E-Companion to “Multi-stage Intermediation in Display Advertising”

Santiago R. Balseiro
Graduate School of Business, Columbia University, srb2155@columbia.edu,

Ozan Candogan
Booth School of Business, University of Chicago, ozan.candogan@chicagobooth.edu,

Huseyin Gurkan
The Fuqua School of Business, Duke University, hg67@duke.edu,

Contents

A. Model and Equilibrium Concept

First, we formally define the set of feasible mechanisms $\mathcal{M}$ for intermediaries, along with the preliminary properties of this set that are discussed in Section 2. Using these properties, we show that truthful bidding is an optimal strategy for buyers $I_b \in \mathcal{B}$, and thus we focus on the game between intermediaries and the seller. Then, we formally define this game and provide the equilibrium concept.

We use symmetry of the **k**-trees to reduce the mechanism of an intermediary to a single pair of reserve price and reporting function per tier. Therefore, all reserves and reporting functions are equal for intermediaries in the same tier. We start with a formal definition for the set of feasible mechanisms $\mathcal{M}$ as follows:

Definition A.1. The set of feasible mechanisms for intermediaries is denoted by $\mathcal{M}$. A mechanism $(r, Y) \in \mathcal{M}$ consists of a nondecreasing bidding function $Y : \mathbb{R}_+ \to \mathbb{R}_+$ and a nonnegative reserve price $r \in \mathbb{R}_+$. Let intermediary $I_\ell \in \mathcal{I}$ select a mechanism $(r_\ell, Y_\ell) \in \mathcal{M}$. When I_ℓ receives a set of downstream reports $\{w_c\}_{c:I_c \in \mathcal{C}(I_\ell)}$, she submits $Y_\ell\left(\max_{c:I_c\in\mathcal{C}(I_\ell)} w_c\right)$, whenever $\max_{c:I_c\in\mathcal{C}(I_\ell)} w_c \geq r_\ell$, otherwise she reports 0. In case of winning the impression, I_ℓ allocates it to her downstream agent $I_c \in \mathcal{C}(I_\ell)$ (with the maximum report w_c), and charges the minimum amount that guarantees winning which is given by $\inf \Theta_c$. Here, the set Θ_c is given by

$$\Theta_c \triangleq \left\{ w \geq 0 : w \geq r_\ell, w \geq \max_{c':I_{c'}\in\mathcal{C}(I_\ell)\setminus\{I_c\}} w_{c'}, Y_\ell(w) \in \Theta_\ell \right\},$$

where $I_u \in \mathcal{U}(I_\ell)$ is the upstream agent of I_ℓ and Θ_ℓ is the set of winning reports of agent I_ℓ in the mechanism of I_u. If an intermediary I_c bids at the seller's auction, this set is similarly given by

$$\Theta_c \triangleq \left\{ w \geq 0 : w \geq r_S, w \geq \max_{c':I_{c'}\in\mathcal{C}(I_S)\setminus\{I_c\}} w_{c'} \right\},$$

where r_S is the reserve price of the seller.

A.1. Preliminary Results

In this section, we characterize the payment scheme under $\mathcal{M}$ and use this result in the proof of Lemma 1 in our online appendix where we show that it is weakly dominant for buyers to report values truthfully.

We first define the *random bids* of intermediaries and buyers $\{\bar{W}_\ell : I_\ell \in \mathcal{I} \cup \mathcal{B}\}$ induced by mechanisms $\{(r_\ell, Y_\ell) \in \mathcal{M} : I_\ell \in \mathcal{I}\}$ and buyers' reporting functions $\{(R_\ell : \mathcal{V} \to \mathbb{R}_+) : I_\ell \in \mathcal{B}\}$ as follows:

$$\bar{W}_\ell \triangleq \begin{cases} Y_\ell(\bar{W}_c) 1\{\bar{W}_c \geq r_\ell\} & \text{if } I_\ell \in \mathcal{I} \text{ where } c = \arg\max_{c':I_{c'}\in\mathcal{C}(I_\ell)} \bar{W}_{c'}, \\ R_\ell(V_\ell) & \text{if } I_\ell \in \mathcal{B}, \end{cases} \tag{1}$$

where V_ℓ is the value of the buyer $I_\ell \in \mathcal{B}$.

The random bids $\{\bar{W}_\ell : I_\ell \in \mathcal{I} \cup \mathcal{B}\}$ represent the reports of intermediaries and buyers to their upstream mechanisms when they consistently report according to the mechanisms $\{(r_\ell, Y_\ell) \in \mathcal{M} : I_\ell \in \mathcal{I}\}$ and reporting functions $\{(R_\ell : \mathcal{V} \to \mathbb{R}_+) : I_\ell \in \mathcal{B}\}$. We show that $\bar{W}_\ell$ can equivalently be taken as the random value of buyer $I_\ell \in \mathcal{B}$ because it is optimal for buyers to report their values truthfully when intermediaries implement mechanisms in $\mathcal{M}$ for all $I_\ell \in \mathcal{I}$ and the mechanism of the seller is a second-price auction (see part (B) of Lemma 1).

LEMMA A.1. *Assume that the seller runs a second-price auction with reserve price r_S, intermediaries in $\mathcal{I}$ implement mechanisms $\{(r_\ell, Y_\ell) \in \mathcal{M} : I_\ell \in \mathcal{I}\}$ and buyers use the reporting functions $\{(R_\ell : \mathcal{V} \to \mathbb{R}_+) : I_\ell \in \mathcal{B}\}$. Let $\{\bar{w}_\ell : I_\ell \in \mathcal{I} \cup \mathcal{B}\}$ denote the realizations of the random bids. In case of allocating the impression, the seller charges her downstream agent with the largest report the maximum of her reserve price and the second largest report she received. In case of winning, intermediary I_ℓ charges her downstream agent $I_c \in \mathcal{C}(I_\ell)$ with the largest report, the amount*

$$P_c = \max\left(r_\ell, Y_\ell^{-1}(P_\ell), \max_{c': I_{c'} \in \mathcal{C}(I_\ell) \setminus \{I_c\}} \bar{w}_{c'}\right),$$

where P_ℓ is the payment of I_ℓ to her upstream agent $\mathcal{U}(I_\ell)$. Additionally, the set of winning reports of agent I_c is $\Theta_c = \{w \geq 0 : w \geq P_c\}$.

Proof. We prove this lemma by induction.

Base case. Since the seller runs a second-price auction with reserve price r_S, the set Θ_ℓ for intermediaries $I_\ell \in \mathcal{C}(I_S)$ connected to the seller is given by:

$$\Theta_\ell = \left\{w \geq 0 : w \geq r_S, w \geq \max_{c': I_{c'} \in \mathcal{C}(I_S) \setminus \{I_\ell\}} \bar{w}_{c'}\right\}.$$

Thus, we can write the payment P_ℓ in case of winning as

$$P_\ell \triangleq \inf \Theta_\ell = \max\left(r_S, \max_{c': I_{c'} \in \mathcal{C}(I_S) \setminus \{I_\ell\}} \bar{w}_{c'}\right).$$

Now, consider the set of winning reports for an intermediary $I_c \in \mathcal{C}(I_\ell)$ that is positioned in tier $n-1$. This is given by:

$$\Theta_c = \left\{w \geq 0 : w \geq r_\ell, w \geq \max_{c': I_{c'} \in \mathcal{C}(I_\ell) \setminus \{I_c\}} \bar{w}_{c'}, Y_\ell(w) \in \Theta_\ell\right\}.$$

First, notice that $Y_\ell(w) \in \Theta_\ell$ if and only if $Y_\ell(w) \geq P_\ell$. Thus, by using the monotonicity of the reporting function Y_ℓ, the payment of agent I_c is given by:

$$P_c \triangleq \inf \Theta_c = \max\left(r_\ell, \max_{c': I_{c'} \in \mathcal{C}(I_\ell) \setminus \{I_c\}} \bar{w}_{c'}, Y_\ell^{-1}(P_\ell)\right).$$

Inductive step. Assume that, in case of winning, the payment of intermediary I_ℓ (is positioned in tier t) to her upstream agent $I_u = \mathcal{U}(I_\ell)$ (positioned in tier $t+1$) is given by P_ℓ. By definition of $\mathcal{M}$, we know that P_ℓ is the minimum amount which guarantees winning for intermediary I_ℓ. For intermediary $I_c \in \mathcal{C}(I_\ell)$ to win the impression, her report $\bar{w}_c$ has to be large enough to guarantee that intermediary I_ℓ wins when she submits a bid to the upstream mechanism on behalf of I_c, i.e., $Y_\ell(\bar{w}^c) \geq \inf \Theta_\ell = P_\ell$. Moreover, intermediary I_c should have a large enough report to win in the mechanism of I_ℓ, i.e., $\bar{w}_c \geq \max\left(r_\ell, \max_{c' : I_{c'} \in \mathcal{C}(I_\ell)\setminus\{I_c\}} \bar{w}_{c'}\right)$.

Combining these, the set of reports which guarantees winning for intermediary I_c is given by:

$$\Theta_c = \left\{ w \geq 0 : w \geq \max\left(r_\ell, \max_{c' : I_{c'} \in \mathcal{C}(I_\ell)\setminus\{I_c\}} \bar{w}_{c'}\right), Y_\ell(w) \geq P_\ell \right\}.$$

Therefore, using monotonicity of Y_ℓ, the payment of intermediary I_c is given as follows:

$$P_c = \inf \Theta_c = \left(\max\left(r_\ell, Y_\ell^{-1}(P_\ell), \max_{c' : I_{c'} \in \mathcal{C}(I_\ell)\setminus\{I_c\}} \bar{w}_{c'}\right)\right).$$

Q.E.D.

A.2. Equilibrium Concept

Consistently with the timing of events described in Section 2 when buyers report their values truthfully, this game can be formulated as a Stackelberg game where each intermediary chooses an optimal mechanism after observing the mechanisms of upstream agents and anticipating that the downstream agents will react accordingly. Intermediaries in the same tier choose their mechanisms simultaneously. Thus, to state our equilibrium concept formally, following the approach in Osborne and Rubinstein (1994), we rely on the concept of the player function $\mathcal{P}$ which species the set of players who (simultaneously) move after a given history.

Definition A.2 formally defines the game we study in this section. In order to make the histories and the order of moves clear, we use the notations I_ℓ and $I_{(t,j)}$ interchangeably to refer to an intermediary. The latter notation makes explicit that the intermediary is in tier t.

Definition A.2. The game among intermediaries and the seller is an extensive form game $\Gamma = \langle \mathcal{I} \cup \{I_S\}, \mathcal{H}, \mathcal{S}, \{u_\ell : I_\ell \in \mathcal{I}\} \cup \{u_{n+1}\}, \mathcal{P}\rangle$ where

- The set of players is $\mathcal{I} \cup \{I_S\}$.
- The set of histories is $\mathcal{H} = \cup_{t=1}^n \mathcal{H}_t$ where $\mathcal{H}_t$ is the set of all possible tier t histories and is given by:

$$\mathcal{H}_t = \{H_t : H_t = \{(r_{t'}, Y_{t'}) \in \mathcal{M}, t < t'\} \cup \{r_{n+1}\}\}.$$

Here, history H_t consists of the mechanisms $\{(r_{t'}, Y_{t'}) : t < t'\}$ chosen by upstream intermediaries $\{I_{(t,k)} \in \mathcal{I}, t < t'\}$ and the seller's reserve price r_{n+1}. Following the seller, intermediaries move sequentially from tier n to tier 1 after observing upstream mechanisms.

- The set of pure strategies for intermediary $I_{(t,j)}$ is $\mathcal{S}_{(t,j)} = \{s | s : \mathcal{H}_t \to \mathcal{M}\}$, and for the seller I_S is $\mathcal{S}_{n+1} = \mathcal{V}$. Then, the set of pure strategy profiles is $\mathcal{S} = \prod_{I_{(t,j)} \in \mathcal{I}} \mathcal{S}_{(t,j)} \times \mathcal{S}_{n+1}$.
- The utility functions, $u_\ell : \mathcal{S} \to \mathbb{R}$, for intermediaries $I_\ell \in \mathcal{I}$ in tier t are given by

$$u_\ell(s = \{s_{\ell'} : I_{\ell'} \in \mathcal{I}\} \cup \{s_{n+1}\}) = \sum_{c: I_c \in \mathcal{C}(I_\ell)} \mathbb{E}\left[(P_c - P_\ell) 1\{I_c \text{ wins}\}\right],$$

where P_c and P_ℓ are the payments of agents $I_c \in \mathcal{C}(I_\ell)$ and I_ℓ, respectively, in case of I_c winning the item. The utility function $u_{n+1} : \mathcal{S} \to \mathbb{R}$ for the seller is

$$u_{n+1}(s = \{s_\ell : I_\ell \in \mathcal{I}\} \cup \{s_{n+1}\}) = \sum_{c: I_c \in \mathcal{C}(I_S)} \mathbb{E}\left[P_c 1\{I_c \text{ wins}\}\right],$$

where P_c is the payment of intermediary $I_c \in \mathcal{C}(I_S)$.

- The player function $\mathcal{P} : \mathcal{H} \to 2^{\mathcal{I} \cup \{I_S\}}$ is $\mathcal{P}(H_t) = \{I_{(t,j)} : I_{(t,j)} \in \mathcal{I}\}$ and $\mathcal{P}(\emptyset) = I_S$.

In this game, the histories observed by the agents consist of the set of mechanisms which are implemented by all upstream agents. For instance, H_t is the set of mechanisms implemented by the seller and the intermediaries positioned between the seller and the tier t of the intermediation network. Note that agents in the same tier observe the same history even if they connect to different agents at upstream. Also observe that the only players in this game are intermediaries and the seller, since as indicated by part (B) of Lemma 1, buyers always report their values truthfully when we restrict attention to mechanisms in $\mathcal{M}$ for intermediaries and second-price auctions for the seller. Note that we could alternatively focus on an extensive form game of incomplete information, where the buyers are also players and the type of each buyer is unknown to the remaining players (as in Section 2 of our paper), and analyze the perfect Bayesian equilibrium (PBE) of the strategic interaction among the intermediaries and the buyer. A PBE is a behavioral strategy profile and a system of beliefs such that the strategy profile is sequentially rational given the belief system, and the belief system is consistent with the strategy profile in terms of Bayesian updates. In this case, it can be shown that any subgame perfect equilibrium captured in Definition A.2 together with a truthful bidding strategy for the buyer constitutes a PBE, with the prior distribution of the buyers' type as the supporting belief system. In that game, intermediaries do not update their belief at any stage because observed actions of upstream intermediaries do not reveal any new information about the type of the buyer. Thus the prior distribution would be the supporting beliefs for all intermediaries.

The utility of intermediary I_ℓ is her expected profit, which is the difference between the payment of the winning downstream agent and her payment to the upstream agent. In Definition A.1, it is stated that in case of winning, intermediary I_ℓ allocates the impression to her downstream agent $I_c \in \mathcal{C}(I_\ell)$ with the highest report. In other words, intermediary I_ℓ allocates the impression to one of her downstream agents if and only if she acquires it from her upstream agent. Therefore, the term $1\{I_c \text{ wins}\}$ corresponds to the event that I_ℓ wins the impression from upstream and allocates it to I_c. In the expectation, $P_c 1\{I_c \text{ wins}\}$ represents the downstream payment, which is collected only if the item is allocated to downstream agent I_c. The second term $P_\ell 1\{I_c \text{ wins}\}$ represents the payment of the intermediary to her upstream agent, which is made only if the impression is acquired by intermediary I_ℓ. Since, the seller is the initial owner of the impression, she does not incur a cost, thus implying her utility is given by the payments of her downstream agents. The equilibrium concept for this extensive form game is subgame perfect equilibrium (SPE), which we define next.

