and *strictly (global) monotone attractor*. See Appendix A.2 for precise definitions.

For some local stability properties of system equilibria, we assume a regularity condition (which is different from the notion of regular distributions) that allows us to analyze the projected dynamical system (TR-3) locally as the classical dynamical system without the projection operator. We say a solution s^* to (TR-1) is *regular* if 1) s^* is in the interior $\operatorname{int}\mathcal{S}$, or 2) s^* is on the boundary $\partial\mathcal{S}$ and the strict complementarity holds in (TR-1), that is, if one inequality is an equality then the other is a strict inequality.[1] In practical situations of interest, system equilibria would likely be regular since it is unlikely that a system equilibrium is on the boundary and the budgets are met exactly; slightly perturbing the budgets can make any equilibrium a regular one. If s^* is a regular solution to (TR-1), we can use the eigenvalues of the Jacobian matrix J to determine stability: if all the eigenvalues have strictly negative real parts, the equilibrium is stable, and if at least one eigenvalue has a strictly positive real part, the equilibrium is unstable (see, e.g., Theorems 1 and 2 in Section 2.9 of Chapter 2 in Perko (2008)).

Truncated Parameter Spaces Note that budgets are positive and this, in particular, implies that an equilibrium s^* cannot be on an "upper" boundary surface; equivalently, there is no i such that $s_i^* = \bar{s}_i$. Furthermore, for the budget management mechanisms, we will truncate the buyers' parameter spaces on the lower bound such that buyer i's parameter space is, for example, $\mathcal{S}_i = [\underline{s}_i + \epsilon, \bar{s}_i]$ for some arbitrarily small $\epsilon > 0$.[2] With this slight modification, we can avoid the partial derivatives $\frac{\partial G_i}{\partial s_i}$ evaluating to 0 on the boundaries of the now-truncated parameter space. The modification simplifies the technical details (only on the uniqueness property) and does not change the nature of our results as far as the implementation of budget management mechanisms is concerned; in particular, a system equilibrium always exists as before. We conjecture that our results hold without the truncation. We will use the parameter space and its truncated version interchangeably in what follows.

TR.3 Uniqueness and Stability in the Independent Setting

We analyze the uniqueness and stability properties of system equilibria of the budget management mechanisms in Table 1 of the main paper. We use some sufficient conditions for the uniqueness of solutions to (TR-1) and the stability of solutions to (TR-3) which are presented in Appendix A. The proofs of the following theorems are presented in Appendix B.

TR.3.1 Reserve Pricing

For reserve pricing, we provide a nearly complete characterization. We show that the mechanism admits a unique system equilibrium and the system equilibrium is stable if it is regular. We assume the truncated parameter space for the following result.

Theorem TR.1. *For any independent problem instance with $n \geq 2$ buyers and the value distributions that have strictly increasing GFRs or are strictly regular, reserve pricing admits a unique system equilibrium and it is asymptotically stable if it is regular when the parameter spaces $\mathcal{S}_i$ are suitably restricted (as in Theorem 3.2 of the main paper) and truncated.*

We provide some intuition for our result. Fix a buyer i. Note that only the reserve prices of other buyers *with higher* reserve prices than buyer i can impact the allocation and payment of buyer i.

[1] This is different from the regularity notion for the buyers' distributions defined in Section 2 of the main paper. Which notion of regularity is being used will be clear from the context.

[2] The assumption of positive budgets implies that system equilibria cannot be on the upper boundary. The system equilibria can be either in the interior or on the lower boundary of the parameter space $\mathcal{S}$. The truncation will shift the lower boundary so that system equilibria are now in the interior of the original parameter space.

This follows because the bids of buyers with lower reserve prices than buyer i compete with buyer i only when their bids are above the buyer i's reserve price. Therefore, the residual Jacobian is upper triangular and its eigenvalues are the entries on its main diagonal, which are positive because the residual expenditure of a buyer is always increasing in its parameter. See Figure TR.1a for an example showing the uniqueness and stability properties of reserve pricing.

It is worth pointing out that reserve pricing has received considerable attention in the last few years (cf. Paes Leme et al. 2016). Two different implementations of reserve pricing are considered in the literature: in the lazy version, the winner is first determined, and then reserve prices are applied; and in the eager version, buyers not meeting their reserve prices are discarded, and then the winner is determined among the remaining buyers. In this paper, we consider the eager version, which dominates the lazy version on revenue whenever the buyers' values are independent. All our results translate to the lazy version in a straightforward fashion: in the lazy version, the allocation and payment of a buyer are only affected by his parameter, and the residual expenditure function is always monotone.

TR.3.2 Multiplicative Boosting

For multiplicative boosting, we can provide a complete characterization based on the monotonicity of the corresponding residual expenditure function $\bar{G}$. More specifically, we show multiplicative boosting admits a unique and stable system equilibrium. We do not need to truncate the parameter space.

Theorem TR.2. *For any independent problem instance with $n \geq 2$ buyers, multiplicative boosting admits a unique system equilibrium and it is a strictly global monotone attractor.*

In multiplicative boosting, the allocation and payment of a buyer are only affected by his own parameter. Because the expenditure of each buyer is monotone in his parameter, the residual Jacobian matrix is diagonal with positive entries and the residual expenditure function is monotone. This implies that the system equilibrium is unique and a strictly global monotone attractor. See Figure TR.1b for an example showing the uniqueness and stability properties of multiplicative boosting.

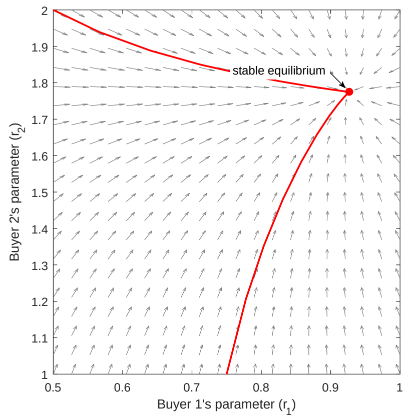
TR.3.3 Alternative Multiplicative Boosting

We show that alternative multiplicative boosting admits a unique system equilibrium which is a strictly global monotone attractor. As in the case of multiplicative boosting, the allocation and payment of a buyer are only affected by his own parameter, and the residual Jacobian matrix is diagonal. The proof of the following theorem is similar to that of Theorem TR.2 and is provided in Appendix B.3. We do not truncate the parameter space to remove the lower boundary on which derivatives can vanish. See Figure TR.1c for an example showing the uniqueness and stability properties of alternative multiplicative boosting.

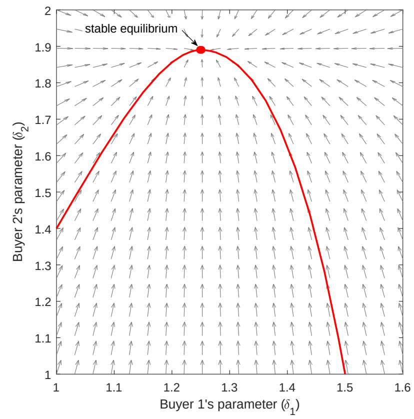
Theorem TR.3. *For any independent problem instance with $n \geq 2$ buyers and the value distributions that have strictly increasing GFRs or are strictly regular, alternative multiplicative boosting admits a unique system equilibrium and it is a strictly global monotone attractor when the parameter spaces $\mathcal{S}_i$ are suitably restricted (as in Theorem 3.2 of the main paper).*

TR.3.4 Throttling

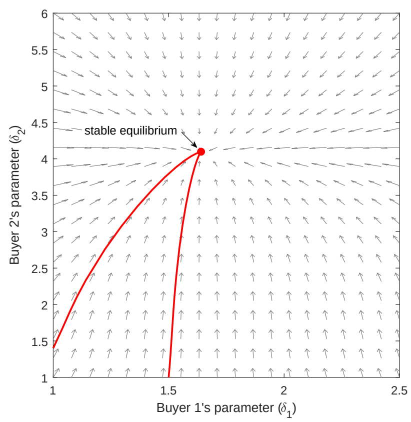
We show that throttling admits a unique system equilibrium when there are $n = 2$ buyers, and that the unique system equilibrium is asymptotically stable if it is regular. Proving uniqueness and stability in the case of throttling is more involved because increasing the parameter of buyer i has



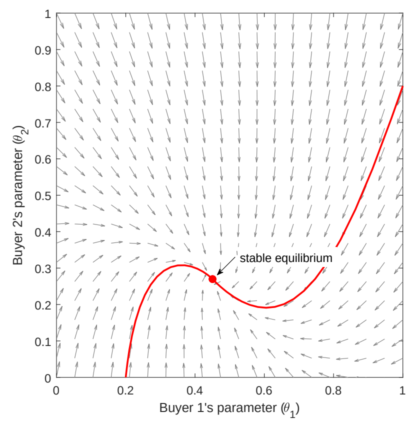
(a) Reserve pricing



(b) Multiplicative boosting



(c) Alternative mult. boosting



(d) Throttling

Figure TR.1: Phase portraits showing the unique and stable system equilibria of reserve pricing, multiplicative boosting, alternative multiplicative boosting, and throttling. Red curves show trajectories for different initial system parameters converging to the unique stable system equilibrium. In this example, the opportunity cost is $c = 0.2$ and there are two buyers with values uniformly distributed according to $U[0, 1]$ and $U[0, 2]$ and budgets $B_1 = 0.06$ and $B_2 = 0.2$, respectively.

two opposite effects on the expenditures of his competitors. Recall that when the parameter of buyer i increases, he participates in fewer auctions. When buyer i has the highest bid and he is throttled, some competitors would win the auction, increasing their expenditures. When buyer i has the second-highest bid, excluding him reduces the payment of the competitors. In the case of two buyers, we can carefully balance these effects and show the stability of the unique system equilibrium.