Definition A.3. A strategy profile $s^* \in \mathcal{S}$ constitutes an SPE if and only if

- for any intermediary $I_{(t,j)}$ and any history $H_t \in \mathcal{H}_t$, we have

$$u_{(t,j)}\left(s^*_{(t,j)}, s^*_{-(t,j)} | H_t\right) \geq u_{(t,j)}\left(s_{(t,j)}, s^*_{-(t,j)} | H_t\right) \;\; \forall s_{(t,j)} \in \mathcal{S}_{(t,j)}\,,$$

- for the seller I_S, we have

$$u_{n+1}\left(s^*_{n+1}, s^*_{-(n+1)} | \emptyset\right) \geq u_{n+1}\left(s_{n+1}, s^*_{-(n+1)} | \emptyset\right) \;\; \forall s_{n+1} \in \mathcal{S}_{n+1}\,.$$

In this definition $s^*_{-(t,j)}$ denotes the strategies of the agents other than $I_{(t,j)}$, and $u_{(t,j)}\left(s^*_{(t,j)}, s^*_{-(t,j)} | H_t\right)$ again denotes the payoff of the intermediary $I_{(t,j)}$ after observing history $H_t \in \mathcal{H}_t$. For the seller, $s^*_{-(n+1)}$ denotes the strategies of the agents other than I_S, and $u_{n+1}\left(s^*_{n+1}, s^*_{-(n+1)} | \emptyset\right)$ again denotes the payoff of the seller I_S at the beginning of the game.

B. Extension of Lemma 2 to Distributions with an Atom at Zero

Lemma B.1. *Assume that the seller runs a second-price auction with a nonnegative reserve price* $r_S \leq \sup \mathcal{V}$ *and a random competing bid* D *within support* $\mathcal{D}$*. Suppose that buyers' values are drawn independently from the probability distribution of a random variable* V*, which has an atom at zero but absolutely continuous elsewhere, and its strictly positive part has a strictly increasing virtual value function. Consider intermediary* I_ℓ *as shown in Figure 3. An optimal mechanism for this intermediary in* $\mathcal{M}$ *is given by* (r^*, Y^*) *where*

$$Y^*(v) = \psi_V(v)\,,$$

$$r^* = z_V\,.$$

Proof. We consider that the random variable V has an atom at zero and it is absolutely continuous elsewhere. Then we can rewrite problem (2) by taking the expectation over V by separately considering the cases when $V=0$ and $V>0$.

Let M be the number of buyers with strictly positive values. M is distributed as a binomial random variable with m trials and success probability $\mathbb{P}(V>0)$. The intermediary's problem is equivalent to a problem with M random bidders with values $V|V>0$. The result follows because the optimal reserve price in a second-price auction is independent of the number of bidders and $\phi_{V|V>0}(x)=\phi_V(x)$ for $x>0$ (see Theorem 1 in Laffont and Maskin 1980). Q.E.D.

C. Proof of Lemma 3

We prove each part of this lemma separately.

C.1. Part (A)

Proof. In this part of the lemma, we restrict attention to **k**-trees with $k_t=1$ for $t=1,\dots,n+1$, i.e., a chain of intermediaries. Since we have a single agent in each tier, in the remainder of this document, we drop the second index from our notation, $I_{(t,j)}$, for agents and denote by $I_{(t)}$ an agent in tier t of the intermediation chain. We start by defining anticipated reports for intermediation chains (see Definition 3). In a chain of intermediaries, the anticipated report of buyer $I_{(0)}$ is $W_0=V$ and the anticipated report of an intermediary $I_{(t)}$ is reduced to $W_t=\psi_{W_{t-1}}(W_{t-1})$ because an intermediary has one downstream agent.

Suppose the buyer's value V is a GPD with parameters (ξ,σ,μ), where $\xi<1$ and $\mu=0$. We first derive the distribution of random variable $W=\psi_V(V)$.

We start by considering $\mathcal{W}$, the support of W. Since $\psi_V(\cdot)$ is a nondecreasing function (strictly increasing for all v such that $\phi_V^{-1}(0)\le v<\sup\mathcal{V}$ and constant elsewhere), we know that $\inf\mathcal{W}=\psi_V(0)$ and $\sup\mathcal{W}=\psi_V(\sup\mathcal{V})$. Note that $z_V=\phi_V^{-1}(0)=\sigma/(1-\xi)>0$ so $\psi_V(0)=0$ and $\psi_V(\sup\mathcal{V})=\sup\mathcal{V}$ by definition. Thus, the support of W is given by

$$\mathcal{W}=\begin{cases}[0,\sup\mathcal{V}] & \text{if } \sup\mathcal{V}<\infty\,,\\ [0,\infty) & \text{otherwise.}\end{cases}$$

This observation reveals that the supports of W and V are the same. We can write the c.d.f. of W as follows:

$$G_W(w)=\begin{cases}G_V(\phi_V^{-1}(w)) & w\ge 0\,,\\ 0 & \text{otherwise.}\end{cases}$$

Since the virtual value function is always below the 45 degree line and $\mu=0$, the set $\{v\in\mathcal{V}|\phi_V(v)<0\}$ has positive measure, thereby W takes value of zero whenever $\phi_V(V)<0$. Hence, W has an atom at zero. We write the cumulative distribution function of conditional random variable $W|W>0$ as follows

$$G_{W|W>0}(w)=\frac{G_W(w)-G_W(0)}{1-G_W(0)}=\begin{cases}\frac{1-\exp\{-\frac{w+\sigma-\mu}{\sigma}\}-1+\exp\{-\frac{\sigma-\mu}{\sigma}\}}{1-1+\exp\{-\frac{\sigma-\mu}{\sigma}\}} & \text{if } \xi=0\\ \frac{1-(1+\xi(\frac{w+\sigma-\mu}{(1-\xi)\sigma}))^{-1/\xi}-1+(1+\xi(\frac{\sigma-\mu}{(1-\xi)\sigma}))^{-1/\xi}}{1-1+(1+\xi(\frac{\sigma-\mu}{(1-\xi)\sigma}))^{-1/\xi}} & \text{if } \xi\neq 0\end{cases}$$

$$=\begin{cases}1-\frac{\exp\{-\frac{w+\sigma-\mu}{\sigma}\}}{\exp\{-\frac{\sigma-\mu}{\sigma}\}} & \text{if } \xi=0\\ 1-\frac{(1+\xi(\frac{w+\sigma-\mu}{(1-\xi)\sigma}))^{-1/\xi}}{(1+\xi(\frac{\sigma-\mu}{(1-\xi)\sigma}))^{-1/\xi}} & \text{if } \xi\neq 0\end{cases}=\begin{cases}1-\exp\{-\frac{w}{\sigma}\} & \text{if } \xi=0\,,\\ 1-(1+\frac{\xi(w)}{\sigma-\xi\mu})^{-1/\xi} & \text{if } \xi\neq 0.\end{cases}$$

Observe that $W|W>0$ has the same distribution as $V|V>0$ and hence the same virtual value function, which implies

$$\phi_{W|W>0}(w)=(1-\xi)w-\sigma\,.$$

Proceeding similarly, it can be showed that $W_t|W_t>0$ has the same distribution as $V|V>0$ for all $t=1,\dots,n$. This observation implies first $\mathbb{E}[W_t|W_t>0]<\infty$ because $\mathbb{E}[V|V>0]<\infty$. Moreover, the random variable $W_t|W_t>0$ has a continuous and positive density function because GPDs have continuous and positive density function in their domains except 0. Finally, $\phi_{W_t|W_t>0}(w)=(1-\xi)w-\sigma$ is strictly increasing when $\xi<1$. Therefore, Assumption 1 is satisfied. Q.E.D.

C.2. Part (B)

Proof. We provide some preliminary results before proving part (B) of Lemma 3.

LEMMA C.1. *Define the following functions:*

1. $h_1(x,k,\xi)\triangleq[(x-1)^k[x(1+\xi)-\xi+k]-x^k[x(1+\xi)-\xi-k]]$
2. $h_2(x,k,\xi)\triangleq\Big[x^k\Big(k(k+\xi)-x(1+\xi)(k+1)+x^2(1+\xi)\Big)-(1+\xi)(x-1)^{k+1}x\Big]$

The functions $h_1(x,k,\xi)$ and $h_2(x,k,\xi)$ are positive when $x>1$, $\xi<1$ and k is an integer greater or equal than 1.

We defer the proof of this lemma to the end of this proof. We are now in a position to prove part (B) of Lemma 3. We prove this result in three steps whose first step is for $\phi_{W_1|W_1>0}$ and second step is for $\phi_{W_2|W_2>0}$ when $\xi\neq 0$, and in the third step, we prove the same when $\xi=0$.

Step 1 (W_1 for $\xi \neq 0$): In this step, we first show that W_1 has finite mean, and $W_1|W_1>0$ has continuous and strictly positive density function. Recall that $W_1=\psi_V(V^{\text{FH}(k_1)})$. The conditional p.d.f. of $W_1|W_1>0$ is given as follows:

$$g_{W_1|W_1>0}(x)=\frac{k_1\left[1-\left(\frac{\xi x+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}}\right]^{k_1-1}\left(\frac{\xi x+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}-1}\frac{1}{(1-\xi)\sigma}}{1-\left[1-(1-\xi)^{\frac{1}{\xi}}\right]^{k_1}}.$$

This is a positive and continuous function for all $x\in(0,\sup\mathcal{W}_1]$ (in cases where $\sup\mathcal{W}_1=\infty$, e.g., shifted Pareto distribution, we consider $x\in(0,\sup\mathcal{W}_1)$) where $\sup\mathcal{W}_1=\sup\mathcal{V}$ because $\phi_V(\sup\mathcal{V})=\sup\mathcal{V}$. Since the (projected) virtual value function is below the 45 degree line, it follows that $\mathbb{E}[W_1]\le\mathbb{E}[V^{\text{FH}(k_1)}]$. Moreover, we know that $\mathbb{E}[V^{\text{FH}(k_1)}]<\infty$ because $\mathbb{E}[V^{\text{FH}(k_1)}]\le\mathbb{E}[\sum_{i=1}^{k_1}V_i]=k_1\mathbb{E}[V]$ where V_i's are i.i.d. copies of V, and $\mathbb{E}[V]<\infty$.

We next show that $\phi_{W_1|W_1>0}(\cdot)$ is strictly increasing. The function $\phi_{W_1|W_1>0}(x)$ is given by

$$\phi_{W_1|W_1>0}(x)=x+\frac{(1-\xi)\sigma}{k_1}\left(\frac{\xi x+\sigma}{(1-\xi)\sigma}\right)^{\frac{1+\xi}{\xi}}\left[1-\left(\frac{\xi x+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}}\right]\left(1-\left[1-\left(\frac{\xi x+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}}\right]^{-k_1}\right).$$

In this case, the derivative of $\phi_{W_1|W_1>0}(x)$ is given by $\phi'_{W_1|W_1>0}(x)=\dfrac{h_1(\tau_x,k_1,\xi)}{(\tau_x-1)^{k_1}k_1}$ where $\tau_x\triangleq\left(\frac{\xi x+\sigma}{(1-\xi)\sigma}\right)^{1/\xi}$. Because $\tau_x>1$ for all $x\in\mathcal{V}$, the first item in Lemma C.1 implies the result.

Step 2 (W_2 for $\xi\neq 0$): We first show that W_2 has finite mean, and $W_2|W_2>0$ has continuous and strictly positive density function. The conditional p.d.f. of $W_2|W_2>0$ is given as follows:

$$g_{W_2|W_2>0}(x)=\frac{k_1k_2\left[1-\left(\frac{\xi\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}}\right]^{k_1k_2-1}\left(\frac{\xi\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}-1}}{(1-\xi)\sigma\phi'_{W_1|W_1>0}(\phi^{-1}_{W_1|W_1>0}(x))\left\{1-\left[1-\left(\frac{\xi\phi^{-1}_{W_1|W_1>0}(0)+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}}\right]^{k_1k_2}\right\}}.$$

This is a continuous and positive function of x in the support of W_2. Specifically, because $W_1|W_1>0$ has finite mean and absolutely continuous, the second item in Lemma G.1 implies that the projection point $\phi^{-1}_{W_1|W_1>0}(0)$ is finite and $\phi^{-1}_{W_1|W_1>0}(0)<\sup\mathcal{W}_1$. Note that the term $1-\left(\frac{\xi\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}}$ is equal to $G_V(\phi_V^{-1}(\phi^{-1}_{W_1|W_1>0}(x)))$. Because $\phi_V^{-1}(\phi^{-1}_{W_1|W_1>0}(x))>0$ for all $x\in\mathcal{W}_2$, it follows that the term $\left[1-\left(\frac{\xi\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}}\right]$ is positive. Moreover, the term $\left(\frac{\xi\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}-1}$ is also positive, hence it follows that the numerator is positive. Next, we consider the denominator. The functions $\phi'_{W_1|W_1>0}$ and $\phi^{-1}_{W_1|W_1>0}$ are continuous and positive. The term in the curly brackets is equal to $1-\left(G_V(\phi_V^{-1}(\phi^{-1}_{W_1|W_1>0}(0)))\right)^{k_1k_2}$, and $\phi_V^{-1}(\phi^{-1}_{W_1|W_1>0}(0))<\sup\mathcal{V}$, thus the denominator is also positive. Therefore, it follows that $g_{W_2|W_2>0}(x)$ is continuous and positive.

Because $W_2 = \max\left(0, \phi_{W_1|W_1>0}(W_1^{\text{FH}(k_2)})\right)$ and $\phi_{W_1|W_1>0}(x) \le x$, it follows that $\mathbb{E}[W_2] \le \mathbb{E}[W_1^{\text{FH}(k_2)}]$. Since $W_1 \le V^{\text{FH}(k_1)}$ almost everywhere, it follows that $\mathbb{E}[W_2] \le \mathbb{E}[W_1^{\text{FH}(k_2)}] \le \mathbb{E}[V^{\text{FH}(k_1 k_2)}] < \infty$.

We next show that the virtual value function of $W_2|W_2>0$ is strictly increasing. The virtual value function is given by:

$$\phi_{W_2|W_2>0}(x) = x + \frac{(1-\xi)\sigma\left\{1-\left[1-\left(\frac{\xi\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}}\right]^{-k_1k_2}\right\}\phi'_{W_1|W_1>0}(\phi^{-1}_{W_1|W_1>0}(x))}{k_1k_2\left[1-\left(\frac{\xi\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}}\right]^{-1}\left(\frac{\xi\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{(1-\xi)\sigma}\right)^{-\frac{1}{\xi}-1}}.$$

Here, instead of $\phi_{W_2|W_2>0}$, we show that the function $\upsilon(x) \triangleq \phi_{W_2|W_2>0}(\phi_{W_1|W_1>0}(x))$ is increasing, which is sufficient because $\phi_{W_1|W_1>0}(x)$ is increasing. We show that the derivative of $\upsilon(x)$ is positive. The derivative is given by:

$$\upsilon'(x) = \frac{\phi'_{W_1|W_1>0}(x)h_1(\tau_x, k_1k_2, \xi)}{k_1k_2[\tau_x-1]^{k_1k_2}} + \frac{[\tau_x^{k_1k_2}-(\tau_x-1)^{k_1k_2}]\,h_2(\tau_x, k_1, \xi)}{(\tau_x-1)^{2k_1k_2-1}k_1^2k_2},$$

where τ_x is define as before. Because $\tau_x > 1$, Lemma C.1 implies that $\upsilon'(x) > 0$.