Theorem TR.4. *For any independent problem instance with $n = 2$ buyers, throttling admits a unique system equilibrium and it is asymptotically stable if it is regular when the parameter spaces $\mathcal{S}_i$ are truncated.*

We conjecture that the uniqueness and stability properties of throttling mechanism also hold in the independent setting with $n > 2$ buyers. To shed some light on the case of more than two players, we look at the symmetric independent setting where all buyers have the same budget and all bids are drawn i.i.d. from the same distribution. Proposition 4.1 of the main paper implies that there exists a unique symmetric system equilibrium. We study the local stability at the unique symmetric system equilibrium when the restriction that buyers' parameters are equal is relaxed for the updates in the projected dynamical system (TR-3). In particular, we can show it is a strictly monotone attractor. We refer to Figure TR.1d for a phase portrait showing the unique and stable system equilibrium of throttling .

Theorem TR.5. *For any symmetric independent problem instance with $n > 2$ buyers, the unique symmetric system equilibrium of throttling is a strictly monotone attractor.*

To show a system equilibrium is stable in the symmetric independent setting, we show that the corresponding Jacobian matrix J has only strictly negative real eigenvalues. At the symmetric system equilibrium, J is symmetric and, hence, negative definite. By the continuity of J, we can find a neighborhood around the symmetric system equilibrium where J is negative definite and show the stability property.

TR.3.5 Bid Shading

For bid shading, the uniqueness and stability properties of system equilibria may depend on the buyers' distributions. For several distributions in the independent setting with $n = 2$ buyers, we can show the uniqueness and stability properties hold. More specifically, the diagonal entries and determinant of $\bar{J}$ are positive over the parameter space and, therefore, the properties follow.

We provide the proof and additional details in Appendix B.5.

Theorem TR.6. *For any independent problem instance with $n = 2$ buyers and the buyers' distributions and opportunity cost c as described below, bid shading admits a unique system equilibrium and, if regular, it is asymptotically stable:*

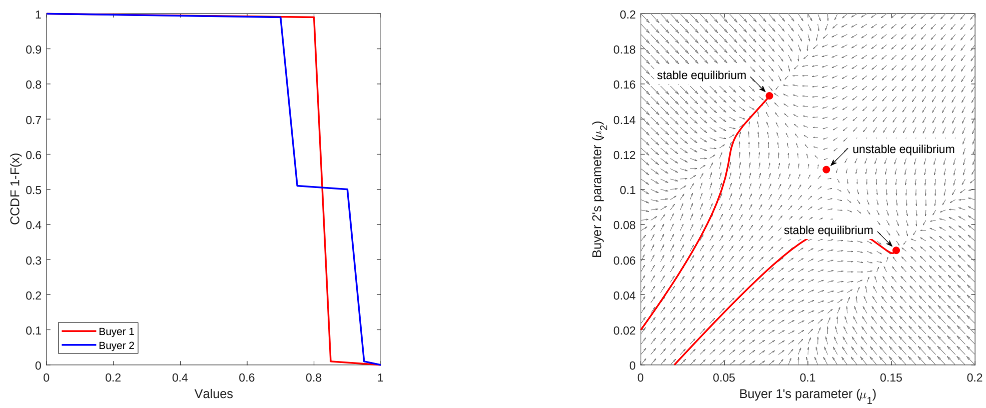
1. *Both exponential distributions with potentially different parameters and sufficiently small $c > 0$;*
2. *Both Rayleigh distributions with potentially different parameters and sufficiently small $c > 0$;*
3. *Both Weibull distributions with the same shape parameter r and sufficiently small $c > 0$;*
4. *Both uniform distributions with potentially different ranges and any $c > 0$.*

Furthermore, in the symmetric independent setting with $n \geq 2$ buyers, we show that the unique symmetric system equilibrium of bid shading is a strictly monotone attractor.

Theorem TR.7. *For any symmetric independent problem instance with $n \geq 2$ buyers, the unique symmetric system equilibrium of bid shading is a strictly monotone attractor.*

The following example shows that even with $n = 2$ buyers, bid shading can admit multiple system equilibria. In this example, the distribution of bids, while continuous, mimics a discrete distribution in which bids are concentrated around a few values.[3] This behavior has been recently observed by Conitzer et al. (2017), who show that bid shading can lead to multiple and unstable equilibria in a model in which bids are discrete and potentially correlated across buyers. Moreover, they show that a simple best-response procedure may cycle and fail to converge to an equilibrium. We conjecture that bid shading has a unique and stable equilibrium when the distributions of bids are sufficiently smooth.

[3] Strictly speaking, the probability density functions are continuous except at finite points. Our stability results still hold with appropriate modifications with this weaker condition instead of continuity. In this sense, the example still demonstrates that bid shading can admit multiple and unstable equilibria.



(a) Complementary CDF's of buyers' distributions

(b) A phase portrait

Figure TR.2: An example showing multiple system equilibria of bid shading in an asymmetric setting with two buyers. See Remark TR.1.

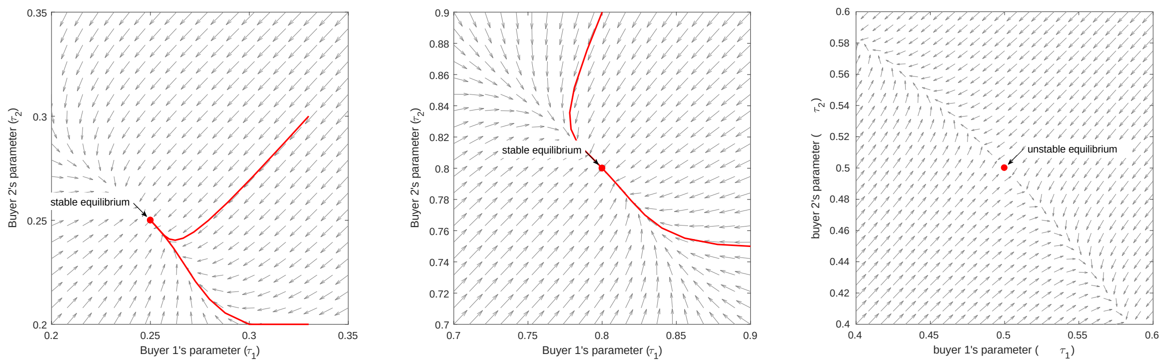
Remark TR.1. *In the independent setting with $n = 2$ buyers, bid shading can admit multiple system equilibria. Consider the following problem instance where the buyers' cumulative distribution functions are piecewise linear. Let $\epsilon = 0.01$. Representing each piecewise linear function as a sequence of breakpoints at which the slope changes, buyer 1's distribution is given by $[(0,0),(0.8,\epsilon),(0.85,1-\epsilon),(1,1)]$ and buyer 2's distribution is given by $[(0,0),(0.7,\epsilon),(0.75,0.5-\epsilon),(0.9,0.5),(0.95,1-\epsilon),(1,1)]$. Let buyers' budgets be $B_1 = 0.3191$ and $B_2 = 0.3727$ and the opportunity cost be $c = 0.00001$. Solving the projected dynamical system numerically, we find three equilibria: $\mu^* = (0.0770, 0.1530)$, $\mu^* = (0.1530, 0.0652)$, and $\mu^* = (0.111, 0.111)$. The first two system equilibria are stable, but the last one is unstable. See Figure TR.2.*

TR.3.6 Thresholding

We show that thresholding can admit multiple system equilibria and that its system equilibria can be unstable even in the symmetric independent setting. To show a system equilibrium is unstable, it suffices to show the residual Jacobian matrix $\bar{J}$ has a negative real eigenvalue or, equivalently, the Jacobian matrix J has a positive real eigenvalue. The following result characterizes the stability properties of the unique symmetric system equilibrium of thresholding. For the proof, we refer to Appendix B.6.

Theorem TR.8. *For any symmetric independent problem instance with $n \geq 2$ buyers, the unique symmetric system equilibrium of thresholding can be stable or unstable depending on the buyers' budgets. In particular, the unique symmetric system equilibrium $\tau_1^* \vec{\mathbf{1}}$ is a strictly monotone attractor if $\tau_1^* \bar{F}_1(\tau_1^*) < c$ and an unstable equilibrium if $\tau_1^* \bar{F}_1(\tau_1^*) > c$.*

The previous theorem shows that the stability of thresholding can depend on the buyers' budgets, the distribution of bids and the opportunity cost. In particular, when the opportunity cost is arbitrarily close to 0, the unique symmetric system equilibrium of thresholding is unstable. Otherwise, the stability property depends on the buyers' budgets. To see this, recall that when budgets are binding, the symmetric system equilibrium satisfies $G_1(\tau_1^*) = B_1$. Because $G_1(\cdot)$ is decreasing and the function $\tau_1 \bar{F}_1(\tau_1)$ is quasi-concave in τ_1 when the bid distribution has an increasing GFR, we obtain that thresholding is stable when budgets are low or high and unstable for moderate budgets.