Step 3 ($\xi = 0$): The continuity and the positivity of the density functions can be proved as in steps 1 and 2, and hence are omitted. We directly focus on the virtual value functions. If $\xi = 0$, the function $\phi_{W_1|W_1>0}(x)$ is given by

$$\phi_{W_1|W_1>0}(x) = x + \frac{\sigma}{k_1}\exp\left(\frac{x+\sigma}{\sigma}\right)\left[1-\left(1-\exp\left(-\frac{x+\sigma}{\sigma}\right)\right)^{-k_1}\right]\left[1-\exp\left(-\frac{x+\sigma}{\sigma}\right)\right].$$

then the derivative of $\phi_{W_1|W_1>0}(x)$ is given by $\phi'_{W_1|W_1>0}(x) = \frac{h_1(\tau_x,k_1,0)}{k_1(\tau_x-1)^{k_1}}$ where, this time, $\tau_x = \exp(x/\sigma+1)$. Following Lemma C.1, the result follows. We next consider $\phi_{W_2|W_2>0}$ for $\xi = 0$, which is given by

$$\phi_{W_2|W_2>0}(x) = x + \frac{\sigma\left\{1-\left[1-\exp\left(-\frac{\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{\sigma}\right)\right]^{-k_1k_2}\right\}\phi'_{W_1|W_1>0}(\phi^{-1}_{W_1|W_1>0}(x))}{k_1k_2\left[1-\exp\left(-\frac{\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{\sigma}\right)\right]^{-1}\exp\left(-\frac{\phi^{-1}_{W_1|W_1>0}(x)+\sigma}{\sigma}\right)}.$$

Let $\upsilon(x) \triangleq \phi_{W_2|W_2>0}(\phi_{W_1|W_1>0}(x))$. The derivative of $\upsilon(x)$ is given as follows:

$$\upsilon'(x) = \frac{\phi'_{W_1|W_1>0}(x)h_1(\tau_x, k_1k_2, 0)}{(\tau_x-1)^{k_1(k_2+1)}k_1^2k_2} + \frac{(\tau_x^{k_1k_2}-(\tau_x-1)^{k_1k_2})h_2(\tau_x, k_1, 0)}{(\tau_x-1)^{k_1(k_2+1)}k_1^2k_2}.$$

Because $\tau_x > 1$, Lemma C.1 implies that $\upsilon'(x) > 0$. Q.E.D.

Proof of Lemma C.1. We prove each item separately.

1. We show that $h_1(x,k,\xi)$ is positive by induction over k for fixed $\xi < 1$ and $x > 1$. First, note that $h_1(x,1,\xi) = (1-\xi)(x-1) > 0$ for all $x > 1$ and $\xi < 1$. Now suppose that $h_1(x,k-1,\xi)$ is positive. Simple algebra yields that

$$\frac{\partial h_1(x,k,\xi)}{\partial x} = (k+1)h_1(x,k-1,\xi) + x^{k-1}(1-\xi)\,. \tag{2}$$

The induction hypothesis implies $h_1(x,k,\xi)$ is increasing in x. This observation together with the fact that $h_1(1,k,\xi) = k-1 \geq 0$ implies $h_1(x,k,\xi)$ is positive for all $x > 1$ and $\xi < 1$.

2. We show that $h_2(u+1,k,\xi)$ is positive for $u > 0$. Thus, we substitute $x = u+1$, and using binomial expansions, we derive an equivalent expression in the following steps:

$$\begin{aligned}
h_2(u+1,k,\xi) &= \Big[(u+1)^k\Big(k(k+\xi) - (u+1)(1+\xi)(k+1) + (u+1)^2(1+\xi)\Big) - (1+\xi)((u))^{k+1}(u+1)\Big] \\
&= \Big[(u+1)^k\Big((u+1)[(1+\xi)u] - k(u+1)(1+\xi) + k(k+\xi))\Big) - (1+\xi)u^{k+1}(u+1)\Big] \\
&= \sum_{i=0}^{k-1}\binom{k+1}{i+2}u^{k-i}(1+\xi) - k(1+\xi)\sum_{i=0}^{k}\binom{k+1}{i+1}u^{k-i} + k(k+\xi)\sum_{i=0}^{k}\binom{k}{i}u^{k-i} \\
&= k^2 - k + \sum_{i=0}^{k-1}u^{k-i}\frac{k!}{(k-i)!(i+2)!}\Big[\underbrace{(i+1)^2k^2 - 2ik - k - i}_{(a_i)} - \xi\underbrace{((i+1)k^2 - i^2k - k(i+1) + i)}_{(b_i)}\Big] \\
&= k^2 - k + \sum_{i=0}^{k-1}u^{k-i}\frac{k!}{(k-i)!(i+2)!}\Big[a_i - \xi b_i\Big] = k^2 - k + \sum_{i=0}^{k-1}u^{k-i}\frac{k!}{(k-i)!(i+2)!}\Big[a_i - b_i + b_i(1-\xi)\Big]\,.
\end{aligned}$$

By assumption $1-\xi > 0$, therefore, we next argue that b_i is positive and smaller than a_i, which implies the result. The first claim follows because $k(i+1)(k-1) > i(ik-1)$. The second claim follows because the difference between these terms is $a_i - b_i = (k+1)((i+1)k-2)i \geq 0$. Q.E.D.

D. Equilibrium Characterization for Chain of Intermediaries

In this section, we restrict attention to **k**-trees with $k_t = 1$ for $t = 1, \ldots, n+1$, i.e., chain of intermediaries. We start by defining anticipated reports for intermediation chains. In a chain of intermediaries, the anticipated report of buyer $I_{(0)}$ is $W_0 = V$ and the anticipated report of an intermediary $I_{(t)}$ is reduced to $W_t = \psi_{W_{t-1}}(W_{t-1})$ because an intermediary has one downstream agent.

The equilibrium characterization for a chain of intermediaries follows from Theorem 1.

PROPOSITION D.1. *Suppose that the buyer's value V is a GPD with (ξ, σ, μ) such that $\xi < 1$ and $\mu = 0$. Then, at the SPE given in Theorem 1, the seller and intermediary $I_{(t)}$ select a reserve price $r^* = \frac{\sigma}{(1-\xi)}$. Moreover, each intermediary $I_{(t)}$ bids upstream according to*

$$Y_t^*(w) = \begin{cases} w(1-\xi) - \sigma & \text{if } r^* \leq w\,, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose the buyer's value V is a GPD with (ξ, σ, μ) such that $\xi < 1$ and $\mu = 0$. Then, part (A) of Lemma 3 shows that Assumption 1 is satisfied. Hence, Theorem 1 provides an equilibrium where

$$Y_t^*(w) = \psi_{W_{t-1}}(w)\,,$$
$$r_t^* = z_{W_{t-1}} > 0\,.$$

We leverage these result to prove the proposition. The reserve price of intermediary $I_{(t)}$ is defined as the projection point $z_{W_{t-1}}$ of the anticipated report W_{t-1}. In the proof of part (A) of Lemma 3, we also show that the strictly positive part of the anticipated report of an intermediary $I_{(t)}$, $W_t|W_t > 0$ is distributed as $V|V > 0$. Thus, it follows that

$$\phi_{W_t|W_t>0}(w) = (1-\xi)w - \sigma\,,$$

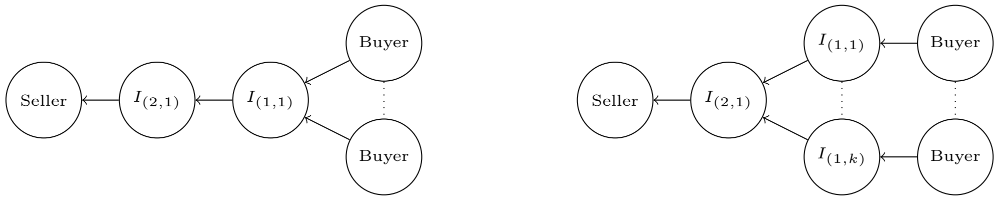
and, the reserve price r_t is $\phi^{-1}_{W_t|W_t>0}(0) = \sigma/(1-\xi)$ for the intermediaries. For the seller, the equilibrium reserve price is $r_S = z_{W_n}$ (see Theorem 1), thus $r_S = \sigma/(1-\xi)$. The reporting function $Y_t^*(\cdot)$ is the projected virtual value function $\psi_{W_{t-1}}(\cdot)$. By definition, the projected virtual value function is equal to the virtual value function of the associated random variable for the values greater than the projection point and zero elsewhere. Note that in the proof of part (A) of Lemma 3, we show that the support of the anticipated reports are bounded by $\sup \mathcal{V}$. Hence, observing a report realization w such that $w \geq \sup \mathcal{V}$ has measure of zero. Thus, we do not consider the values $w \geq \sup \mathcal{V}$ in the reporting function. Hence, the reporting function is given by

$$Y_t^*(w) = \begin{cases} w(1-\xi) - \sigma & \text{if } \sigma/(1-\xi) \leq w\,, \\ 0 & \text{otherwise.} \end{cases}$$

Q.E.D.

E. Impact of Network Structure on Seller's Revenue

In this section, we consider the impact of the intermediation network structure on the profit of the seller. To this end, we consider two alternative network configurations with two intermediation tiers and the same total number of buyers, who share the same value distribution. The first network's downstream intermediation tier is *consolidated* ($\mathbf{k} = (1,1,k)$), that is, there is a single downstream intermediary with k captive buyers and this downstream intermediary is connected with an upstream intermediary. The second network's downstream intermediation tier is *fragmented* ($\mathbf{k} = (1,k,1)$), that is, there are k downstream intermediaries each with a single captive buyer and these downstream intermediaries are connected with the same upstream intermediary. These networks are illustrated in Figure 1. The following result compares the equilibrium reports of the upstream intermediary to the seller in both networks.



(a) Tree 1: Consolidated Network, $\mathbf{k} = (1,1,k)$ (b) Tree 2: Fragmented Network, $\mathbf{k} = (1,k,1)$

Figure 1 Two alternative tree networks with k buyers and two tiers of intermediation.

PROPOSITION E.1. *Suppose that Assumption 1 holds, and consider the SPE of Theorem 1 for the fragmented and consolidated network shown in Figure 1. Let W_2^F and W_2^C be the random reports from the upstream intermediary $I_{(2,1)}$ to the seller in the fragmented and consolidated networks, respectively. W_2^F first-order stochastically dominates W_2^C, that is, $\mathbb{P}\{W_2^F \le w\} \le \mathbb{P}\{W_2^C \le w\}$ for all $w \ge 0$.*

Proposition E.1 shows it is more likely for the seller in the fragmented network to receive higher downstream reports. The expected revenue of a seller with a single downstream agent is given by $r\mathbb{P}\{W_2 \ge r\}$ when she selects reserve price r, and the distribution of W_2 is independent of r because at the SPE given in Theorem 1 intermediaries do not change their reporting strategies depending on the reserve price of the seller. Thus, it follows that the optimal expected revenue in the fragmented network is larger because, for every reserve price, the revenue of the seller in the fragmented network is higher. This observation suggests that the network structure, rather than the total number of intermediaries, plays a key role in determining the seller's revenue.

Note that in both networks the upstream intermediary $I_{(2,1)}$ receives the same highest bid from the downstream tier. In the consolidated network (Tree 1), the downstream intermediary $I_{(1,1)}$ first ranks the buyers' reports and then submits the projected virtual value associated with the highest value to the upstream intermediary $I_{(2,1)}$, i.e., $\psi_V(\max_i v_i)$, where $\{v_i\}_{i=1,\dots,k}$ denote the realizations of the buyers' valuations.

Meanwhile, in the fragmented network (Tree 2), each downstream intermediary $I_{(1,i)}$ submits $\psi_V(v_i)$ to the upstream intermediary $I_{(2,1)}$. Due to Assumption 1, virtual values are increasing, hence, in both networks the upstream intermediary $I_{(2,1)}$ has access to the same highest possible downstream report $\max_i \psi_V(v_i) = \psi_V(\max_i v_i)$. Thus, it can be seen that the lower revenue of the seller in the consolidated network is driven by a more aggressive bid shading strategy employed by intermediary $I_{(2,1)}$.

Why does the upstream intermediary shade bids more aggressively in the consolidated network? In the fragmented network, unlike the consolidated one, the winning intermediary in the first tier may need to pay upstream the report of another first tier intermediary, thereby leading to larger profits for the upstream intermediary when the aforementioned report is large. In contrast, in the consolidated network the fact that the upstream intermediary has no access to the second-highest bid from the buyers leads to lower profits, if the intermediary uses the same mechanism. Because of the second-price-like payment rule of the intermediaries' mechanisms, in this network the upstream intermediary optimally improves these lower profits by shading bids more aggressively. This translates to lower payments made to the seller, and higher charges to the downstream intermediary in case of winning the impression, while possibly reducing the probability of winning it. As a consequence of the upstream intermediary shading bids more aggressively in the consolidated network, payments to the seller are lower despite the smaller total number of intermediaries.

E.1. Proof of Proposition E.1

Proof. We prove this results in two steps. In the first step, we show that the maximum reports from the downstream intermediaries are same in both the fragmented and the consolidated trees when we take a fixed realizations of the buyers' values. Thus it suffices to compare the reporting functions of the upstream intermediary as a function of the highest downstream report. In the second step, we show that the reporting function of the upstream intermediary is point-wise larger in the fragmented network than in the consolidated network. By a standard coupling argument, these observations suffice to prove the first-order stochastic dominance.

Step 1. We proceed by coupling the realization of values in both networks, that is, we assume that the realizations of buyer i's value v_i for $i = 1, \ldots, k$ are the same in the fragmented and consolidated trees. In the consolidated tree, by Theorem 1, the report of downstream intermediary $I_{(1,1)}$ is given by $W_1^C = \max_{i=1,\ldots,k} \psi_V(V_i)$, where we used that all values V_i are i.i.d. copies of the random variable V. Let $X_i \triangleq \psi_V(V_i)$

and denote by $X^{\mathrm{FH}(k)} \triangleq \max_{i=1,\dots,k} X_i$ the highest-order statistic. Then we can write the report of $I_{(1,1)}$ as $W_1^C = X^{\mathrm{FH}(k)}$. The report of the upstream intermediary $I_{(2,1)}$ is given by

$$W_2^C = \psi_{W_1^C}\left(W_1^C\right) = \psi_{X^{\mathrm{FH}(k)}}\left(X^{\mathrm{FH}(k)}\right). \tag{3}$$

Similarly, in the fragmented tree the report of downstream intermediary $I_{(1,i)}$ is given by $W_1^F = \psi_V(V_i) = X_i$. The report of the upstream intermediary $I_{(2,1)}$ is given by

$$W_2^F = \max_{i=1,\dots,k} \psi_{W_1^F}\left(W_1^F\right) = \max_{i=1,\dots,k} \psi_X\left(X_i\right) = \psi_X\left(\max_{i=1,\dots,k} X_i\right) = \psi_X\left(X^{\mathrm{FH}(k)}\right), \tag{4}$$

where the second equality follows from the fact that the variables X_i are i.i.d. copies of the random variable $X \triangleq \psi_V(V)$, and the third from the fact that the function $\psi_X(\cdot)$ is nondecreasing. Because the realization of values are coupled, by comparing equations (3) and (4), it suffices to show

$$\psi_X(x) \geq \psi_{X^{\mathrm{FH}(k)}}(x),$$

for all $x \geq 0$ to conclude that $W_2^F \geq W_2^C$.