(a) $B_1 = B_2 = 0.1781$ (high budgets) **(b)** $B_1 = B_2 = 0.0493$ (low budgets) **(c)** $B_1 = B_2 = 0.1333$ (moderate budgets)

Figure TR.3: Phase portraits around stable and unstable symmetric system equilibria of thresholding for different budget levels in the symmetric independent setting with two buyers with uniform distribution $U[0,1]$ and opportunity cost $c = 0.2$. See Remark TR.2.

Intuitively speaking, an unstable system equilibrium can arise when the net effect of the concurrent updates of buyers' system parameters leads the system parameters farther away from the equilibrium. This is possible if the cross effects captured by partial derivatives $\frac{\partial G_i}{\partial \tau_j}$ are greater in magnitude and work in the opposite direction of the direct effects captured by partial derivatives $\frac{\partial G_i}{\partial \tau_i}$ when system parameters are updated simultaneously. For example, consider an instance with two buyers and $c = 0$ (or arbitrarily close to 0). Under thresholding, increasing (decreasing) the parameter of a buyer decreases (increases) both his expenditure and the expenditure of his competitor. Assume that buyer 1's current expenditure is slightly above his budget and buyer 2's current expenditure is slightly below his budget. The updates are determined unilaterally: the system parameter of buyer 1 will be increased and that of buyer 2 will be decreased, say, by the same magnitude. When these updates occur simultaneously, buyer 1's expenditure will decrease due to his own parameter increasing but will also increase due to buyer 2's parameter decreasing. Overall, the net effect is such that buyer 1's expenditure will increase and, similarly, buyer 2's expenditure will decrease, leading both buyers farther away from a system equilibrium.

Remark TR.2. *We provide specific instances for both cases in Theorem TR.8 by varying the budget levels when the buyers' distributions are uniform distribution $U[0,1]$ and the opportunity cost is $c = 0.2$. See Figure TR.3 for corresponding phase portraits. For a stable symmetric system equilibrium, consider $B_1 = B_2 = 0.1781$ (high budgets) or $B_1 = B_2 = 0.0493$ (low budgets). For the first set of budgets, the symmetric system equilibrium is $\tau_1^* = 0.25$. For the second set, the corresponding symmetric system equilibrium is $\tau_1^* = 0.8$. For an unstable symmetric system equilibrium, consider $B_1 = B_2 = 0.1333$ (moderate budgets). The symmetric system equilibrium is $\tau_1^* = 0.5$ and satisfies $c < \tau_1^* \bar{F}_1(\tau_1^*)$.*

Thresholding can admit multiple system equilibria even in the symmetric independent setting as the following example shows.

Remark TR.3. *In the symmetric independent setting with $n = 2$ buyers, thresholding can admit multiple system equilibria. When the opportunity cost is $c = 0$, budgets are $B = (0.05, 0.05)$ and buyers' distributions are both uniform distribution $U[0,1]$, three system equilibria arise: $\tau^* = (0.6367, 0.6367)$ is the unique symmetric system equilibrium, and $\tau^* = (0.0, 0.8879)$ and $\tau^* = (0.8879, 0.0)$ are also*

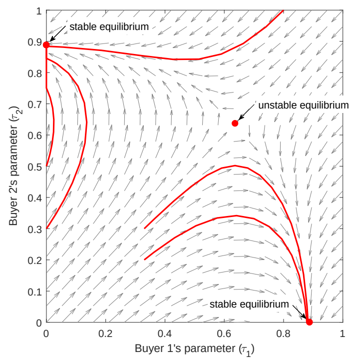


Figure TR.4: A phase portrait showing multiple system equilibria of thresholding in the symmetric independent setting with opportunity cost $c = 0$ and two buyers with uniform distribution $U[0, 1]$ and budgets $B_1 = B_2 = 0.05$. See Remark TR.3.

system equilibria which can be found via a numerical simulation of the projected dynamical system (TR-3)*. The symmetric system equilibrium is unstable and the other two system equilibria are stable. We obtain similar plots for $c > 0$ and the stable system equilibria come off the boundaries as c increases. See Figure TR.4.*

References

Balseiro, S., A. Kim, M. Mahdian, and V. Mirrokni (2017). Budget management strategies in repeated auctions. In *Proceedings of the 26th International Conference on World Wide Web*, WWW '17, Republic and Canton of Geneva, Switzerland, pp. 15–23. International World Wide Web Conferences Steering Committee.

Borgs, C., J. Chayes, N. Immorlica, K. Jain, O. Etesami, and M. Mahdian (2007). Dynamics of bid optimization in online advertisement auctions. In *Proceedings of the 16th International Conference on World Wide Web*, WWW '07, New York, NY, USA, pp. 531–540. ACM.

Conitzer, V., C. Kroer, E. Sodomka, and N. E. Stier (2017). Multiplicative pacing equilibria in auction markets. Working paper.

Facchinei, F. and J.-S. Pang (2003). *Finite-Dimensional Variational Inequalities and Complementarity Problems, Volume 1*. Springer Series in Operations Research and Financial Engineering. Springer.

Jiang, C., C. L. Beck, and R. Srikant (2014, April). Bidding with limited statistical knowledge in online auctions. *SIGMETRICS Perform. Eval. Rev. 41*(4), 38–41.

Moré, J. and W. Rheinboldt (1973). On p- and s-functions and related classes of n-dimensional nonlinear mappings. *Linear Algebra and its Applications 6*(Supplement C), 45 – 68.

Paes Leme, R., M. Pal, and S. Vassilvitskii (2016). A field guide to personalized reserve prices. In *Proceedings of the 25th International Conference on World Wide Web*, WWW '16, pp. 1093–1102. International World Wide Web Conferences Steering Committee.

Perko, L. (2008). *Differential Equations and Dynamical Systems*. Texts in Applied Mathematics. Springer New York.

Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society*, 520–534.

Zhang, D. and A. Nagurney (1995). On the stability of projected dynamical systems. *Journal of Optimization Theory and Applications 85*(1), 97–124.

A Sufficient Conditions on Uniqueness and Stability

We provide additional definitions and technical results that describe rather general sufficient conditions for the uniqueness and stability of system equilibria. We use the alternative characterizations of system equilibria as solutions to the nonlinear complementarity problem (TR-1) and solutions to the projected dynamical systems (TR-3). We present sufficient conditions for the uniqueness of solutions to (TR-1) and the stability of solutions to (TR-3).

A.1 Uniqueness

We can show the uniqueness property of solutions to what is known as the variational inequality problem (cf. Facchinei and Pang (2003); Zhang and Nagurney (1995)) and, hence, of equilibria of (TR-3) when the residual expenditure function is a P-function. We say that $\bar{G}$ is a P_0-*function* if for any $s \neq s'$ in its domain, there is an index $i = i(s, s')$ such that

$$s'_i \neq s_i \quad \text{and} \quad (s'_i - s_i)(\bar{G}_i(s') - \bar{G}_i(s)) \geq 0\,.$$

$\bar{G}$ is a P-*function* if there exists an index $i = (s, s')$ such that the inequality holds strictly.

The non-linear complementarity problem (TR-1) admits at most one equilibrium if the residual expenditure function is a P-function on its domain.

Proposition TR.1. *Suppose $\bar{G}$ is a P-function on $\mathcal{S}$. Then, the non-linear complementarity problem* (TR-1) *admits at most one solution.*

Proof. For the sake of contradiction, assume $s \neq s' \in \mathcal{S}$ are two solutions to (TR-1). Then, for each i,

$$\begin{aligned}(s'_i - s_i)\left(\bar{G}_i(s') - \bar{G}_i(s)\right) &= \left((s'_i - \underline{s}_i) - (s_i - \underline{s}_i)\right)\left(\bar{G}_i(s') - \bar{G}_i(s)\right) \\ &= -(s_i - \underline{s}_i)\bar{G}_i(s') - (s'_i - \underline{s}_i)\bar{G}_i(s) \\ &\leq 0\,,\end{aligned}$$

where the second equality follows from the complementarity conditions. On the other hand, since $\bar{G}$ is a P-function, there exists some j such that $(s'_j - s_j)\left(\bar{G}_j(s') - \bar{G}_j(s)\right) > 0$. This is a contradiction. □

We can show that $\bar{G}$ is a P-function in two different ways: 1) showing that the residual Jacobian $\bar{J}$ is a P-matrix; and 2) showing that $\bar{G}$ is strictly monotone, for which the positive definiteness of $\bar{J}$ is sufficient. Essentially, both relies on showing some specific properties of the residual Jacobian $\bar{J}$. We describe the two approaches below.

We say a matrix A is a P_0-*matrix* (P-*matrix*) if every principal minor of the matrix is nonnegative (strictly positive). Given that the parameter space $\mathcal{S}$ of the budget management mechanisms under consideration is a Cartesian product of n closed intervals, we have the following sufficient condition[4]:

Proposition TR.2 (Theorem 5.2 in Moré and Rheinboldt (1973))**.** *Suppose $\bar{G}$ is differentiable and $\bar{J}(s)$ is a P-matrix for all $s \in \mathcal{S}$. Then, $\bar{G}$ is a P-function on $\mathcal{S}$.*

We now define monotone functions which will also be used to prove the stability properties of system equilibria in Section A.2. We say $\bar{G}$ is *locally monotone* at $s \in \mathcal{S}$ if there is a neighborhood $\mathcal{B}(s)$ of s such that

$$\langle \bar{G}(s') - \bar{G}(s), s' - s \rangle \geq 0, \quad \forall s' \in \mathcal{B}(s)\,.$$

[4]More generally, Proposition TR.2 holds for any Cartesian product of intervals where each interval can be open, closed or semi-open, and may be unbounded.