Step 2. Since the projected virtual value functions are determined by the virtual value functions and the supports coincide, it suffices to show $\phi_{X|X>0}(x) \geq \phi_{X^{\mathrm{FH}(k)}|X^{\mathrm{FH}(k)}>0}(x)$ for all $x>0$ since the projection point is strictly positive by Lemma G.1. Because virtual values are invariant to truncation from below, it suffices to show

$$\phi_X(x) \geq \phi_{X^{\mathrm{FH}(k)}}(x), \quad \forall x > 0. \tag{5}$$

Furthermore, we can express the virtual value function of a random variable in terms of the hazard rate function $\lambda(x)$ as $\phi(x) = x - \frac{1}{\lambda(x)}$. Thus, in order to compare two virtual value functions as in (5), it suffices to show that the hazard rate functions satisfy $\lambda_X(x) \geq \lambda_{X^{\mathrm{FH}(k)}}(x)$. In Reliability Theory, it is a well-known result that the hazard rate of the highest-order statistic of i.i.d. random variables is smaller than the hazard rate of each single random variable (see, e.g., Rinne 2014, p. 38). Therefore, we obtain that $\psi_X(x) \geq \psi_{X^{\mathrm{FH}(k)}}(x)$ for all $x \geq 0$, which in turn implies by step 1 that W_2^F first-order stochastically dominates W_2^C.

Note that the stochastic dominance between W_2^F and W_2^C implies that the seller has a downstream customer in the fragmented network who values the impression more than the one in the consolidated network. Therefore, it follows that the seller has more revenue in the fragmented network.

We can formally show this result. Let r_S^F and r_S^C denote the optimal reserve prices of the seller in both network configurations. By the first order stochastic dominance, we get $r_S^C \mathbb{P}(W_2^F \geq r_S^C) \geq r_S^C \mathbb{P}(W_2^C \geq r_S^C)$.

Since, the r_S^F is optimal reserve for the seller in the fragmented network, we also get $r_S^F\mathbb{P}(W_2^F \geq r_S^F) \geq r_S^C\mathbb{P}(W_2^F \geq r_S^C)$. These observations together imply that

$$\underbrace{r_S^F\mathbb{P}(W_2^F \geq r_S^F)}_{\text{The seller revenue in the fragmented network}} \geq \underbrace{r_S^C\mathbb{P}(W_2^C \geq r_S^C)}_{\text{The seller revenue in the consolidated network}}. \qquad \text{Q.E.D.}$$

F. Merger of Intermediaries

In this section, we provide the proofs of the results in Section 5.1 and their extensions. As in Section 5.1, we separately consider two types of mergers.

F.1. Vertical Merger

We start this section by providing a proposition for vertical mergers of intermediaries in a chain.

PROPOSITION F.1. *Consider an intermediation chain with $n+2$ tiers (n tiers of intermediation) as shown in Figure 4. Assume that the buyer's value is a GPD with parameters ($\xi < 1, 0 < \sigma$). Let $\Pi_{(t)}$ and $\Pi_{(t+1)}$ denote the expected profits of intermediaries $I_{(t)}, I_{(t+1)}$, and $\Pi_{(m)}$ denote the expected profit of the consolidated intermediary $I_{(m)}$. Under the equilibrium characterization in Theorem 1, the ratio of the expected profits is given by*

$$\frac{\Pi_{(t)} + \Pi_{(t+1)}}{\Pi_{(m)}} = (2-\xi)(1-\xi)^{(1-\xi)/\xi}.$$

Proof. In this proof, we use the closed form expression provided in Proposition D.1. If the intermediaries $I_{(t)}$ and $I_{(t+1)}$ in a chain with n intermediation tiers merge, the consolidated intermediary would be $I_{(t)}$ in a chain with $n-1$ tier of intermediation. Therefore, the profits in case of winning are determined in this manner by using the payment of the intermediary $I_{(t)}$ provided in the proof of Proposition D.1. In both configurations, intermediaries profit only when the buyer acquires the impression. This event is represented by the report of the intermediary connected to the seller is higher than the reserve price of the seller. Here, we use the composition of the reporting functions given the first item in Proposition D.1. Let r_S denote the equilibrium reserve price of the seller, i.e., $r_S = \sigma/(1-\xi)$. Note that the seller selects the same reserve price independent of the number of tiers (see Proposition D.1).

The expected profits are given by

$$\begin{aligned}\Pi_{(m)} &= \mathbb{E}\left[1\{\text{The buyer acquires the impression}\}(P_{t-1} - P_t)\right] \\ &= \mathbb{E}\left[1\{(V+\sigma/\xi)(1-\xi)^{n-1} - \sigma/\xi \geq r_S\}\left(\frac{r_S - \frac{\sigma}{\xi}[(1-\xi)^{n-t} - 1]}{(1-\xi)^{n-t}} - \frac{r_S - \frac{\sigma}{\xi}[(1-\xi)^{n-1-t} - 1]}{(1-\xi)^{n-1-t}}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left(V \geq \frac{\xi r_S + \sigma(1-(1-\xi)^{n-1})}{\xi(1-\xi)^{n-1}}\right)\frac{\xi r_S + \sigma}{(1-\xi)^{n-t}} \\
&= \left(1 + \frac{\xi}{\sigma}\frac{\xi r_S + \sigma(1-(1-\xi)^{n-1})}{\xi(1-\xi)^{n-1}}\right)^{-1/\xi}\frac{\xi r_S + \sigma}{(1-\xi)^{n-t}} \\
&= \left(\frac{\xi r_S + \sigma}{\sigma(1-\xi)^{n-1}}\right)^{-1/\xi}\frac{\xi r_S + \sigma}{(1-\xi)^{n-t}}\,.
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{(t+1)} + \Pi_{(t)} &= \mathbb{E}\left[1\{\text{The buyer acquires the impression}\}(P_{t-1} - P_{t+1})\right] \\
&= \mathbb{E}\left[1\{(V+\sigma/\xi)(1-\xi)^n - \sigma/\xi \geq r_S\}\left(\frac{r_S - \frac{\sigma}{\xi}[(1-\xi)^{n-t+1}-1]}{(1-\xi)^{n-t+1}} - \frac{r_S - \frac{\sigma}{\xi}[(1-\xi)^{n-t-1}-1]}{(1-\xi)^{n-t-1}}\right)\right] \\
&= \mathbb{P}\left(V \geq \frac{\xi r_S + \sigma(1-(1-\xi)^{n})}{\xi(1-\xi)^{n}}\right)\frac{(2-\xi)(\xi r_S + \sigma)}{(1-\xi)^{n-t+1}} \\
&= \left(1 + \frac{\xi}{\sigma}\frac{\xi r_S + \sigma(1-(1-\xi)^{n})}{\xi(1-\xi)^{n}}\right)^{-1/\xi}\frac{(2-\xi)(r_S\xi + \sigma)}{(1-\xi)^{n-t+1}} \\
&= \left(\frac{\xi r_S + \sigma}{\sigma(1-\xi)^{n}}\right)^{-1/\xi}\frac{(2-\xi)(r_S\xi + \sigma)}{(1-\xi)^{n-t+1}}\,.
\end{aligned}
$$

Using these expressions, the ratio of the expected profits is given by

$$\frac{\Pi_{(t)} + \Pi_{(t+1)}}{\Pi_{(m)}} = (2-\xi)(1-\xi)^{(1-\xi)/\xi}\,.$$ Q.E.D.

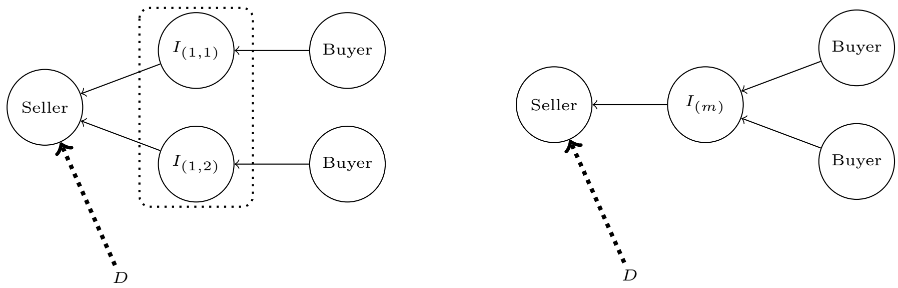
The previous result shows that, for any consecutive pair of intermediaries in a chain, the ratio of the sum of expected profits of the intermediaries before the merger to the expected profit of the consolidated intermediary is always less than one and the same regardless of the intermediation position (see Proposition F.1). Therefore, consecutive intermediaries can increase their combined profits by vertically merging. Note that the profits of intermediaries in different tiers depend on the value distribution as discussed in Proposition 1. These two observations together imply that when intermediaries in tiers t and $t+1$ merge, the benefit of merging (the difference of expected profits) is highest for the intermediaries in the most profitable position of the chain.

F.2. Horizontal Merger

We start this section by providing an extension for Proposition 3 and its proof. The proof of Proposition 3 follows from this general result.

F.2.1. Extension of Proposition 3 In this section, we extend Proposition 3 to more general networks. As shown in Figure 2, we again consider the bilateral merger decision of intermediaries $I_{(1,1)}$ and $I_{(1,2)}$, however, there is no restriction on the configuration of the remaining part, i.e., competing part. This is modeled by introducing an exogenous bidder reporting an arbitrary amount to the seller. We denote by D

the bid of this exogenous bidder, with support $\mathcal{V}$. We prove the result point-wise for every realization of D, which captures the case where the competing report is random. Note that Proposition 3 is a special case where D is 0.



(a) No Merger (b) Horizontal Merger

Figure 2 **Tree networks before and after the merger of $I_{(1,1)}$ and $I_{(1,2)}$ into $I_{(m)}$. The remaining part of the network is assumed to be arbitrary, yet, equal in both cases.**

Before the main result of this section, we explain the equilibrium strategies for the agents in the networks shown in Figure 2. First, note that the seller runs a second-price auction and intermediaries select their mechanisms from $\mathcal{M}$. Thus, Lemma 1 implies that truthful reporting is a weakly dominant strategy for buyers. Second, the equilibrium mechanisms for intermediaries provided in Theorem 1 are independent of the reserve price of the seller when the seller runs a second-price auction. Therefore, intermediaries use the mechanisms characterized in Theorem 1. Finally, consider the seller. In Theorem 1, the seller receives reports only from intermediaries, and there is no exogenous bidder. However, in the networks shown in Figure 2, the seller receives a report, D, from an exogenous bidder. This report can be thought of as the seller's cost for allocating the impression to intermediaries because the seller could have obtained an amount of D by selling the impression to the exogenous bidder. Therefore, the seller's optimal reserve price for intermediaries is $\psi_{W_1}^{-1}(D)$ (see Equation 5.7 in Myerson 1981, p. 67) where W_1 are the reports of intermediaries $I_{(1,1)}$ and $I_{(1,2)}$ in the no merger scenario or intermediary $I_{(m)}$ in the merger scenario. In particular, the seller evaluates the virtual value functions of the downstream reports, and the reserve price is the inverse of these virtual value functions evaluated at D.

In this section, we focus on these equilibrium strategies for agents, and the following proposition states our main result.

Proposition F.2. *Consider the network structures shown in Figure 2 and suppose that the buyers' value distribution is exponential, standard uniform and shifted Pareto (with $\xi = 0.5$ and $\sigma = 1$). The total expected profits of the intermediaries is larger than the expected profit of the consolidated intermediary, i.e.,*

$$\Pi_{(1,1)} + \Pi_{(1,2)} \geq \Pi_{(m)}$$

Proof. We first evaluate the expected profits. In the no merger case (see Figure 2a), the seller's reserve price is $\psi_{W_1}^{-1}(D)$ where $W_1 = \psi_V(V)$. The reporting function and the reserve price of the intermediaries are $\psi_V(\cdot)$ and z_V, respectively. Let $\Pi_{(1,1)} + \Pi_{(1,2)}$ denote the sum of the expected profits of intermediaries $I_{(1,1)}$ and $I_{(1,2)}$. We have:

$$\begin{aligned}\Pi_{(1,1)} + \Pi_{(1,2)} &= \mathbb{E}\Big[1\{\psi_V(V^{\text{FH}(2)}) \geq \max(\psi_{W_1}^{-1}(D), \psi_V(z_V))\}\Big(\psi_V^{-1}(\max(\psi_V(V^{\text{SH}(2)}), \psi_{W_1}^{-1}(D), \psi_V(z_V)))\\ &\quad - \max(\psi_V(V^{\text{SH}(2)}), \phi_{W_1|W_1>0}^{-1}(D))\Big)\Big]\\ &= \mathbb{E}\Big[1\{\phi_V(V^{\text{FH}(2)}) \geq \phi_{W_1|W_1>0}^{-1}(D)\}\Big(\max(V^{\text{SH}(2)}, \phi_V^{-1}(\phi_{W_1|W_1>0}^{-1}(D)))\\ &\quad - \max(\phi_V(V^{\text{SH}(2)}), \phi_{W_1|W_1>0}^{-1}(D))\Big)\Big],\end{aligned}$$

where the second equation follows from $\psi_V(\cdot) = \phi_V(\cdot)$ and $\psi_{W_1}(\cdot) = \phi_{W_1|W_1>0}(\cdot)$, together with the fact that $\psi_V(z_V) = 0$ and $\phi_{W_1|W_1>0}^{-1}(D) > 0$. The indicator in the expected profit corresponds to the event that the impression is acquired by a buyer connected to the intermediaries, and the term in parenthesis corresponds to the intermediation profit.

In the merger case (see Figure 2b), the seller's reserve price is $\psi_{W_1}^{-1}(D)$, however here, $W_1 = \psi_V(V^{\text{FH}(2)})$ because intermediary $I_{(m)}$ is connected to two buyers. Let $\Pi_{(m)}$ denote the expected profit of intermediary $I_{(m)}$ in the consolidated network. We have:

$$\begin{aligned}\Pi_{(m)} &= \mathbb{E}\Big[1\{\psi_V(V^{\text{FH}(2)}) \geq \max(\psi_{W_1}^{-1}(D), \psi_V(z_V))\}\Big(\max(V^{\text{SH}(2)}, \psi_V^{-1}(\psi_{W_1}^{-1}(D)), z_V) - \psi_{W_1}^{-1}(D)\Big)\Big]\\ &= \mathbb{E}\Big[1\{\phi_V(V^{\text{FH}(2)}) \geq \phi_{W_1|W_1>0}^{-1}(D)\}\Big(\max(V^{\text{SH}(2)}, \phi_V^{-1}(\phi_{W_1|W_1>0}^{-1}(D))) - \phi_{W_1|W_1>0}^{-1}(D)\Big)\Big].\end{aligned}$$

For the distributions in the statement of the proposition, the reporting function and the reserve can be derived using Definition 2 and Definition 4. We separately consider the probability distributions given in the statement of the proposition.

Case 1. Exponential Distribution

We start with the exponential distribution. Since $\psi_V(x) = x - \sigma$ and $z_V = \sigma$ for exponential distribution, the expected profits are given as follows:

$$\Pi_{(1,1)} + \Pi_{(1,2)} = \mathbb{E}[1\{V^{\text{FH}(2)} \geq D + 2\sigma\}\sigma],$$
$$\Pi_{(m)} = \mathbb{E}[1\{V^{\text{FH}(2)} \geq \phi^{-1}_{W_1|W_1>0}(D) + \sigma\}[\max(V^{\text{SH}(2)}, \phi^{-1}_{W_1|W_1>0}(D) + \sigma) - \phi^{-1}_{W_1|W_1>0}(D)]].$$

Note that we can alternatively express the expected profits as follows:

$$\Pi_{(1,1)} + \Pi_{(1,2)} = \sum_{i=1}^{2} \mathbb{E}[1\{V_i \geq V_{-i}\}1\{V_i \geq D + 2\sigma\}\sigma],$$
$$\Pi_{(m)} = \sum_{i=1}^{2} \mathbb{E}[1\{V_i \geq V_{-i}\}1\{V_i \geq \phi^{-1}_{W_1|W_1>0}(D) + \sigma\} \max(V_{-i} - \phi^{-1}_{W_1|W_1>0}(D), \sigma)].$$

We show that $\Pi_{(1,1)} + \Pi_{(1,2)} \geq \Pi_{(m)}$ for the exponential distribution by comparing the terms in the summation for each network configuration. We consider $i = 1$ without loss of generality, and then show that the following inequality holds:

$$\underbrace{\mathbb{E}[1\{V_1 \geq V_2\}1\{V_1 \geq D + 2\sigma\}\sigma]}_{=:(LHS)} \geq \underbrace{\mathbb{E}[1\{V_1 \geq V_2\}1\{V_1 \geq \phi^{-1}_{W_1|W_1>0}(D) + \sigma\} \max(V_2 - \phi^{-1}_{W_1|W_1>0}(D), \sigma)]}_{=:(RHS)}. \quad (6)$$

The right-hand side can be written as

$$(RHS) = \sigma \exp\left\{-\frac{\phi^{-1}_{W_1|W_1>0}(D) + \sigma}{\sigma}\right\}\left(1 - \frac{1}{4}\exp\left\{-\frac{\phi^{-1}_{W_1|W_1>0}(D) + \sigma}{\sigma}\right\}\right),$$

while the left-hand side can be written as

$$(LHS) = \sigma \exp\left\{-\frac{D + 2\sigma}{\sigma}\right\}\left(1 - \frac{1}{2}\exp\left\{-\frac{D + 2\sigma}{\sigma}\right\}\right).$$

Therefore, the inequality in (6) can be equivalently expressed as follow:

$$\exp\left\{\frac{P - D}{\sigma} - 1\right\} - \frac{1}{2}\exp\left\{\frac{2(P - D)}{\sigma} - \frac{P + \sigma}{\sigma} - 2\right\} \geq 1 - \frac{1}{4}\exp\left\{-\frac{P + \sigma}{\sigma}\right\},$$

where $P = \phi^{-1}_{W_1|W_1>0}(D)$. For exponential distribution, the corresponding $\phi_{W_1|W_1>0}(\cdot)$ is given by:

$$\phi_{W_1|W_1>0}(x) = x - \frac{\sigma}{2}\frac{2 - e^{-\frac{x+\sigma}{\sigma}}}{1 - e^{-\frac{x+\sigma}{\sigma}}}.$$

Therefore using this definition, it follows that

$$\phi_{W_1|W_1>0}(P) = P - \frac{\sigma}{2}\frac{2 - e^{-\frac{P+\sigma}{\sigma}}}{1 - e^{-\frac{P+\sigma}{\sigma}}} = D \Rightarrow \frac{P - D}{\sigma} = \frac{1}{2}\left(1 + \frac{1}{1 - e^{-\frac{P+\sigma}{\sigma}}}\right).$$

We first rewrite the inequality by replacing $\phi^{-1}_{W_1|W_1>0}(D)$ with P, and then we replace $P-D$ by the expression derived above. We obtain that:

$$\exp\left\{\frac{e^{-\frac{P+\sigma}{\sigma}}}{2(1-e^{-\frac{P+\sigma}{\sigma}})}\right\}-\frac{1}{2}\exp\left\{\frac{e^{-\frac{P+\sigma}{\sigma}}}{1-e^{-\frac{P+\sigma}{\sigma}}}-\frac{P+\sigma}{\sigma}\right\}\geq 1-\frac{1}{4}\exp\left\{-\frac{P+\sigma}{\sigma}\right\}.$$

We further simplify the inequality by introducing $\mathsf{T}=e^{-\left(\frac{P+\sigma}{\sigma}\right)}$.

$$h_E(\mathsf{T})\triangleq\exp\left\{\frac{\mathsf{T}}{2\,(1-\mathsf{T})}\right\}-\frac{1}{2}\exp\left\{\frac{\mathsf{T}}{1-\mathsf{T}}\right\}\mathsf{T}+\frac{1}{4}\mathsf{T}-1\geq 0\,.$$

Since D takes an arbitrary value in $\mathcal{V}$, T is in the interval $[0,e^{-1})$ because it is obtained by monotone transformations. Thus, it is sufficient to show that $h_E(\mathsf{T})\geq 0$ for $\mathsf{T}\in[0,e^{-1})$ in the following steps.

1. First, we derive a lower bound for $h_E(\mathsf{T})$ by using the fact that $e^x\geq 1+x$ for $x\geq 0$. We denote this lower bound function by $\underline{h}_E(\mathsf{T})$ which is given as follows:

$$\begin{aligned}\underline{h}_E(\mathsf{T})&=1+\frac{\mathsf{T}}{2(1-\mathsf{T})}-\frac{1}{2}\exp\left\{\frac{\mathsf{T}}{1-\mathsf{T}}\right\}\mathsf{T}+\frac{1}{4}\mathsf{T}-1\\&=\frac{\mathsf{T}}{2(1-\mathsf{T})}-\frac{1}{2}\exp\left\{\frac{\mathsf{T}}{1-\mathsf{T}}\right\}\mathsf{T}+\frac{1}{4}\mathsf{T}\,.\end{aligned}$$

   In the remainder of the proof, we show that $\underline{h}_E(\mathsf{T})\geq 0$ for $\mathsf{T}\in[0,e^{-1})$.

2. Note that $\underline{h}_E(0)=0$ and $\underline{h}_E(e^{-1})\approx 0.0537$. Therefore, showing that $\underline{h}_E(\mathsf{T})$ is a concave function for all $\mathsf{T}\in[0,e^{-1})$, i.e., $\underline{h}''_E(\mathsf{T})\leq 0$, implies that $\underline{h}_E(\mathsf{T})\geq 0$ for $\mathsf{T}\in[0,e^{-1})$.

$$\underline{h}''_E(\mathsf{T})=\frac{1}{2}\frac{(\mathsf{T}-2)\exp\left\{\frac{\mathsf{T}}{1-\mathsf{T}}\right\}+2-2\mathsf{T}}{(1-\mathsf{T})^4}\,.$$

   Since, the denominator is positive, we can directly consider the numerator. We can show that the numerator is negative as follows. Replacing $\exp\left\{\frac{\mathsf{T}}{1-\mathsf{T}}\right\}$ by its lower bound $1+\mathsf{T}/(1-\mathsf{T})$, we obtain that the numerator is negative when $\mathsf{T}\leq 5/2$. Hence, we get the concavity, and the result follows for exponential distribution.

Case 2. Standard Uniform and Shifted Pareto Distributions

For these two distributions, we first evaluate the expected profits in closed form (for arbitrary $\xi\neq 0$, $\xi<1$ and $0<\sigma$), and derive the inequality corresponding to the expected profit comparison. Then, we use the exact parameter values given in the statement of the proposition to show that the inequality holds. Recall that standard uniform distribution corresponds to $\xi=-1$ and $\sigma=1$, and we consider the shifted Pareto distribution with parameters $\xi = 0.5$ and $\sigma = 1$. For these two distributions, the expected profits are given as follows:

$$\Pi_{(1,1)} + \Pi_{(1,2)} = 2\sigma \left(\frac{\xi D + \sigma}{\sigma(1-\xi)^2}\right)^{-\frac{1}{\xi}+1} \left[1 - \left(\frac{1-\xi}{2-\xi}\right)\left(\frac{\xi D+\sigma}{\sigma(1-\xi)^2}\right)^{-1/\xi}\right],$$
$$\Pi_{(m)} = \sigma \left(\frac{\xi P + \sigma}{\sigma(1-\xi)}\right)^{-\frac{1}{\xi}+1} \left[2 - \left(\frac{1-\xi}{2-\xi}\right)\left(\frac{\xi P+\sigma}{\sigma(1-\xi)}\right)^{-1/\xi}\right],$$

where $\phi^{-1}_{W_1|W_1>0}(D) = P$. In particular, by using the definition of the virtual value functions, we get

$$D = P - \frac{(\xi P + \sigma)\left(2 - \left(\frac{\xi P+\sigma}{\sigma(1-\xi)}\right)^{-\frac{1}{\xi}}\right)}{2\left(1 - \left(\frac{\xi P+\sigma}{\sigma(1-\xi)}\right)^{-\frac{1}{\xi}}\right)}. \tag{7}$$

We make the following change of variables,

$$\mathsf{T} = \left(\frac{\xi P+\sigma}{\sigma(1-\xi)}\right)^{-\frac{1}{\xi}}, \tag{8}$$

and so

$$\frac{\xi D + \sigma}{\xi P + \sigma} = \frac{2(1-\xi) - \mathsf{T}(2-\xi)}{2(1-\mathsf{T})}.$$

Eventually, $\Pi_{(1,1)} + \Pi_{(1,2)} \geq \Pi_{(m)}$ can be equivalently expressed as follows:

$$2\left(\frac{2(1-\xi)-\mathsf{T}(2-\xi)}{2(1-\mathsf{T})(1-\xi)}\right)^{-\frac{1}{\xi}+1}\left[1 - \left(\frac{1-\xi}{2-\xi}\right)\left(\frac{2(1-\xi)-\mathsf{T}(2-\xi)}{2(1-\mathsf{T})(1-\xi)}\right)^{-\frac{1}{\xi}}\mathsf{T}\right] \geq \left[2 - \left(\frac{1-\xi}{2-\xi}\right)\mathsf{T}\right]. \tag{9}$$

Case 2a. We start with standard uniform distribution where $\xi = -1$ and $\sigma = 1$. Due to these parameter values and the fact that D is in $\mathcal{V} = [0,1]$, we get $P \in [\underline{P}, 1]$ where $\underline{P} \approx 0.535$ is obtained by solving (7) for $D = 0$, and $\mathsf{T} \in [0, \bar{T}]$ where $\bar{T} \approx 0.232$ is obtained by solving (8) for $P = \underline{P}$. The inequality in (9) is given by

$$h_U(\mathsf{T}) \triangleq \left(\frac{4-3\mathsf{T}}{4(1-\mathsf{T})}\right)^2\left[1 - \left(\frac{2}{3}\right)\left(\frac{4-3\mathsf{T}}{4(1-\mathsf{T})}\right)\mathsf{T}\right] - \left[1 - \frac{\mathsf{T}}{3}\right] \geq 0.$$

We show that $h_U(\mathsf{T}) \geq 0$ for all $\mathsf{T} \in [0, \bar{T}]$ in the following steps.

1. $h_U(0) = 0$
2. $\tilde{h}_U(\mathsf{T}) \triangleq h_U(\mathsf{T})[4(1-\mathsf{T})]^3/\mathsf{T} \geq 0$ for all $\mathsf{T} \in (0, \bar{T}]$ where the function $\tilde{h}_U(\mathsf{T})$ is

$$\tilde{h}_U(\mathsf{T}) = \frac{32}{3} - 28\mathsf{T} + 20\mathsf{T}^2 - \frac{10}{3}\mathsf{T}^3.$$

   We know that $\tilde{h}_U(\bar{T}) \approx 5.2$ and it is a decreasing function. In particular, the derivative of $\tilde{h}_U$ with respect to T is given by $\tilde{h}'_U(\mathsf{T}) = -10\mathsf{T}^2 + 40\mathsf{T} - 28$, and is negative in $(0, \bar{T})$.

Therefore, it follows that $h_U(\mathsf{T}) \geq 0$ for all T in $[0, \bar{T}]$ and implying that the inequality holds.

Case 2b. We continue with shifted Pareto distribution with parameters $\xi = 0.5$ and $\sigma = 1$, and follow the same steps. Similarly, we first get that $P \in [\underline{P}, \infty)$ where $\underline{P} \approx 2.128$ is obtained by solving (7) for $D = 0$, and $\mathsf{T} \in [0, \bar{T})$ where $\bar{T} \approx 0.058$ is obtained by solving (8) for $P = \underline{P}$. The inequality in (9) reduces to

$$h_{Par}(\mathsf{T}) = \frac{0.01\mathsf{T}(-534 + 2000\mathsf{T} - 2000\mathsf{T}^2 + 367\mathsf{T}^3)}{(3\mathsf{T}^3 - 2)}.$$

We show that the function h_{Par} is nonnegative by the following steps.

1. $h_{Par}(0) = 0$.
2. $\tilde{h}_{Par}(\mathsf{T}) \triangleq h_{Par}(\mathsf{T})(2 - 3\mathsf{T}^3)/\mathsf{T} \geq 0$ for all $\mathsf{T} \in [0, \bar{T})$ where the function $\tilde{h}_{Par}$ is given as follows:

$$\tilde{h}_{Par}(\mathsf{T}) = 53.4 - 200\mathsf{T} + 200\mathsf{T}^2 - 36.7\mathsf{T}^3.$$

Note that $\tilde{h}_{Par}(\mathsf{T})$ is decreasing in $[0, \bar{T})$ and $\tilde{h}_{Par}(\bar{T}) \approx 42.46$.

These two observations imply that the result also follows for shifted Pareto distribution with parameters $\xi = 0.5$ and $\sigma = 1$. Q.E.D.

G. Properties of the Projected Virtual Value Function

LEMMA G.1. *Let X be a random variable with c.d.f. $G_X(\cdot)$, and continuous and positive p.d.f. $g_X(\cdot)$. Suppose that $\mathbb{E}[X]$ is finite and $\phi_X(\cdot)$ is strictly increasing. Then,*

1. $\lim_{t \to \sup \mathcal{X}} \phi_X(t) = \sup \mathcal{X}$.
2. *The projection point z_X is finite, positive, and satisfies $\phi_X(z_X) = 0$.*
3. *$\psi_X(\cdot)$ is continuous.*

Proof. We provide the proof of each item separately.

Item 1. Because the virtual value function lies below the 45° line we have $\limsup_{t \to \sup \mathcal{X}} \phi_X(t) \leq \sup \mathcal{X}$. We prove this item by contradiction. Assume the contrary, i.e. $\limsup_{t \to \sup \mathcal{X}} \phi_X(t) < \sup \mathcal{X}$. Then, there exists some x_0 and M with $\sup \mathcal{X} > x_0 > M$ such that $\phi_X(x) \leq M$ for all $x_0 \leq x \leq \sup \mathcal{X}$. This implies

$$\lambda_X(x) = \frac{1}{x - \phi_X(x)} \leq \frac{1}{x - M}, \quad \forall x_0 \leq x \leq \sup \mathcal{X}.$$

We know that for any continuous and nonnegative random variable (see, e.g., Ross 1996) the complement of c.d.f. can be written in terms of the hazard rate as:

$$\bar{G}_X(x) = \bar{G}_X(x_0) \exp\left\{-\int_0^x \lambda_X(t) \mathrm{d}t\right\}, \forall x \geq x_0.$$

Using the bound on the hazard rate function for the values $x \geq x_0$, we can derive an upper bound for the complement of c.d.f. $G_X(x)$ as follows:

$$\bar{G}_X(x) \geq \bar{G}_X(x_0) \exp\left\{\int_{x_0}^{x} \frac{1}{M-t} \mathrm{d}t\right\}$$
$$= \bar{G}_X(x_0) \exp\left\{\ln \frac{x_0 - M}{x - M}\right\} = \bar{G}_X(x_0) \frac{x_0 - M}{x - M}\,.$$

Now we consider whether $\sup \mathcal{X}$ is finite or infinite, and then show that $\limsup_{t\to\infty} \phi_X(t) \leq M < \sup \mathcal{X}$ results in a contradiction.

First, assume that $\sup \mathcal{X} < \infty$. Using our previous bound we obtain:

$$\bar{G}_X(\sup \mathcal{X}) \geq \bar{G}_X(x_0) \frac{x_0 - M}{\sup \mathcal{X} - M} > 0\,,$$

because $\sup \mathcal{X} > x_0 > M$ and $\bar{G}_X(x_0) > 0$ since the density is positive in its domain. This results in a contradiction because $\bar{G}_X(\sup \mathcal{X}) = 0$.