$\bar{G}$ is *locally strictly monotone* if the above inequality holds strictly for all $s' \in \mathcal{B}(s) \setminus \{s\}$. Similarly, we say that $\bar{G}$ is *monotone* at $s \in \mathcal{S}$ if the above inequality holds for all $s' \in \mathcal{S}$, and that $\bar{G}$ is *strictly monotone* if the above inequality holds strictly for all $s' \in \mathcal{S} \setminus \{s\}$. Furthermore, we say $\bar{G}$ is (strictly) monotone *on its domain* if it is (strictly) monotone at every $s \in \mathcal{S}$. By the definitions, we have:

Remark TR.4. *If $\bar{G}$ is (strictly) monotone on its domain then it is also a (P-function) P_0-function.*

We say that a matrix A is *positive definite* if $z^\intercal Az > 0$ for all nonzero z. A is *positive semidefinite* if the above inequality holds weakly. A sufficient condition for the strict monotonicity of a residual expenditure function is that the residual Jacobian matrix is positive definite on its domain. This condition is too restrictive for our setting as it does not necessarily hold when the parameter of a buyer lies on the lower boundary (before truncation), i.e., there exists i such that $s_i^* = \underline{s}_i$, and the corresponding residual Jacobian matrix may be singular. Thus motivated, we introduce a weaker condition imposing that the residual Jacobian is positive definite for parameters in the interior of the domain. For a subset $\mathcal{I} \subseteq [n]$, let $\operatorname{int} \mathcal{S}_\mathcal{I} \times \mathcal{S}_{-\mathcal{I}}$ be the product space of buyers' parameter spaces and their interiors where buyer i's parameter is restricted to the interior $\operatorname{int} \mathcal{S}_i = (\underline{s}_i, \bar{s}_i)$ if $i \in \mathcal{I}$, and is not restricted otherwise. Let $\bar{J}_\mathcal{I}$ be the square submatrix obtained by removing rows and columns not in $\mathcal{I}$. The following connections can be proved using the mean value theorem.

Proposition TR.3. *Suppose $\bar{G}$ is differentiable. Then:*

1. *If there is a neighborhood $\mathcal{B}(s)$ such that residual Jacobian $\bar{J}(s')$ is positive semidefinite for all $s' \in \mathcal{B}(s)$, then $\bar{G}(s)$ is locally monotone at s.*

2. *If there is a neighborhood $\mathcal{B}(s)$ such that residual Jacobian $\bar{J}_\mathcal{I}(s')$ is positive definite for all $s' \in \mathcal{B}(s)$ with $\mathcal{I} = \{i \in [n] : s'_i > 0\}$, then $\bar{G}(s)$ is locally strictly monotone at s.*

3. *If the residual Jacobian $\bar{J}(s)$ is positive semidefinite for all $s \in \mathcal{S}$, then $\bar{G}$ is monotone on its domain.*

4. *If the residual Jacobian $\bar{J}_\mathcal{I}(s)$ is positive definite for all $s \in \operatorname{int} \mathcal{S}_\mathcal{I} \times \mathcal{S}_{-\mathcal{I}}$ and $\mathcal{I} \subseteq [n]$, then $\bar{G}$ is strictly monotone on its domain.*

Proof. We prove (2). The other results follow similarly. Let $s' \in \mathcal{B}(s)$ with $s' \neq s$ and consider $\phi(\alpha) = \langle \bar{G}(s + \alpha(s' - s)), s' - s\rangle$ for $\alpha \in [0, 1]$. Note

$$\begin{aligned}\phi(1) - \phi(0) &= \langle \bar{G}(s'), s' - s\rangle - \langle \bar{G}(s), s' - s\rangle \\ &= \langle \bar{G}(s') - \bar{G}(s), s' - s\rangle \,.\end{aligned}$$

Since $\phi(\alpha)$ is continuously differentiable, by the mean value theorem, there exists $\xi \in (0, 1)$ such that $\phi(1) - \phi(0) = \phi'(\xi)$. Then,

$$\phi(1) - \phi(0) = \phi'(\xi) = (s' - s)^\intercal \cdot \bar{J}(s + \xi(s' - s)) \cdot (s' - s) \,.$$

Let $z = s + \xi(s' - s) = (1 - \xi)s + \xi s' \in \mathbb{R}^n_+$ and $\mathcal{I} = \{i \in [n] : z_i > 0\}$. Note that if $z_i = 0$, then $s'_i = s_i = 0$ because $\xi \in (0, 1)$, and, hence, that $\mathcal{I}$ is not empty. Furthermore, note that $(s' - s)_\mathcal{I}$ is not 0, otherwise it would follow that $s = s'$. Putting everything together, we obtain that

$$\langle \bar{G}(s') - \bar{G}(s), s' - s\rangle = (s' - s)^\intercal_\mathcal{I} \cdot \bar{J}_\mathcal{I}(z) \cdot (s' - s)_\mathcal{I} > 0 \,.$$

where the last inequality follows because $\bar{J}_\mathcal{I}(z)$ is positive definite since $z \in \mathcal{B}(s)$. □

A.2 Stability

We first define the following stronger notions of stability. Recall $\mathcal{B}(s, r)$ denotes the ball with radius r centered at s intersected with $\mathcal{S}$ for $s \in \mathcal{S}$ and $r > 0$.

Definition TR.1. *An equilibrium point s^* is* asymptotically stable *if it is stable and there exists a $\delta > 0$ such that, for all $s \in \mathcal{B}(s^*, \delta)$, $\lim_{t\to\infty} s(t) = s^*$.*

Definition TR.2. *An equilibrium point s^* is* exponentially stable *if there exists a neighborhood $\mathcal{B}(s^*)$ of s^* and constants $\alpha > 0$ and $\beta > 0$ such that $\|s(t) - s^*\| \leq \alpha \|s - s^*\| e^{-\beta t}$ for all $t \geq 0$ and $s \in \mathcal{B}(s^*)$.*

Definition TR.3. *An equilibrium point s^* is a* monotone attractor *if there exists a $\delta > 0$ such that $d(t) := \|s(t) - s^*\|$ is a nonincreasing function of $t \geq 0$ for all $s \in \mathcal{B}(s^*, \delta)$. Similarly, s^* is a* strictly monotone attractor *if $d(t)$ is strictly decreasing to 0 in $t \geq 0$ for all $s \in \mathcal{B}(s^*, \delta)$.*

Definition TR.4. *An equilibrium point s^* is a* global monotone attractor *if $d(t) := \|s(t) - s^*\|$ is a nonincreasing function of $t \geq 0$ for all $s \in \mathcal{S}$. Similarly, s^* is a* strictly global monotone attractor *if $d(t)$ is strictly decreasing to 0 in $t \geq 0$ for all $s \in \mathcal{S}$.*

Following the work of Zhang and Nagurney (1995), we can show that a system equilibrium is stable in two ways. First, we can show stability properties of solutions to the variational inequality problem (cf. Facchinei and Pang (2003); Zhang and Nagurney (1995)) and, hence, of equilibria of (TR-3) when the residual expenditure function is monotone as follows. Note we defined the monotone functions in Section A.1:

Proposition TR.4 (Theorems 4.1 and 4.2 in Zhang and Nagurney (1995))**.** *Suppose s^* is a solution to* (TR-1)*. Then, the following hold:*

1. *If $\bar{G}$ is locally monotone at s^*, then s^* is a monotone attractor.*
2. *If $\bar{G}$ is locally strictly monotone at s^*, then s^* is a strictly monotone attractor.*
3. *If $\bar{G}$ is monotone at s^*, then s^* is a global monotone attractor.*
4. *If $\bar{G}$ is strictly monotone at s^*, then s^* is a strictly global monotone attractor.*

Second, when equilibria satisfy a regularity condition, we can show local stability properties of the projected dynamical system (TR-3) by considering the following classical dynamical system without the projection operator:

$$\dot{s}(t) = \epsilon \cdot (G(s(t)) - B)\,. \tag{TR-4}$$

Recall that a solution s^* to (TR-1) is *regular* if 1) s^* is in the interior $\operatorname{int}\mathcal{S}$, or 2) s^* is on the boundary $\partial\mathcal{S}$ and the strict complementarity holds in (TR-1). We use the following result in this approach:

Proposition TR.5 (Theorem 3.4 in Zhang and Nagurney (1995))**.** *Suppose s^* is a regular solution to* (TR-1)*. Then, the following hold:*

1. *If s^* is a stable equilibrium for* (TR-4)*, then it is a stable equilibrium for* (TR-3)*.*
2. *If s^* is an asymptotically stable equilibrium for* (TR-4)*, then it is an asymptotically stable equilibrium for* (TR-3)*.*

For instance, in the stability analysis of the (unprojected) dynamical system (TR-4), we can often use the eigenvalues of the Jacobian matrix J at the equilibrium in consideration to determine its stability. If all the eigenvalues have strictly negative real parts, then the equilibrium is exponentially stable and, hence, also asymptotically stable. If at least one eigenvalue has a strictly positive real part, then the equilibrium is unstable. If s^* is a regular solution to (TR-1), we can similarly use the eigenvalues to determine its stability as a solution to the projected dynamical system (TR-3).