Second, assume that $\sup \mathcal{X} = \infty$. We can write the expected value of V in terms of the complement of the c.d.f. as follows:

$$\mathbb{E}[X] = \int_0^{\infty} (1 - G_X(z)) \mathrm{d}z \geq \int_{x_0}^{\infty} \bar{G}_X(x_0) \frac{x_0 - M}{z - M} \mathrm{d}z$$
$$\geq \bar{G}_X(x_0)(x_0 - M) \left[\lim_{z\to\infty} \ln(z - M) - \ln(x_0 - M)\right] = \infty\,,$$

where the first equation follows from discarding the integral in the interval $[0, x_0]$ and using our previous bound on the complement of the c.d.f. This is a contradiction because $\mathbb{E}[X] < \infty$.

Item 2. First note that $\lim_{t\to 0} \phi_X(t) = \lim_{t\to 0} -1/g_X(t) < 0$ and $\lim_{t\to \sup \mathcal{X}} \phi_X(t) = \sup \mathcal{X} > 0$ from item 1. Because $\phi_X(\cdot)$ is continuous we obtain that $z_X = \phi_X^{-1}(0)$ is finite, positive and satisfies $\phi_X(z_X) = 0$ by the Intermediate Value Theorem.

Item 3. The projected virtual value function $\psi_X(\cdot)$ is defined as a piece-wise function which consists of continuous functions. Therefore, for continuity it is sufficient to look at the break-points (the projection point and $\sup \mathcal{X}$). It is clear that at the projection point, $\psi_X(\cdot)$ is continuous because $\phi_X(\cdot)$ is continuous and (1) implies that $\lim_{t\to \psi_X(t)} = \sup \mathcal{X}$, i.e., continuity at $\sup \mathcal{X}$. Q.E.D.

H. Ratio of Intermediaries' Margin

Leveraging the characterization of the equilibrium in Theorem 1 we can provide a succinct expression for the ratio of profit margins between two consecutive intermediaries in the chain. Observe that since intermediaries

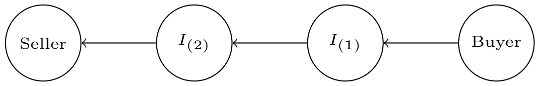


Figure 3 A chain of two intermediaries

earn nonzero profits only when the buyer receives the impression, we restrict attention to such cases. In these cases denoting by P_t the payment of $I_{(t)}$ to the upstream agent $I_{(t+1)}$, we express the profit margin of an intermediary $I_{(t)}$ as follows:

$$\pi_t \triangleq P_{t-1} - P_t\,, \tag{10}$$

where for $t = n$ the payment P_n made to the seller is $P_n = r_S$ where r_S denotes the reserve price at the seller's auction. The object of study in this section is ratio of profit margins of two consecutive intermediaries $I_{(t)}$ and $I_{(t-1)}$, which is given by:

$$\frac{\pi_t}{\pi_{t-1}} = \frac{P_{t-1} - P_t}{P_{t-2} - P_{t-1}}\,.$$

The ratio of margins has been previously studied in the double-marginalization literature (Bresnahan and Reiss 1985) and captures the division of profits between two firms. In our particular setting it provides a measure of an intermediary's inclination for being closer to the supply or demand source. In particular, if the ratio of margins between two consecutive intermediaries is greater than one, then the upstream intermediary makes more profit in comparison to her downstream intermediary, thereby suggesting that being closer to the seller is more profitable. On the other hand, if this measure is smaller than one, then the downstream intermediation (and hence being close to the buyer) is more profitable.

Note that in case of two intermediaries (given in Figure 3), it suffices to check the ratio of margins for the two intermediaries to understand whether being closer to the seller or to the buyer is more profitable. The following lemma provides a closed form expression for this quantity in terms of the buyer's payment, the virtual value function, and a closely related function: the hazard rate of the buyer's valuation. It follows from Theorem 1 that the payment of the buyer P_0 is given by $\phi_V^{-1}(\phi^{-1}_{W_{(1)}|W_{(1)}>0}(r_S))$ where r_S is the reserve price at the seller's auction. Formally, the hazard rate function associated with the buyer's value V, denoted by $\lambda_V(\cdot)$, is given by $\lambda_V(v) = g_V(v)/(1 - G_V(v))$.

Lemma H.1. *In a chain with two intermediaries, under Assumption 1, the ratio of margins between intermediaries $I_{(2)}$ and $I_{(1)}$ is given by*

$$\frac{\pi_2}{\pi_1} = \phi'_V(P_0) = 1 + \frac{\lambda'_V(P_0)}{\lambda^2_V(P_0)}\,, \tag{11}$$

where P_0 denotes the payment the buyer makes to $I_{(1)}$ in case of winning.

Proof. The ratio of margins is considered only when all intermediaries have non zero profits, i.e., when the buyer acquires the impression through the intermediation chain. The ratio is given by:

$$\frac{\pi_2}{\pi_1} = \frac{P_1 - P_2}{P_0 - P_1}.$$

We start by providing the corresponding expression of the ratio of margins under the equilibrium characterized by Theorem 1 in terms of the buyer's payment P_0 and the virtual value functions of the anticipated reports $W_0 = V$ and W_1. We conclude the proof by determining the virtual value function of the anticipated report W_1.

Step 1. The ratio of intermediaries' margins is examined only in case of winning, i.e., when the payments are such that $P_0 \geq r_1$, $P_1 \geq r_2$ and $P_2 = r_S$. In particular, it follows from Lemma A.1 that the payments are given by $P_1 = \max(r_2, Y_2^{-1}(P_2))$ and $P_0 = \max(r_1, Y_1^{-1}(P_1))$. We next write all payments in terms of the buyer's payment P_0. Because $r_1 = z_V$ and $Y_1(\cdot) = \psi_V(\cdot)$ we have

$$\phi_V(P_0) = \psi_V(P_0) = \psi_V\left(\max(z_V, \psi_V^{-1}(P_1))\right) = \max(0, P_1) = P_1,$$

where the first equation follows because $z_V \leq P_0 \leq \sup \mathcal{V}$ when the impression is won and thus $\phi_V(P_0) = \psi_V(P_0)$; the second equation follows from the recursive formula for payments; the third follows because ψ_V is nondecreasing and $\psi_V(z_V) = 0$; and the last one follows because $P_1 \geq 0$. Similarly, because $r_2 = z_{W_1}$ and $Y_2(\cdot) = \psi_{W_1}(\cdot)$ we have

$$\phi_{W_1|W_1>0}(P_1) = \psi_{W_1}(P_1) = \psi_{W_1}\left(\max(z_{W_1}, \psi_{W_1}^{-1}(P_2))\right) = \max(0, P_2) = P_2,$$

where we used that $\phi_{W_1|W_1>0}(P_1) = \psi_{W_1}(P_1)$ because $z_{W_1} \leq P_1 \leq \sup \Theta_1$ when the impression is won. Therefore, the ratio can be written in terms of virtual values and the payment of the buyer P_0 as follows:

$$\frac{\pi_2}{\pi_1} = \frac{\phi_V(P_0) - \phi_{W_1|W_1>0}(\phi_V(P_0))}{P_0 - \phi_V(P_0)}.$$

Step 2. Denote the cumulative distribution and probability density functions of V as $G_V(\cdot)$ and $g_V(\cdot)$, respectively. Because $W_1 = \phi_V(V)$ when $W_1 > 0$, the distribution of $W_1|W_1 > 0$ is given by:

$$G_{W_1|W_1>0}(w) = G_V(\phi_V^{-1}(w)) \quad \text{and} \quad g_{W_1|W_1>0}(w) = g_V(\phi_V^{-1}(w))\frac{1}{\phi_V'(\phi_V^{-1}(w))}.$$

Note that, the p.d.f. $g_V(\cdot)$ is continuous and this implies that the virtual value function $\phi_V(\cdot)$ is differentiable. Hence, the virtual value function for $W_1|W_1>0$ can be written in terms of $G_V(\cdot)$, $g_V(\cdot)$ and the virtual value function $\phi_V(\cdot)$ of V as:

$$\phi_{W_1|W_1>0}(w) = w - \frac{1-G_{W_1|W_1>0}(w)}{g_{W_1|W_1>0}(w)} = w - \frac{[1-G_V(\phi_V^{-1}(w))]\phi'_V(\phi_V^{-1}(w))}{g_V(\phi_V^{-1}(w))}.$$

By this observation, we get

$$\phi_{W_1|W_1>0}(\phi_V(P_0)) = \phi_V(P_0) - \frac{[1-G_V(P_0)]\phi'_V(P_0)}{g_V(P_0)}.$$

This result immediately implies that the ratio of margins is given by:

$$\frac{\pi_2}{\pi_1} = \frac{\frac{[1-G_V(P_0)]\phi'_V(P_0)}{g_V(P_0)}}{P_0-\phi_V(P_0)} = \frac{\frac{[1-G_V(P_0)]\phi'_V(P_0)}{g_V(P_0)}}{\frac{[1-G_V(P_0)]}{g_V(P_0)}} = \phi'_V(P_0) = 1 + \frac{\lambda'_V(P_0)}{\lambda_V^2(P_0)},$$

where $\lambda_V(\cdot)$ denotes the hazard rate function of random variable V. Q.E.D.

This result shows that the hazard rate of the buyer's value plays a critical role on margins. In particular, if the distribution of the buyer's value has:

- increasing hazard rate, then the margin of $I_{(2)}$ is greater than the margin of $I_{(1)}$,
- constant hazard rate, then the margin of $I_{(2)}$ is equal to the margin of $I_{(1)}$,
- decreasing hazard rate, then the margin of $I_{(2)}$ is less than the margin of $I_{(1)}$.

This result reveals a previously unrecognized connection between the buyer's value distribution and the intermediaries' margins. Previous work in manufacturer/retailer settings showed that the ratio of margins is greater/less than one-half if the demand curve is strictly convex/concave (see, e.g, Bresnahan and Reiss 1985, Lariviere and Porteus 2001). Lemma H.1 establishes for the first time that the increasingness/decreasingness of the hazard rate of the buyer's value is key to understanding whether the ratio of margins is greater/less than one, instead of the convexity/concavity of the demand curve (which can equivalently be expressed as the inverse of the buyer's cumulative value distribution).

Intuitively, the upstream intermediary has a first-mover advantage which allows her to commit to a mechanism and incur lower upstream payments to acquire the impression. The downstream intermediary, however, has the advantage of being closer to the buyer which leads to higher downstream bids since intermediaries sequentially shade bids. Loosely speaking, when the buyer's value has decreasing hazard rate (typically associated with heavy-tailed distributions), with significant probability the buyer's value for the impression is large and hence the downstream intermediary can claim a larger margin. Conversely, when the buyer's value has increasing hazard rate (typically associated with light-tailed distributions), the first-mover advantage prevails and the upstream intermediary's margin is larger due to lower costs of acquiring the impression. This result suggests that depending on the buyer's value distribution, intermediaries may prefer to participate in different tiers of the intermediation process.

This lemma can be extended to multi-stage intermediation settings by considering the ratio of margins between consecutive intermediaries in the line. In particular, when we consider the equilibrium characterized by Theorem 1 under Assumption 1, the ratio of margins between intermediaries $I_{(t)}$ and $I_{(t-1)}$ can be given in terms of the hazard rate function of the reports observed by the downstream intermediary W_{t-2} and the payment received by the downstream intermediary P_{t-2}. The proof and formal claim are analogous to Lemma H.1 and hence are omitted. This last observation can be used to "locally" compare the margins of two consecutive intermediaries in the chain. In Section 4.1, we focus on a class of distributions containing distributions which are commonly used in auction theory literature, e.g., uniform and exponential distributions and instead obtain an explicit comparison of the margins of all intermediaries along the chain.

I. Optimal Mechanism Design for Single-stage Intermediation

We consider the mechanism design problem of intermediaries positioned between a seller who runs a second-price auction and buyers whose values for the item are private. The value distributions of the buyers are common knowledge and satisfy the conditions in Section 3. We also allow for the possibility that other exogenous agents, i.e., agents other than intermediaries and their downstream buyers, can participate in the mechanism of the seller. As in Lemma 2, we model these as random competing bids at the sellers mechanism. The largest of these bids is denoted by the random variable D, with the c.d.f. $F_D(\cdot)$ and the p.d.f. $f_D(\cdot)$ over the support $\mathcal{D}$. In this setting, each intermediary determines their bids to submit to the auction of the seller on behalf of their buyers, the allocation of the item in case of winning and the payments that would be charged to the buyers.

We first consider a simpler setting with one intermediary and multiple buyers (see Figure 4). Lemma I.1 characterizes the optimal mechanism of the intermediary I_ℓ in this setting.

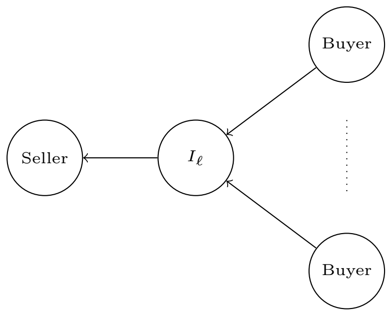


Figure 4 **Single intermediary with multiple buyer**

Lemma I.1. *Consider the network shown in Figure 4 with buyers whose values have increasing virtual value functions. An optimal mechanism for the intermediary I_ℓ is represented by the vector of reserve prices $\mathbf{r}_\ell$ and the vector of reporting functions $\mathbf{Y}_\ell$ where*

$$Y_\ell^i(v) = \psi_{V_i}(v)\,,$$

$$r_\ell^i = z_{V_i}\,,$$

with V_i the random variable that captures the value of buyer i for the impression. The intermediary first ranks buyers according to $\psi_{V_i}(v_i)$ and then reports the maximum $\psi_{V_i}(v_i)$, if greater than zero, to the upstream auction. In case of winning, the intermediary charges a payment to the winning buyer that is equal to the minimum amount which guarantees winning.

Proof. We prove this result in three main steps. In order to prove this lemma, we solve a general mechanism design problem with no restrictions on the set of mechanisms for an intermediary that participates in a second-price auction on behalf of her multiple buyers.

Step 1: Formulating the mechanism design problem. We introduce the notation used for the general mechanism design problem formulation. Let N denote the set of buyers and $V = (V_i)_{i \in N}$ be the random vector denoting the values of the buyers. With some abuse of notation we denote by N the number of buyers, too. We next define the functions which are used to represent a general mechanism for the intermediary. The mechanism of the intermediary consists of a *reporting function* $Y(v) : \mathbb{R}^N \to \mathbb{R}_+$ that maps the reports of the buyers to a bid in the auction of the seller; a *payment function* $X(v,d) : \mathbb{R}^{N+1} \to \mathbb{R}_+^N$, which determines the amount charged to the buyers; and an *allocation function* $Q(v,d) : \mathbb{R}^{N+1} \to [0,1]^N$, which determines the winner of the auction. The intermediary can allocate the impression to the buyer only if she wins at the auction of the seller, so the allocation probability is bounded by the probability of winning the impression for the intermediary. By the Revelation Principle we restrict attention, without loss of generality, to direct mechanisms in which the buyer reports her type truthfully to the intermediary.