In particular, the following results are derived from the above two approaches and are used in the proofs in Appendix B.

Proposition TR.6. *Let $n \geq 2$ and s^* be a symmetric solution (i.e., all s_i^* are the same) to* (TR-1) *in the symmetric independent setting. If the Jacobian matrix J is continuous at s^* and has negative eigenvalues only, then s^* is a strictly monotone attractor.*

Proof. Let $H : \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}$ be defined as $H(A, z) = z^\intercal Az$ for any $A \in \mathbb{R}^{n\times n}$ and $z \in \mathbb{R}^n$, and $h : \mathbb{R}^{n\times n} \to \mathbb{R}$ be defined as $\max_{\|z\|=1} H(A, z)$. Note that $\{z : \|z\| = 1\}$ is compact. By the maximum theorem, h is continuous.

Since J has negative eigenvalues and is symmetric at the symmetric solution s^*, it is negative definite, i.e., $h(J) = \alpha$ for some $\alpha < 0$. By the continuity of h and J, we can find a neighborhood B around s^* such that $h(J) < \frac{\alpha}{2}$; that is, $z^\intercal J' z < 0$ for all $z \in \mathbb{R}^n$ and $J' \in B$. Then, the Jacobian matrix is negative definite or, equivalently, the residual Jacobian is positive definite over the neighborhood. Finally, the stability property follows from Propositions TR.3 and TR.4. □

Proposition TR.7. *Let $n = 2$ and s^* be a regular solution to* (TR-1). *If the Jacobian matrix J at s^* has a negative trace,* $\mathrm{Tr}(J) < 0$*, and a positive determinant,* $\det(J) > 0$*, then s^* is asymptotically stable.*

Proof. Note that the characteristic polynomial for the Jacobian matrix J is $P(z) = z^2 - \mathrm{Tr}(J)z + \det(J)$. The roots of $P(z)$ are the eigenvalues of J and are given by the quadratic formula: $\frac{\mathrm{Tr}(J)\pm\sqrt{\mathrm{Tr}(J)^2-4\det(J)}}{2}$. If $\mathrm{Tr}(J)^2 \geq 4\det(J)$, then we get two negative real eigenvalues. If $\mathrm{Tr}(J)^2 < 4\det(J)$, the we get two imaginary eigenvalues with negative real parts. In both cases, the system equilibrium would be exponentially stable and, hence, asymptotically stable when considered as an equilibrium to the dynamical system (TR-4). By Proposition TR.5, s^* is an asymptotically stable equilibrium for the projected dynamical system (TR-3). □

B Missing Proofs from Section TR.3

We provide the missing proofs from Sections TR.3.1–TR.3.6 on the characterizations of the uniqueness and stability of system equilibria of the budget management mechanisms. We use the results presented in Appendix A to characterize the uniqueness and stability properties. For ease of presentation, we assume $T = 1$ in the proofs to be presented below without loss of generality.

B.1 Reserve Pricing

Proof of Theorem TR.1. Fix any independent problem instance with $n \geq 2$ buyers and the value distributions that have strictly increasing GFRs or are strictly regular. Assume $T = 1$ without loss of generality. We restrict the parameter spaces $\mathcal{S}_i$ as in the proof of Theorem 3.2 of the main paper and truncate it as described in Section TR.2. By Theorem 3.2 of the main paper, reserve pricing admits system equilibria.

We show that each principal submatrix of $\bar{J} = -J$ has only strictly positive real eigenvalues for all $s \in \mathcal{S}$. Then, it would follow that $\bar{J}(s)$ is a P-matrix for all $s \in \mathcal{S}$ and, by Propositions TR.1 and TR.2, the uniqueness property follows. For the stability property, note that $J(s)$ has strictly negative real eigenvalues. By the stability theory of dynamical system, it would follow that a system equilibrium is exponentially stable and, hence, asymptotically stable as a solution to the dynamical system version of (TR-3). By Proposition TR.5, the unique system equilibrium would be also asymptotically stable as a solution to (TR-3) as long as it is regular.

Without loss of generality, we assume $r_1 \leq \cdots \leq r_n$. Note the expected expenditure of buyer i can be written as

$$G_i^{\mathrm{R}}(r_i, r_{-i}) = -\int_{r_i}^{\infty} \left(\bar{F}_i(z) - f_i(z)z\right) H_i^{\mathrm{R}}(z; r_{-i}) dz\,,$$

where $H_i^{\mathrm{R}}(z; r_{-i}) = \mathbb{P}\{\max_{j \neq i: x_j \geq r_j} \leq z\} = \prod_{j \neq i: r_j \geq z} F_j(r_j) \cdot \prod_{j \neq i: r_j < z} F_j(z)$. We compute the partial derivatives and obtain, for all $i \in [n]$ and $j \neq i$:

$$\frac{\partial G_i^{\mathrm{R}}}{\partial r_i} = \left(\bar{F}_i(r_i) - f_i(r_i) r_i\right) H_i^{\mathrm{R}}(r_i; r_{-i}) \quad \text{and}$$

$$\frac{\partial G_i^{\mathrm{R}}}{\partial r_j} = \begin{cases} -\int_{r_i}^{r_j} \left(\bar{F}_i(z) - f_i(z)z\right) f_j(r_j) \\ \qquad \cdot \prod_{k \neq i,j: r_k \geq z} F_k(r_k) \prod_{k \neq i,j: r_k < z} F_k(z) dz, & \text{if } r_i \leq r_j \\ 0, & \text{if } r_i > r_j \end{cases}.$$

Note $-J$ is an upper triangular matrix and its eigenvalues are exactly the diagonal entries. The diagonal entries $-\frac{\partial G_i^{\mathrm{R}}}{\partial r_i}$ are strictly positive for $r_i \in \mathcal{S}_i$. Furthermore, every principal submatrix of $-J$ is again an upper triangular matrix with strictly positive diagonal entries. □

B.2 Multiplicative Boosting

Proof of Theorem TR.2. Fix any independent problem instance with $n \geq 2$ buyers. Assume $T = 1$ without loss of generality. By Theorem 3.2 of the main paper, multiplicative boosting admits system equilibria. In what follows, we show every principal submatrix $-J_{\mathcal{I}}$ is positive definite in $\operatorname{int} \mathcal{S}_{\mathcal{I}} \times \mathcal{S}_{-\mathcal{I}}$. Then, $\bar{G}$ would be strictly monotone on its domain by Proposition TR.3 and, hence, a P-function on $\mathcal{S}$ by Remark TR.4. The uniqueness and stability properties would follow from Propositions TR.1 and TR.4, respectively.

Under multiplicative boosting, the expected expenditure of buyer i is

$$G_i^{\mathrm{M}B}(\delta_i, \delta_{-i}) = c\bar{F}_i(\delta_i c) H_i^{\mathrm{M}B}(c) + \int_c^{\infty} z \bar{F}_i(\delta_i z)\, \mathrm{d}H_i^{\mathrm{M}B}(z)\,,$$

where $H_i^{\mathrm{M}B}(z) = \mathbb{P}\{\max_{j \neq i} x_j \leq z\} = \prod_{j \neq i} F_j(z)$. Then, we compute the partial derivatives and obtain, for all $i \in [n]$ and $j \neq i$,

$$\frac{\partial G_i^{\mathrm{M}B}}{\partial \delta_i} = -c^2 f_i(\delta_i c) H_i^{\mathrm{M}B}(c) - \int_c^{\infty} z^2 f_i(\delta_i z) dH_i^{\mathrm{M}B}(z)$$

and $\frac{\partial G_i^{\mathrm{M}B}}{\partial \delta_j} = 0$. Note $-J$ is a diagonal matrix with strictly positive entries on the diagonal and 0's elsewhere. Its every principal submatrix $-J_{\mathcal{I}}$ is also a diagonal matrix with strictly positive entries on the diagonal and, hence, positive definite for all $s \in \operatorname{int} \mathcal{S}_{\mathcal{I}} \times \mathcal{S}_{-\mathcal{I}}$. Note that diagonal matrices are symmetric. □

B.3 Alternative Multiplicative Boosting

Proof of Theorem TR.3. Fix any independent problem instance with $n \geq 2$ buyers and the value distributions that have strictly increasing GFRs or are strictly regular. Assume $T = 1$ without loss of generality. We restrict the parameter spaces $\mathcal{S}_i$ as in the proof of Theorem 3.2 of the main paper. By Theorem 3.2 of the main paper, alternative multiplicative boosting admits system equilibria. We show $\bar{G}$ is strictly monotone on its domain and, hence, a P-function on $\mathcal{S}$ by Remark TR.4. The uniqueness and stability properties would follow from Propositions TR.1 and TR.4, respectively.

For alternative multiplicative boosting, the expected expenditure of buyer i is

$$\begin{aligned} G_i^{\text{MB2}}(\delta_i, \delta_{-i}) &= \delta_i G_i^{\text{MB}}(\delta_i, \delta_{-i}) \\ &= \delta_i c \bar{F}_i(\delta_i c) H_i^{\text{MB2}}(c) + \int_c^\infty \delta_i z \bar{F}_i(\delta_i z)\, \mathrm{d}H_i^{\text{MB2}}(z)\,, \end{aligned}$$

where $H_i^{\text{MB2}}(z) = \mathbb{P}\{\max_{j \neq i} x_j \leq z\} = \prod_{j \neq i} F_j(z)$. Then, the partial derivatives can be computed as follows, for all $i \in [n]$ and $j \neq i$:

$$\frac{\partial G_i^{\text{MB2}}}{\partial \delta_i} = c\bar{F}_i(\delta_i c)\left(1 - \frac{\delta_i c f_i(\delta_i c)}{\bar{F}_i(\delta_i c)}\right) H_i^{\text{MB2}}(c) + \int_c^\infty z\bar{F}_i(\delta_i z)\left(1 - \frac{\delta_i z f_i(\delta_i z)}{\bar{F}_i(\delta_i z)}\right) dH_i^{\text{MB2}}(z)\,,$$

and $\frac{\partial G_i^{\text{MB2}}}{\partial \delta_j} = 0$. Note the residual Jacobian $\bar{J}$ is diagonal. Furthermore, note $\frac{\partial G_i^{\text{MB2}}}{\partial \delta_i} \leq 0$ if $\delta_i \in \mathcal{S}_i$, with a strict inequality if $\delta_i \in \text{int}(\mathcal{S}_i)$ (i.e., the interior). Consequently, a diagonal entry of $\bar{J}$ is at most 0 and is strictly below 0 if the corresponding buyer's parameter is in the interior of his parameter space. By Part 4 in Proposition TR.3, $\bar{G}$ is strictly monotone on its domain. □

B.4 Throttling

Proof of Theorem TR.4. Fix any independent problem instance with $n = 2$ buyers. Assume $T = 1$ for ease of exposition. By Theorem 3.2 of the main paper, throttling admits a system equilibrium. For the uniqueness property, we show that $\bar{J} = -J$ is a P-matrix for all $s \in \mathcal{S}$. Since $n = 2$, we only need to look at two diagonal entries and the whole 2×2 matrix and apply Proposition TR.1.

For $n = 2$, the expected expenditure of buyers $i \in \{1, 2\}$ is

$$G_i^{\text{TO}}(\theta) = \bar{\theta}_i(1 - \bar{\theta}_j)G_i(1) + \bar{\theta}_i\bar{\theta}_j G_i(2)\,,$$

where $G_i(1)$ is the expected expenditure of buyer i when he is the only buyer and $G_i(2)$ is that of buyer i when both buyers participate in the auction. Note $G_i(1) = c\bar{F}_i(c)$ and $G_i(2) = \int_c^\infty (z f_i(z) - \bar{F}_i(z))$ $F_j(z)dz$. In what follows, the notation is that i is one of the buyers and j is the other buyer; for example, $i = 1$ and $j = 2$. We calculate the partial derivatives and obtain, for $i \in \{1, 2\}$ and $j \neq i$:

$$\begin{aligned} \frac{\partial G_i^{\text{TO}}}{\partial \theta_i} &= -\left((1 - \bar{\theta}_j)G_i(1) + \bar{\theta}_j G_i(2)\right) \qquad \text{and} \\ \frac{\partial G_i^{\text{TO}}}{\partial \theta_j} &= -\bar{\theta}_i\left(G_i(2) - G_i(1)\right)\,. \end{aligned}$$

Clearly, the diagonal entries of $-J$ are strictly positive. The determinant of $-J$ is nonnegative:

$$\begin{aligned}\det(J) &= \frac{\partial G_1^{\mathrm{TO}}}{\partial \theta_1}\frac{\partial G_2^{\mathrm{TO}}}{\partial \theta_2} - \frac{\partial G_1^{\mathrm{TO}}}{\partial \theta_2}\frac{\partial G_2^{\mathrm{TO}}}{\partial \theta_1} \\ &= \left((1-\bar{\theta}_2)G_1(1) + \bar{\theta}_2 G_1(2)\right)\left((1-\bar{\theta}_1)G_2(1) + \bar{\theta}_1 G_2(2)\right) \\ &\quad - \bar{\theta}_2\left(G_2(2) - G_2(1)\right)\bar{\theta}_1\left(G_1(2) - G_1(1)\right) \\ &= G_1(1)G_2(1)\left(1 - \bar{\theta}_1 - \bar{\theta}_2\right) + G_1(1)G_2(2)\bar{\theta}_1 + G_1(2)G_2(1)\bar{\theta}_2 \\ &= G_1(1)G_2(1)\left(1 - \bar{\theta}_1 - \bar{\theta}_2 + \frac{G_2(2)}{G_2(1)}\bar{\theta}_1 + \frac{G_1(2)}{G_1(1)}\bar{\theta}_2\right) \\ &\geq G_1(1)G_2(1)\left(1 - \bar{\theta}_1 - \bar{\theta}_2 + \frac{G_2(2)\bar{\theta}_1 + G_1(2)\bar{\theta}_2}{G_1(2) + G_2(2)}\right) \\ &= \frac{G_1(1)G_2(1)}{G_1(2) + G_2(2)}\left(G_1(2)(1-\bar{\theta}_1) + G_2(2)(1-\bar{\theta}_2)\right) \\ &\geq 0\,,\end{aligned}$$

where the first inequality follows from the fact that the total expenditure of two buyers is always greater than the expenditure of one buyer in isolation. Then, the determinant is strictly positive in the truncated parameter space $\mathcal{S}$ because the last inequality in the above becomes strict.

For the stability property, we use Proposition TR.7. Since the diagonal entries of J are strictly negative, $\mathrm{Tr}(J) < 0$, and the above analysis shows $\det(J) > 0$. It follows that the unique system equilibrium is asymptotically stable. □

Proof of Theorem TR.5. Fix any symmetric independent problem instance. Assume $T = 1$ for ease of exposition. By Proposition 4.1 of the main paper, there exists a unique symmetric system equilibrium $\theta_1^* \vec{\mathbf{1}}$. Note $\theta_1^* < 1$ because the buyers' budgets B_1 is strictly positive. It suffices to show that all the eigenvalues of the Jacobian matrix J are negative when J is evaluated at the unique symmetric system equilibrium in the truncated parameter space $\mathcal{S}$. Note J is continuous and symmetric at the symmetric system equilibrium. By Proposition TR.6, the stability property would follow. We use the notation $\bar{y} := 1 - y$ for $y \in [0,1]$.

Now, we show that the eigenvalues of J are negative as follows. For any $\theta \in \mathcal{S}$, we note

$$G_i^{\mathrm{TO}}(\theta_i, \theta_{-i}) = c\bar{\theta}_i \bar{F}_i(c) H_i^{\mathrm{TO}}(c, \theta_{-i}) + \int_c^\infty z\bar{\theta}_i \bar{F}_i(z)\, dH_i^{\mathrm{TO}}(z; \theta_{-i})\,,$$

where $H_i^{\mathrm{TO}}(z;\theta_{-i}) = \prod_{k\neq i}(1 - \bar{\theta}_k \bar{F}_k(z))$. Equivalently, we can write

$$G_i^{\mathrm{TO}}(\theta_i, \theta_{-i}) = \bar{\theta}_i \int_c^\infty (z f_i(z) - \bar{F}_i(z)) H_i^{\mathrm{TO}}(z;\theta_{-i})\, dz\,.$$

For $\theta_1 \in [0,1]$, let $G_1^{\mathrm{TO}}(\theta_1 \vec{\mathbf{1}}; n)$ be the expected expenditure of one buyer in the symmetric independent setting with n buyers under throttling with symmetric system parameters $\theta_1 \vec{\mathbf{1}}$. Then, $nG_1^{\mathrm{TO}}(\theta_1 \vec{\mathbf{1}}; n)$ is the total expected expenditure by all n buyers; it strictly increases as n increases when $\theta_1 < 1$ is fixed. We compute the diagonal entries of J and evaluate them at the symmetric system equilibrium $\theta_1^* \vec{\mathbf{1}}$ to obtain, for each i:

$$\frac{\partial G_i^{\mathrm{TO}}}{\partial \theta_i}(\theta_1^* \vec{\mathbf{1}}) = -\int_c^\infty (z f_1(z) - \bar{F}_1(z))(1 - \bar{\theta}_1^* \bar{F}_1(z))^{n-1} dz = -\frac{1}{\bar{\theta}_1^*} G_1^{\mathrm{TO}}(\theta_1^* \vec{\mathbf{1}}; n)\,.$$

For off-diagonal entries (i.e., $j \neq i$), we obtain

$$\begin{aligned}\frac{\partial G_i^{\text{TO}}}{\partial \theta_j}(\theta_1^* \vec{\mathbf{1}}) &= \bar{\theta}_1^* \int_c^\infty (z f_1(z) - \bar{F}_1(z)) \bar{F}_1(z)(1 - \bar{\theta}_1^* \bar{F}_1(z))^{n-2} dz \\ &= \int_c^\infty (z f_1(z) - \bar{F}_1(z))(1 - (1 - \bar{\theta}_1^* \bar{F}_1(z))(1 - \bar{\theta}_1^* \bar{F}_1(z))^{n-2} dz \\ &= \frac{1}{\bar{\theta}_1^*}(G_1^{\text{TO}}(\theta_1^* \vec{\mathbf{1}}; n-1) - G_1^{\text{TO}}(\theta_1^* \vec{\mathbf{1}}; n)) \,.\end{aligned}$$