Additionally, we define the interim allocation and payments by

$$q_i(v_i) = \int_{\mathcal{D}} \int_{V_{-i}} Q_i(v_i, v_{-i}) g_{V_{-i}}(v_{-i}) f_D(z) \mathrm{d}v_{-i} \mathrm{d}z \,,$$

$$x_i(v_i) = \int_{\mathcal{D}} \int_{V_{-i}} X_i(v_i, v_{-i}) g_{V_{-i}}(v_{-i}) f_D(z) \mathrm{d}v_{-i} \mathrm{d}z \,,$$

where $g_{V_i}(\cdot)$ denotes the p.d.f. of V_i and $g_{V_{-i}}(\cdot)$ denote the p.d.f. of the induced joint probability distribution for $\{V_j\}_{j \neq i}$. Here $q_i(v_i)$ and $x_i(v_i)$ respectively denote the expected allocation probability and payment of buyer i when she reports her value as v_i. Finally we denote by

$$u(v_i, v_i') = v_i q_i(v_i') - x_i(v_i') \,,$$

the expected payoff of buyer i when her true value is v_i and she reports v_i' to the seller.

The optimal mechanism design problem of the intermediary can now be stated in terms of the payment, allocation functions defined above as follows:

$$\max_{X,Q,Y} \sum_{i \in N} \mathbb{E}_{V_i} \left[x_i(V_i) \right] - \mathbb{E}_{V,D} \left[D 1\{ Y(V) \geq D \} \right] \qquad \text{(Profit of Intermediary)}$$

$$\text{st. } u_i(v_i, v_i) \geq 0 \,, \forall i, v_i \qquad \text{(Individual Rationality)}$$

$$u_i(v_i, v_i) \geq u_i(v_i, v_i'), \forall i, v_i, v_i' \qquad \text{(Incentive Compatibility)}$$

$$0 \leq \sum_{i \in N} Q_i(v, d) \leq 1\{Y(v) \geq d\}, \forall v, d \qquad \text{(Feasible Allocation)}$$

$$0 \leq Y(v), \forall v.$$

Here the first two constraints are the standard IR, IC constraints. The third constrain ensures that the total allocation probability for a realization v of the reports V and a realization d of the exogenous competing bid D is less than 1 if the intermediary wins in the auction of the seller and zero otherwise.

Following an identical approach to Myerson (1981), it can be seen that incentive compatibility and individual rationality constraints can be replaced by the following conditions: $u_i(v_i) \triangleq u_i(v_i, v_i) = u_i(0) + \int_0^{v_i} q_i(t)\mathrm{d}t$ and $q_i(\cdot)$ nondecreasing. Note that this result immediately implies that the expected payment of buyer i when she reports v_i is expressed as follows:

$$x_i(v_i) = x_i(0) - \int_0^{v_i} q_i(t)\mathrm{d}t + v_i q_i(v_i). \tag{12}$$

Observe that the IR constraint implies that $x_i(0) \leq 0$. When the payment $x_i(\cdot)$ is eliminated from the objective function in the optimal mechanism design problem of the intermediary by (12), the objective function is maximized at $x_i(0) = 0$ for all $i \in N$. Therefore, in the remainder of the proof, we set $x_i(0) = 0$.

Step 2: Point-wise optimization. We first relax the feasible allocation constraint by requiring the allocation feasibility constraint in expectation respect to D. Let $\tilde{Q}_i(v) \triangleq \mathbb{E}_D[Q_i(v, D)]$. Specifically we replace the *Feasible Allocation* constraint with the following:

$$0 \leq \sum_{i \in N} \tilde{Q}_i(v) \leq F_D(Y(v)).$$

Note that any vector of allocation functions Q that satisfies the *Feasible Allocation* constraint also satisfies its interim version. Therefore this is a relaxation.

Next, eliminating the payments from the objective by using (12) and integrating by parts, we equivalently state the relaxed optimal mechanism design problem as follows:

$$\begin{aligned} \max_{\tilde{Q}, Y} \mathbb{E}_V & \left\{ \sum_{i \in N} \phi_{V_i}(V_i) \tilde{Q}_i(V) - \int_0^{Y(V)} z f_D(z) \mathrm{d}z \right\} \\ \text{st. } & 0 \leq \sum_{i \in N} \tilde{Q}_i(v) \leq F_D(Y(v)), \forall v \\ & 0 \leq Y(v), \forall v \\ & q_i(\cdot) \text{ nondecreasing.} \end{aligned} \tag{OMDP}$$

Momentarily relaxing the constraint that $q_i(\cdot)$ is nondecreasing and maximizing the integrand point-wise over V, we get

$$\max_{\tilde{Q},Y} \sum_{i\in N} \phi_{V_i}(v_i)\tilde{Q}_i - \int_0^Y z f_D(z)\mathrm{d}z$$
$$\text{st. } 0 \le \sum_{i\in N} \tilde{Q}_i \le F_D(Y)\,, 0 \le Y$$

Define the subproblem for a given Y as

$$T(Y) = \max_{\tilde{Q}} \sum_{i\in N} \phi_{V_i}(v_i)\tilde{Q}_i$$
$$\text{st. } 0 \le \sum_{i\in N} \tilde{Q}_i \le F_D(Y).$$

For this subproblem, the optimal solution can readily be given as follows:

$$\tilde{Q}_i = \begin{cases} F_D(Y) & \text{if } i = \arg\max_{j\in N} \phi_{V_j}(v_j) \text{ and } \phi_{V_i}(v_i) \ge 0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Thus, $T(Y) = F_D\,(Y) \max_{j\in N} \phi^+_{V_j}(v_j)$ and hence the point-wise optimization problem becomes,

$$\max_{Y\ge 0} F_D(Y) \max_{j\in N} \phi^+_{V_j}(v_j) - \int_0^Y z f_D(z)\mathrm{d}z$$

The first-order condition with respect to Y can be expressed as follows:

$$f_D(Y)\left(\max_{j\in N} \phi^+_{V_j}(v_j) - Y\right) = 0\,.$$

The objective function of the point-wise maximization problem is unimodal due to the strictly increasing property of virtual value functions so the first-order condition is sufficient for optimality. Summarizing the optimal solution of the point-wise optimization problem for any given $\{v_j\}_{j\in N}$ is such that $Y^*(v) = \max_{j\in N} \phi^+_{V_j}(v_j)$, and the corresponding allocation function is

$$\tilde{Q}^*_i(v) = \begin{cases} F_D\left(\max_{j\in N} \phi^+_{V_j}(v_j)\right)\,, & \text{if } i = \arg\max_{j\in N} \phi_{V_j}(v_j) \text{ and } \phi_{V_i}(v_i) \ge 0\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{14}$$

Next, we verify that these allocation and reporting functions also satisfy feasibility in (OMDP). Observe that the first two constraints are trivially satisfied by construction (through the point-wise optimization problem). Thus, for the feasibility it suffices to check that the induced $q_i(\cdot)$ is nondecreasing. On the other hand, by the strictly increasing property of virtual value functions and (14), $\tilde{Q}^*_i(v_i, v_{-i})$ is nondecreasing in v_i for every v_{-i}. Observe that this immediately implies that the constructed solution is feasible in (OMDP). Moreover, since it maximizes the integrand point-wise, it follows that this solution is also optimal.

Step 3: Implementation. We next construct a feasible ex-post incentive compatible mechanism which has the same expected profit as the optimal interim mechanism (after taking the expectation on the realization of the highest exogenous bid d). To this end, we define the following allocation rule:

$$Q_i^*(v,d) = 1\left\{\phi_{V_i}(v_i) \geq d, i = \arg\max_{j\in N} \phi_{V_j}(v_j), \phi_{V_i}(v_i) \geq 0\right\}, \tag{15}$$

and payment rule

$$X_i^*(v,d) = v_i Q_i^*(v,d) - \int_0^{v_i} Q_i^*(t, v_{-i}, d)\ \mathrm{d}t\,. \tag{16}$$

Observe that this ex-post mechanism obtains the identical profit since $\tilde{Q}_i^*(v) = \mathbb{E}_D[Q_i^*(v,D)]$ and it is feasible. We next write the payment rule explicitly by letting $P_i = \phi_{V_i}^{-1}\left(\max\left\{0, d, \max_{j\in N\setminus i}\phi_{V_j}(v_j)\right\}\right)$ be the payment of buyer i when she wins the impression (i.e., she wins both at the intermediary's and seller's auction). In this notation we can write the allocation rule as $Q_i^*(v,d) = 1\{v \geq P_i\}$ and the payment rule as

$$\begin{aligned} X_i^*(v,d) &= v_i 1\{v \geq P_i\} - \int_0^{v_i} 1\{t \geq P_i\}\ \mathrm{d}t = v_i 1\{v \geq P_i\} - (v_i - P_i)1\{v \geq P_i\} \\ &= P_i 1\{v \geq P_i\}\,. \end{aligned} \tag{17}$$

Using this explicit characterization of $X_i^*(\cdot,\cdot)$, this mechanism can be expressed as follows: The intermediary first computes $\phi_{V_i}(\cdot)$ for each buyer, and identifies the buyer $i = \arg\max_{t\in N}\phi_{V_t}(v_t)$. She submits $\phi_{V_i}(v_i)$ to the seller if this bid is nonnegative. In case of winning, (i.e., $\phi_{V_i}(v_i) \geq d$) the impression is then assigned to buyer i. Note that this reporting function and allocation is consistent with (15) and can alternatively be expressed in terms of reporting functions and reserve prices:

$$Y_\ell^i(v_i) = \psi_{V_i}(v_i)\,,$$

$$r_\ell^i = z_{V_i}$$

where V_i denotes the value distribution of the buyer i. That is, the allocation in (15) can be supported by a mechanism in second-price mechanisms. From the previous discussion we see that a payment of amount P_i is charged to buyer i only if she obtains the impression. Note that this payment is supported by a mechanism in a second-price mechanism since P_i is exactly the minimum amount buyer i should bid to win the impression.

This mechanism achieves the allocation and the payments in (15) and (17), and hence is optimal. Q.E.D.

In order to capture the simultaneous moves of the intermediaries, we next consider the single stage intermediation with multiple intermediaries. The following proposition formally states that the optimal mechanism in case of a single intermediary in Lemma I.1 is still optimal when there are other intermediaries participating in the same upstream mechanism. This can be explained by the fact that the optimal mechanism of the single intermediary is independent of the competing bids in the upstream mechanism.

PROPOSITION I.1. *Assume that multiple intermediaries, each with multiple buyers whose values have increasing virtual value functions, participate in a second-price auction run by the seller. Consider the game between intermediaries $I_\ell \in \mathcal{I}$ who choose their mechanisms simultaneously. At a Nash equilibrium of the game among intermediaries, the mechanism of $I_\ell \in \mathcal{I}$ is such that:*

$$Y_\ell^i(v_i) = \psi_{V_i}(v_i)\,,$$

$$r_\ell^i = z_{V_i}\,,$$

where V_i is the random variable that captures the value distribution of buyer i connected to intermediary I_ℓ.

Proof. For the proof of this proposition, we invoke Lemma I.1 to characterize the best response functions of intermediaries. The best response of intermediary I_ℓ, assuming other intermediaries use mechanisms given in the statement of the proposition as follows, can be obtained by solving the following optimization problem.

$$\max_{(\mathbf{r}_\ell,\mathbf{Y}_\ell)\in\mathcal{M}_\ell} \sum_{I_i\in\mathcal{C}(I_\ell)} \mathbb{E}_{V,T}\left\{\left[\left(Y_\ell^i\right)^{-1}\left(\max\left(Y_\ell^i(r_\ell^i), \max_{j\neq i} Y_\ell^j(V_j), T\right)\right) - T\right] 1_{E_i}\right\},$$

where $E_i = \left\{V_i \geq (Y_\ell^i)^{-1}\left(\max(Y_\ell^i(r_\ell^i), \max_{j\neq i} Y_\ell^j(V_j), T)\right)\right\}$ represents the event that the buyer i wins the impression and $T = \max\left(D, \max_{I_k\in\mathcal{C}(I_c)\setminus\{I_\ell\}} Y_c^k(V_k)\right)$ represents the competing bid at the auction of the seller. It can be seen that this maximization problem is equivalent to the profit maximization problem of a single intermediary participating in the seller's mechanism with competing bid T. Thus, Lemma I.1 can be applied for characterizing the optimal mechanism of I_ℓ, which is independent of the distribution of T and hence the mechanism of the remaining intermediaries. In particular, Lemma I.1 implies that the best response for I_ℓ is

$$Y_\ell^i(v_i) = \psi_{V_i}(v_i)\,,$$

$$r_\ell^i = z_{V_i}\,,$$

where V_i denotes the random variable of buyer i connected to I_ℓ. Since we consider an arbitrary intermediary, it follows that the strategy profile given in Proposition I.1 constitutes a Nash equilibrium of the game among intermediaries.

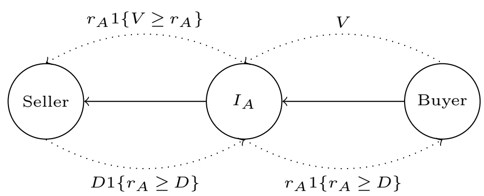
In addition to characterizing an equilibrium, this proposition also shows that focusing on second-price mechanisms is without loss of optimality in single-stage intermediation with symmetric buyers. Specifically, when all buyers have the same value distribution, i.e., $V_i = V$ for all i, the reporting functions and reserve prices in Proposition I.1 are given by $Y_\ell(v) = \psi_V(v)$ and $r_\ell = z_V$, respectively. Therefore, each intermediary first compares the values of her buyers, and reports the projected virtual value evaluated at the maximum report if it is greater than zero to the upstream auction. In case of winning, the payment is determined as the minimum amount which guarantees winning. This mechanism is in the set $\mathcal{M}$.

I.1. A Comparison with Feldman et al. (2010)

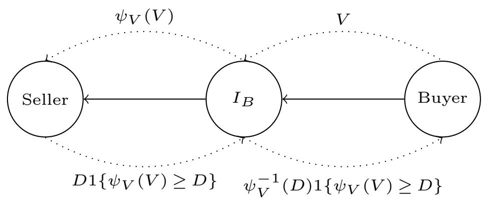
In a related work to ours, Feldman et al. (2010) consider settings where each intermediary i) is restricted to report her reserve price to the upstream auction, and ii) competes by choosing her reserve prices. Moreover, the network structure is very special: there is only one tier of intermediation, and most of the results are for the setting where each intermediary has a single buyer.

The results Lemma I.1 and Proposition I.1 already establish that our mechanism is optimal for the single-stage intermediation setting with symmetric buyers. Here, taking one step further we provide two numerical examples to illustrate that restricting attention to reserve price optimization, and charging the maximum of the reserve price and the second-highest downstream bid is suboptimal for an intermediary. In these examples, the suboptimality is a result of restricting attention to mechanisms where payments charged to the downstream agents are only a function of the bids of the downstream agents. On the other hand, at full generality, the payments charged to downstream can depend on the payment of the intermediary to her upstream agent as well. This is the case in our mechanism, that is, the payment charged to the downstream agents is a function of both the bids of downstream agents and the payment of the intermediary.

Example 1: We consider two scenarios of a setting with one seller, one intermediary and one buyer, which are illustrated in Figure 5. In both scenarios, the seller runs a second-price auction with a random competing bid D that follows a standard uniform distribution over $[0, 1]$ (for simplicity, we assume that the seller has no reserve price), and each intermediary has a single buyer whose value follows a standard uniform distribution over $[0, 1]$. In scenario A (see Figure 5a), the intermediary I_A selects her mechanism from the class of mechanisms in Feldman et al. (2010) by optimizing over the reserve price. We denote by r_A the optimal reserve price of I_A. The intermediary I_A reports her reserve price r_A whenever the report of the buyer is larger than r_A because I_A has only one buyer. If the reserve price r_A is larger than the competing bid D, I_A wins the impression and pays D to the seller. The reserve price r_A is the revenue of I_A in the case of winning in the upstream auction because I_A has a single buyer. In scenario B (see Figure 5b), the intermediary I_B implements the mechanism given in Lemma 2 in our paper. Specifically, I_B reports the projected virtual value evaluated at the buyer's value. In the case of the winning, the buyer is charger the minimum amount that guarantees her winning which is given by $\psi_V^{-1}(D)$.[1] We next evaluate the optimal expected profit of I_A and the expected profit of I_B.