Note J is symmetric and is equal to $J = -(\alpha - \beta) I - \beta \vec{\mathbf{1}} \vec{\mathbf{1}}^{\mathsf{T}}$ where $\alpha = \frac{1}{\bar{\theta}_1^*} G_1^{\text{TO}}(\theta_1^* \vec{\mathbf{1}}; n)$ and $\beta = \frac{1}{\bar{\theta}_1^*}(G_1^{\text{TO}}(\theta_1^* \vec{\mathbf{1}}; n) - G_1^{\text{TO}}(\theta_1^* \vec{\mathbf{1}}; n-1))$. Then, the Jacobian matrix J has eigenvalues $\lambda_1 = -\alpha - (n-1)\beta = -\frac{1}{\bar{\theta}_1^*}(n G_1^{\text{TO}}(\theta_1^* \vec{\mathbf{1}}; n) - (n-1) G_1^{\text{TO}}(\theta_1^* \vec{\mathbf{1}}; n-1))$ with multiplicity 1 and $\lambda_2 = -\alpha + \beta = -\frac{1}{\bar{\theta}_1^*} G_1^{\text{TO}}(\theta_1^* \vec{\mathbf{1}}; n-1)$ with multiplicity $n-1$, both of which are negative. □

B.5 Bid Shading

Proofs for Theorem TR.6. Fix any independent problem instance with $n = 2$ buyers and with the buyers' distributions and opportunity cost c as described in the theorem statement (so, any of the 4 cases). Assume $T = 1$ for ease of exposition. By Theorem 3.2 of the main paper, bid shading admits a system equilibrium. We show that Jacobian matrix J has negative diagonal entries and a positive determinant. Then, residual Jacobian $\bar{J}$ is a P-matrix and, by Propositions TR.2 and TR.1, bid shading admits a unique system equilibrium. Furthermore, by Proposition TR.7, the unique system equilibrium is asymptotically stable if it is regular.

In what follows, the notation is that i is one of the buyers and j is the other buyer; for example, $i = 1$ and $j = 2$. Note the expected expenditure of buyer i under bid shading is given by

$$G_i^{\text{S}}(\mu_1, \mu_2) = c \bar{F}_i((1+\mu_i)c) F_j((1+\mu_j)c) + \int_c^\infty z \bar{F}_i((1+\mu_i)z)(1+\mu_j) f_j((1+\mu_j)z) dz \,.$$

Correspondingly, the Jacobian matrix is given by

$$J = \begin{bmatrix} -\alpha_1 - \gamma_{12} & -\omega_{12} + \gamma_{12} \\ -\omega_{21} + \gamma_{21} & -\alpha_2 - \gamma_{21} \end{bmatrix} ,$$

where, for $i \in \{1, 2\}$ and $j \neq i$,

$$\begin{aligned}\alpha_i &= c^2 (1+\mu_i) f_i((1+\mu_i)c) F_j((1+\mu_j)c) \,; \\ \gamma_{ij} &= \int_c^\infty (1+\mu_i) z^2 f_i((1+\mu_i)z)(1+\mu_j) f_j((1+\mu_j)z) dz \,; \text{ and} \\ \omega_{ij} &= \int_c^\infty \bar{F}_i((1+\mu_i)z)(1+\mu_j) z f_j((1+\mu_j)z) dz \,.\end{aligned}$$

Clearly, the diagonal entries are negative. For the determinant, we find that

$$\begin{aligned}\det J &= \gamma_{12}\omega_{21} + \gamma_{21}\omega_{12} - \omega_{12}\omega_{21} + \alpha_1\alpha_2 + \alpha_1\gamma_{21} + \alpha_2\gamma_{12} \\ &\geq \gamma_{12}\omega_{21} + \gamma_{21}\omega_{12} - \omega_{12}\omega_{21} \,,\end{aligned}$$

since the last three terms are nonnegative. Thus, it remains to show that the last expression is positive. The last expression will be denoted $D := \gamma_{12}\omega_{21} + \gamma_{21}\omega_{12} - \omega_{12}\omega_{21}$. In each of the 4 cases described in the theorem statement, we will show that $D > 0$ for $c = 0$. Since D is continuous in c,

$D > 0$ for sufficiently small $c > 0$. When the buyers' distributions are uniform distributions, we will show $D > 0$ for any $c > 0$.

(Exponential Distributions) Let the buyers' distributions be $F_1(x_1) = 1 - e^{-\sigma_1 x_1}$ and $F_2(x_2) = 1 - e^{-\sigma_2 x_2}$ for $x_1 \geq 0$ and $x_2 \geq 0$, respectively. Evaluating D for the given distributions and $c = 0$, we obtain

$$D = \frac{(1+\mu_1)(1+\mu_2)\sigma_1^3\sigma_2^3}{(\sigma_1\mu_2 + \sigma_2\mu_1 + \sigma_1 + \sigma_2)^4} > 0\,.$$

(Rayleigh Distributions) Assume the buyers' distributions are Rayleigh distributions given by $F_1(x_1) = 1 - e^{-x_1^2/(2\sigma_1^2)}$ and $F_2(x_2) = 1 - e^{-x_2^2/(2\sigma_2^2)}$ for $x_1 \geq 0$ and $x_2 \geq 0$, respectively. For $c = 0$,

$$D = \frac{\pi\sigma_1^4\sigma_2^4(1+\mu_1)^2(1+\mu_2)^2}{(\sigma_1^2\mu_2^2 + \sigma_2^2\mu_1^2 + 2\sigma_1^2\mu_2 + 2\sigma_2^2\mu_1 + \sigma_1^2 + \sigma_2^2)^3} > 0\,.$$

(Weibull Distributions) The buyers' distributions are given by $F_1(x_1) = 1 - e^{-(x_1/\sigma_1)^k}$ and $F_2(x_2) = 1 - e^{-(x_2/\sigma_2)^k}$ for $x_1 \geq 0$ and $x_2 \geq 0$, respectively. Note both distributions have the same shape parameter $k > 0$. We let $c = 0$ and find $D = \frac{\text{NUM}}{\text{DEN}}$ where

$$\begin{aligned}
\text{NUM} &= \Gamma\left(\frac{1+k}{k}\right)^2 k\sigma_1^{2+k}\sigma_2^{2+k}(1+\mu_1)^k(1+\mu_2)^k\left((1+\mu_1)^k\sigma_2^k + (1+\mu_2)^k\sigma_1^k\right)^{2-2/k} \\
\text{DEN} &= \left(\sigma_1^k(1+\mu_2)^k + \sigma_2^k(1+\mu_1)^k\right)^4\,.
\end{aligned}$$

Clearly, $D > 0$.

(Uniform Distributions) Let the buyers' distributions be uniform distributions, $U[0, \bar{v}_1]$ and $U[0, \bar{v}_2]$, respectively. We assume $c < \frac{\bar{v}_i}{1+\mu_i}$ for both buyers. If $c \geq \frac{\bar{v}_i}{1+\mu_i}$, then buyer i does not get allocated at all and his expenditure would be 0, reducing the problem instance to the one-buyer case.

Without loss of generality, we assume $\frac{\bar{v}_1}{1+\mu_1} \leq \frac{\bar{v}_2}{1+\mu_2}$. Then, we evaluate D and substitute $\sigma_i = \frac{\bar{v}_1}{c(1+\mu_i)}$ for each buyer i:

$$D = \frac{c^2(\sigma_1 - 1)^2(\sigma_1^3\sigma_2 + 2\sigma_1^2\sigma_2 - 4\sigma_1^2 + 5\sigma_1\sigma_2 - 4\sigma_1 + 4\sigma_2 - 4)}{12\sigma_1^2\sigma_2^2}\,.$$

Note $1 < \sigma_1 \leq \sigma_2$. Since the numerator is increasing in σ_2, we can set $\sigma_1 = \sigma_2$ in the numerator:

$$D \geq \frac{c(\sigma_1 - 1)^2(\sigma_1^4 + 2\sigma_1^3 + \sigma_1^2 - 4)}{12\sigma_1^2\sigma_2^2}\,.$$

The last expression is greater than 0 because $\sigma_1 > 1$. Hence, $D > 0$ for any $c > 0$. □

Proof of Theorem TR.7. Fix any symmetric independent problem instance with $n \geq 2$ buyers. Assume $T = 1$ for ease of exposition. By Proposition 4.1 of the main paper, there exists a unique symmetric system equilibrium $\mu_1^*\vec{\mathbf{1}}$. We show that all the eigenvalues of the Jacobian matrix J are negative when J is evaluated at the unique symmetric system equilibrium. Since it is symmetric, J would be negative definite. By Proposition TR.6, the stability property would follow.

Note, for $\mu \in \mathcal{S}$, the expected expenditure of buyer i is

$$G_i^{\text{S}}(\mu_i, \mu_{-i}) = -\int_c^\infty \left(\bar{F}_i((1+\mu_i)z) - (1+\mu_i)zf_i((1+\mu_i)z)\right) H_i^{\text{S}}(z;\mu_{-i})dz\,,$$

where $H_i^{\mathrm{S}}(z;\mu_{-i}) = \mathbb{P}\{\max_{j\neq i} x_j/(1+\mu_j) \leq z\} = \prod_{j\neq i} F_j((1+\mu_j)z)$. We compute the diagonal and off-diagonal entries of the Jacobian matrix J and obtain, for $i \in [n]$ and $j \neq i$:

$$\frac{\partial G_i^{\mathrm{S}}}{\partial \mu_i} = -c^2 f_i((1+\mu_i)c)H_i^{\mathrm{S}}(c;\mu_{-i}) - \sum_{j\neq i} \frac{1}{1+\mu_i}\gamma_{ij} \quad \text{and}$$
$$\frac{\partial G_i^{\mathrm{S}}}{\partial \mu_j} = -\frac{\omega_{ij}}{1+\mu_j} + \frac{\gamma_{ij}}{1+\mu_j}\,,$$

where $H_{ij}^{\mathrm{S}}(z;\mu_{-ij}) = \prod_{k\neq i,j} F_k((1+\mu_k)z)$ and

$$\omega_{ij} = \int_c^\infty \bar{F}_i((1+\mu_i)z)(1+\mu_j)zf_j((1+\mu_j)z)H_{ij}^{\mathrm{S}}(z;\mu_{-ij})dz\,; \text{ and}$$
$$\gamma_{ij} = \int_c^\infty (1+\mu_i)zf_i((1+\mu_i)z)(1+\mu_j)zf_j((1+\mu_j)z)H_{ij}^{\mathrm{S}}(z;\mu_{-ij})dz\,.$$

In the symmetric independent setting and at the unique symmetric system equilibrium $\mu_1^*\vec{\mathbf{1}}$, the Jacobian matrix is symmetric and is given by, for $i \in [n]$ and $j \neq i$:

$$\frac{\partial G_i^{\mathrm{S}}}{\partial \mu_i}(\mu_1^*\vec{\mathbf{1}}) = -c^2 f_1((1+\mu_1^*)c)F_1^{n-1}((1+\mu_1^*)c) - \frac{(n-1)\gamma}{1+\mu_1^*}$$
$$\frac{\partial G_i^{\mathrm{S}}}{\partial \mu_j}(\mu_1^*\vec{\mathbf{1}}) = \frac{-\omega}{1+\mu_1^*} + \frac{\gamma}{1+\mu_1^*}\,,$$

where $\omega = \int_c^\infty \bar{F}_1((1+\mu_1^*)z)(1+\mu_1^*)zf_1((1+\mu_1^*)z)F_1^{n-2}((1+\mu_1^*)z)dz$ and $\gamma = \int_c^\infty \left[(1+\mu_1^*)zf_1((1+\mu_1^*)z)\right]^2 F_1^{n-2}((1+\mu_1^*)z)dz$. Then,

$$J = -(\alpha-\beta)I - \beta\vec{\mathbf{1}}\vec{\mathbf{1}}^{\intercal}\,,$$

where $\alpha = c^2 f_1((1+\mu_1^*)c)F_1^{n-1}((1+\mu_1^*)c) + \frac{(n-1)\gamma}{1+\mu_1^*}$ and $\beta = \frac{\omega}{1+\mu_1^*} - \frac{\gamma}{1+\mu_1^*}$. The Jacobian matrix has eigenvalues $\lambda_1 = -\alpha-(n-1)\beta = -c^2 f_1((1+\mu_1^*)c)F_1^{n-1}((1+\mu_1^*)c) - \frac{(n-1)\omega}{1+\mu_1^*}$ with multiplicity 1 and $\lambda_2 = -\alpha+\beta = -c^2 f_1((1+\mu_1^*)c)F_1^{n-1}((1+\mu_1^*)c) + \frac{\omega}{1+\mu_1^*} - \frac{n\gamma}{1+\mu_1^*}$ with multiplicity $n-1$. Clearly, $\lambda_1 < 0$.

We show $\lambda_2 < 0$. We perform a change of variables in the integrals by letting $y = (1+\mu_1^*)z$ and substituting $r = (1+\mu_1^*)c$ and obtain:

$$\lambda_2 = -\frac{1}{(1+\mu_1^*)^2}\left(r^2 f_1(r)F_1^{n-1}(r) + n\int_r^\infty y^2 f_1^2(y)F_1^{n-2}(y)dy - \int_r^\infty \bar{F}_1(y)yf_1(y)F_1^{n-2}(y)dy\right).$$

Let $A = \int_r^\infty y^2 f_1^2(y)F_1^{n-2}(y)dy$ and $B = \int_r^\infty \bar{F}_1(y)yf_1(y)F_1^{n-2}(y)dy$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} B &= \int_r^\infty \bar{F}_1(y)yf_1(y)F_1^{n-2}(y)dy \\ &\leq \sqrt{\int_r^\infty y^2 f_1^2(y)F_1^{n-2}(y)dy \cdot \int_r^\infty \bar{F}_1^2(y)F_1^{n-2}(y)dy} \\ &= \sqrt{A\cdot\int_r^\infty \bar{F}_1^2(y)F_1^{n-2}(y)dy}\,. \end{aligned}$$

Furthermore,

$$
\begin{aligned}
\int_r^\infty \bar{F}_1^2(y)F_1^{n-2}(y)dy &= \bar{F}_1^2(y)F_1^{n-2}(y)y\Big|_r^\infty - \int_r^\infty \frac{d}{dy}\left[\bar{F}_1^2(y)F_1^{n-2}(y)\right] ydy \\
&= -\bar{F}_1^2(r)F_1^{n-2}(r)r + 2\int_r^\infty yf_1(y)\bar{F}_1(y)F_1^{n-2}(y)dy - \int_r^\infty y\bar{F}_1^2(y)(n-2)F_1^{n-3}(y)f_1(y)dy \\
&\le 2\int_r^\infty yf_1(y)\bar{F}_1(y)F_1^{n-2}(y)dy \\
&= 2B\,,
\end{aligned}
$$

where the first equality follows from integration by parts and the second one follows from $\lim_{y\to\infty}\bar{F}_1^2(y)y = 0$ which is a consequence of Proposition A.1 in Appendix **??** of the main paper.

Putting the above inequalities together, we obtain $B \le \sqrt{2AB}$ or, equivalently, $B \le 2A$. Consequently,

$$
\lambda_2 = -\frac{1}{(1+\mu_1^*)^2}\left(r^2 f_1(r)F_1^{n-1}(r) + nA - B\right) \le -\frac{1}{(1+\mu_1^*)^2}\left(r^2 f_1(r)F_1^{n-1}(r) + nA - 2A\right) < 0\,.\square
$$

B.6 Thresholding

Proof of Theorem TR.8. Fix any symmetric independent problem instance with $n \ge 2$ buyers. Assume $T = 1$ for ease of exposition. By Proposition 4.1 of the main paper, there exists a unique symmetric system equilibrium $\tau_1^*\vec{\mathbf{1}}$. Note that, for $\tau \in \mathcal{S}$,

$$
G_i^{\mathrm{T}}(\tau_i,\tau_{-i}) = c\bar{F}_i(\tau_i)H_i^{\mathrm{T}}(\min_j \tau_j;\tau_{-i}) + \int_{\min_j \tau_j} z\bar{F}_i(z\vee\tau_i)dH_i^{\mathrm{T}}(z;\tau_{-i})\,,
$$

where $H_i^{\mathrm{T}}(z;\tau_{-i}) = \prod_{j\ne i}F_j(z\vee\tau_j)$.

In the symmetric independent setting, the Jacobian matrix J at the symmetric system equilibrium $\tau_1^*\vec{\mathbf{1}}$ has the following diagonal and off-diagonal entries, for all $i\in[n]$ and $j\ne i$:

$$
\begin{aligned}
&\frac{\partial G_i^{\mathrm{T}}}{\partial \tau_i}(\tau_1^*\vec{\mathbf{1}}) = -cf_1(\tau_1^*)F_1^{n-1}(\tau_1^*) \quad \text{and} \\
&\frac{\partial G_i^{\mathrm{T}}}{\partial \tau_j}(\tau_1^*\vec{\mathbf{1}}) = -(\tau_1^*-c)\bar{F}_1(\tau_1^*)f_1(\tau_1^*)F_1^{n-2}(\tau_1^*)\,.
\end{aligned}
$$

Then, $J = -(\alpha-\beta)I - \beta\vec{\mathbf{1}}\vec{\mathbf{1}}^\intercal$ and J has eigenvalues $\lambda_1 = -\alpha-(n-1)\beta$ with multiplicity 1 and $\lambda_2 = -\alpha+\beta$ with multiplicity $n-1$, where $\alpha = cf_1(\tau_1^*)F_1^{n-1}(\tau_1^*)$ and $\beta = (\tau_1^*-c)\bar{F}_1(\tau_1^*)f_1(\tau_1^*)F_1^{n-2}(\tau_1^*)$.

Clearly, $\lambda_1 < 0$. Note λ_2 can be both positive and negative depending on the budget level B_1 which, in turn, affects the symmetric system equilibrium $\tau_1^*\vec{\mathbf{1}}$. Since $\lambda_2 = -(\alpha-\beta) = -f_1(\tau_1^*)F_1^{n-2}(\tau_1^*)(c - \tau_1^*\bar{F}_1(\tau_1^*))$, we have $\lambda_2 < 0$ if $\tau_1^*\bar{F}_1(\tau_1^*) < c$ and $\lambda_2 > 0$ if $\tau_1^*\bar{F}_1(\tau_1^*) > c$. Both cases are possible as we can achieve a different symmetric system equilibrium $\tau_1^*\vec{\mathbf{1}}$ for a different budget B_1. When both eigenvalues are negative, J is negative definite and, consequently, Proposition TR.6 implies that the unique symmetric system equilibrium is a strictly monotone attractor. On the other hand, when there is a positive eigenvalue, the unique symmetric system equilibrium is unstable even in the unprojected dynamical system without the projection map $\Pi_{\mathcal{S}}$ and, hence, in the projected dynamical system (TR-3). $\square$