(a) The intermediary I_A reports her reserve price r_A to the upstream auction of the seller whenever the value of the buyer V is larger than r_A. In the case of winning in the upstream auction, I_A charges r_A to her buyer.



(b) The intermediary I_B reports the projected virtual value to the upstream auction of the seller. In the case of winning, the payment of the buyer is determined as the minimum amount that guarantees the winning of the buyer.

Figure 5 **In this figure, we illustrate a setting with one seller, one intermediary and one buyer. The seller runs a second-price auction with a random competing bid D that follows a standard uniform distribution over $[0,1]$. For simplicity, the seller has no reserve price, i.e., her reserve price $r_S = 0$. The value of the buyer follows a standard uniform distribution over $[0,1]$. We denote by $1\{\cdot\}$ the indicator function.**

The optimal expected profit of I_A is found by solving the following optimization problem. Because the value of the buyer takes value in $[0,1]$, we can constrain the reserve price optimization to $[0,1]$:

$$\max_{r\in[0,1]} \mathbb{E}\left[1\{r \geq D\}1\{V \geq r\}(r - D)\right] = \max_{r\in[0,1]} \frac{(1-r)r^2}{2} = \frac{2}{27}\,.$$

Here, the first equality is obtained by evaluating the expected value with respect to the distributions of random variables D and V. The second equality follows from the fact that the function $(1-r)r^2/2$ is a unimodal function of r in $[0,1]$ whose peak point is at $r = 2/3$, i.e., $r_A = 2/3$.

[1] In the mechanism of Lemma 2 in our paper, the reserve price of I_B is z_V that is the projection point of the virtual value function $\phi_V(\cdot)$. Because $D \geq 0$ and the virtual value function of the standard uniform distribution is strictly increasing, we can represent the expected profit of I_B without using the reserve price z_V. Specifically, the winning event $\{V \geq z_V\} \cup \{\psi_V(V) \geq D\}$ is equal to the event $\{\psi_V(V) \geq D\}$, and the payment of the buyer $\max(\psi_V^{-1}(0), \psi_V^{-1}(D), z_V)$ is equal to $\psi_V^{-1}(D)$.

Evaluating the expected profit of I_B, $\mathbb{E}[1\{\psi_V(V) \geq D\}(\psi_V^{-1}(D) - D)]$, we compare the optimal expected profit of I_A and the expected profit of I_B as follows:

$$\underbrace{\frac{1}{12}}_{\text{The expected profit of } I_B} \geq \underbrace{\frac{2}{27}}_{\text{The optimal expected profit of } I_A}.$$

This numerical example illustrates that the class of mechanisms studied in Feldman et al. (2010) is suboptimal.

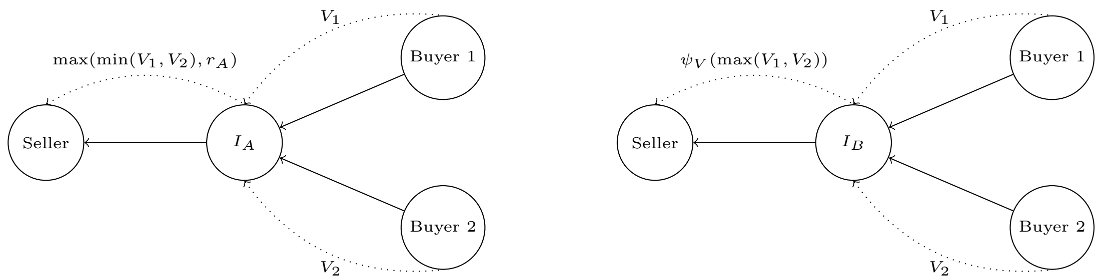
Example 2: We next consider two scenarios of a setting where, this time, each intermediary has two buyers. In Figure 6, we illustrate the intermediation networks. As in Example 1, the seller runs a second-price auction with a random competing bid D that follows a standard uniform distribution over $[0, 1]$ in both scenarios (for simplicity, we assume that the seller has no reserve price). Differently from Example 1, each intermediary has two buyers. The value distribution of buyers follow a standard uniform distribution over $[0, 1]$. In scenario A (see Figure 6a), the intermediary I_A selects her mechanism from the class of mechanisms in Feldman et al. (2010) by optimizing over the reserve price. We denote by r_A the optimal reserve price of I_A. The intermediary I_A reports the maximum of r_A and the second-highest bid submitted by buyers, i.e., $\max(\min(V_1, V_2), r_A)$ whenever the highest bid is greater than r_A, i.e., $\max(V_1, V_2) \geq r_A$. If the report of I_A is larger than the competing bid D, I_A wins the impression in the auction of the seller and pays D. The intermediary I_A allocates the impression to the buyer with the highest bid and charges her $\max(\min(V_1, V_2), r_A)$. In scenario B (see Figure 6b), the intermediary I_B implements the mechanism given in Lemma 2 in our paper, she reports the projected virtual value function evaluated at the highest bid of the buyers, $\psi_V(\max(V_1, V_2))$. In the case of winning in the auction of the seller, I_B allocates the impression to the buyer with the highest bid and charges the minimum amount that guarantees the winning of that buyer $\max(\psi_V^{-1}(D), \min(V_1, V_2))$.

The optimal expected profit of I_A is evaluated by solving the following optimization problem:

$$\max_{r \in [0,1]} \mathbb{E}\left[1\{\max(\min(V_1, V_2), r) \geq D\} 1\{\max(V_1, V_2) \geq r\}(\max(\min(V_1, V_2), r) - D)\right]$$

We compute the optimal expected profit of I_A and the optimal reserve price r_A by creating a grid of possible reserve prices, $[0, 1]$. For each r in this grid, we compute the expected profit of I_A via Monte Carlo simulation. We also compute the expected profit of I_B,

$$\mathbb{E}\left[1\{\psi_V(\max(V_1, V_2)) \geq D\}(\max(\psi_V^{-1}(D), \min(V_1, V_2)) - D)\right]$$



(a) The intermediary I_A reports the maximum of the second-highest bid, $\min(V_1,V_2)$, and her reserve price, r_A, to the upstream auction of the seller whenever $\max(V_1,V_2)$ is larger than r_A. In the case of winning in the upstream auction, I_A charges $\max(\min(V_1,V_2),r_A)$ to her buyer with the highest bid.

(b) The intermediary I_B reports the projected virtual value function evaluated at $\max(V_1,V_2)$ to the auction of the seller. In the case of winning, I_B allocates the impression to the buyer with the highest bid and charges her the minimum amount that guarantees her winning, i.e., $\max(\psi_V^{-1}(D),\min(V_1,V_2))$.

Figure 6 **In this figure, we illustrate a setting with one seller and one intermediary connected to two buyers. The seller runs a second-price auction with a random competing bid D that follows a standard uniform distribution over $[0,1]$. For simplicity, the seller has no reserve price, i.e., her reserve price $r_S=0$. The value V_i of the buyer i follows a standard uniform distribution over $[0,1]$ for $i=1,2$. The random variable V follows a standard uniform distribution over $[0,1]$.**

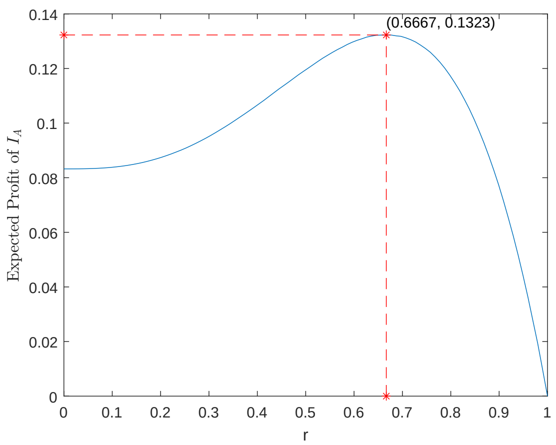


Figure 7 **In this figure, we illustrate with the blue curve how the expected profit of I_A changes depending on the value of reserve price r. This figure reveals that the optimal reserve price r_A of I_A is 0.6667, and the optimal expected revenue of I_A is 0.1323. The dashed red lines are used to indicate the coordinates of the optimal expected profit and the optimal reserve price**

via Monte Carlo simulation. As a result, we compare the optimal expected profit of I_A and the expected profit of I_B as follows:

$$\underbrace{0.1459}_{\text{The eexpected profit of } I_B} \geq \underbrace{0.1323}_{\text{The optimal expected profit of } I_A}.$$

This numerical example also illustrates that the class of mechanisms in Feldman et al. (2010) is suboptimal.

We compute the difference between the expected profit of the optimal mechanism within the class of Feldman et al. (2010) and our mechanism, and divide this quantity by the optimal expected profits (obtained by our mechanism) to obtain optimality gaps. The optimality gaps turn out to be quite substantial: 11% in Example 1 and 9% in Example 2. Note that these gaps are not optimized by considering the network structure or the probability distribution of the exogenous competing bid D. We consider the simplest cases to illustrate the optimality gaps. While we do not provide the worst-case bounds for the optimality gap, we anticipate that the profit loss due to restricting attention to the class of mechanisms studied in Feldman et al. (2010) can be much larger (than the already substantial amount observed in our examples).

J. Asymmetric Networks and More General Mechanisms

In this section, we consider multi-stage intermediation in general network structures when the intermediaries are restricted to choose their mechanisms from the set of strategy-proof mechanisms.

J.1. Network Model and Feasible Mechanisms

In comparison to the single-stage intermediation, the network structure and the timing of events are more complex in multi-stage intermediation. Therefore, in this section we start by providing the notation for the network structure and the timing of events. Moreover, we formally define the set of strategy-proof mechanisms which are appropriately adopted to this network structure and the timing of events, and provide examples of the mechanisms in this set.

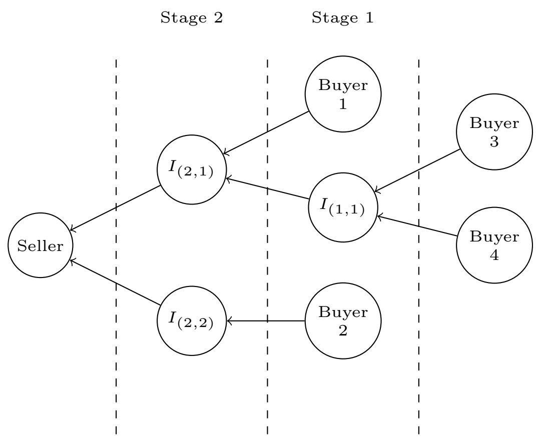


Figure 8 A tree of intermediaries. Here, three intermediaries and four buyers constitute the tree network. The intermediaries $I_{(2,1)}$ and $I_{(2,2)}$ in the upstream tier (tier 2) choose their mechanisms simultaneously. Subsequently, intermediary $I_{(1,1)}$ in tier 1 chooses her mechanism, after observing the mechanisms of $I_{(2,1)}$ and $I_{(2,2)}$. Finally, the values of buyers are realized, and they submit bids to the upstream mechanisms they face.

Network structure and timing of events. A tree network originates from a strategic seller. All leaf nodes in this network correspond to different buyers. Intermediaries are located between buyers and the seller. Each intermediary connects a set of downstream agents (either other intermediaries or buyers) to a single upstream agent which is either the seller or another intermediary. A **k**-tree introduced in Section 2 is a special case in which all agents in a tier have the same number of downstream agents and buyers' values are identical.

Here, the value of buyer $I_\ell \in \mathcal{B}$ is denoted by the random variable V_ℓ. As in Section 2, we assume that this random variable is absolutely continuous and has strictly positive and continuous density. However, now we allow buyers to have different value distributions. Therefore, V_ℓ is associated with a cumulative distribution function $G_{V_\ell}(\cdot)$, and a probability density function $g_{V_\ell}(\cdot)$ that is strictly positive over $\mathcal{V}_\ell$, the support of V_ℓ. The lower bound of the support is assumed to be at 0, and V_ℓ has finite expected values, i.e., $\mathbb{E}[V_\ell] < \infty$. With some abuse of notation we denote by $\mathcal{C}_\ell = \mathcal{C}(I_\ell)$ the set of downstream agents connecting to I_ℓ and $C_\ell = |\mathcal{C}(I_\ell)|$ the total number of downstream agents.

The game among intermediaries, the seller and buyers corresponds to an extensive form incomplete information game, where intermediaries move sequentially from upstream to downstream by choosing their mechanisms following the seller, and buyers select their reporting functions. The timing of events is as follows:

1. The seller I_s determines her mechanism.
2. Intermediaries $\{I_{(t,j)}\}_j$ in tier t simultaneously choose their mechanisms, sequentially from tier $t = n$ down to $t = 1$. Specifically, intermediaries in a tier simultaneously choose their mechanisms after observing the mechanisms chosen by upstream tiers.
3. Each buyer privately draws her type and chooses her bid.
4. If an agent $\{I_{(t,j)}\}_j$ in tier t is:
   (a) a buyer, she submits her bid,
   (b) an intermediary, she receives bids from her downstream agents, and submits a report to the upstream tier as determined by her mechanisms,

   sequentially from tier $t = 0$ up to tier $t = n$.
5. The seller, I_s, determines a winner, allocates the impression (if won) and charges payments according to her mechanism.
6. The winning intermediary in tier t, determines a downstream winner, allocates the impression, and charges payments as determined by her mechanism, from $t = n$ down to $t = 1$.

Feasible mechanism. In our problem, we restrict the set of available mechanisms for intermediaries (and the seller) to the set of the strategy-proof mechanisms (appropriately adjusted to our setting) denoted by $\mathcal{M}^{\mathrm{SP}}$. Nisan et al. (2007, p.218) define strategy-proof mechanisms, and discuss that truthful reporting is a dominant strategy under strategy-proof mechanisms (see Nisan et al. 2007, p. 244).

In a setting without intermediation, an agent is not better off reporting any value other than her true "type" when the seller implements a strategy-proof mechanism. When agents implement strategy-proof mechanism in an intermediation network, it is still the case that buyers are better off reporting their true type for the item. Since intermediaries do not inherently value impressions, a priori there is no straightforward interpretation for the "type" of an intermediary and what it means for an intermediary to report her type truthfully. Instead, in intermediation networks, an intermediary's "type" is endogenously determined, at equilibrium, based on the potential revenue she can extract from auctioning the item to a downstream agent. For example, as discussed in Section 3, we could interpret the willingness-to-pay of the intermediary as her "type." The mechanisms we consider in this setting naturally extend strategy-proof mechanisms to intermediation networks, allowing one to consider more general mechanisms than the set $\mathcal{M}$ introduced in Section 2.

We proceed by introducing the notation that will be used in the remainder of this report, and formally defining the strategy-proof mechanisms for the intermediaries and the seller.

A mechanism for intermediary I_ℓ is given by the triple $(\mathbf{X}_\ell, \mathbf{Q}_\ell, Y_\ell) \in \mathcal{M}^{\text{sp}}$. As before Y_ℓ denotes the *reporting function* the intermediary uses to map the direct downstream reports to an upstream report. In addition, in the general setting considered in here, we allow the intermediary to choose a *payment function* $\mathbf{X}_\ell$, and an *allocation function* $\mathbf{Q}_\ell$, which possibly do not satisfy the second-price structure introduced in Section 2. Note that due to the multi-tier intermediation structure, the intermediary can allocate the item downstream if she acquires it from upstream. Thus, her allocation, and possibly the payments, are not only a function of the direct reports the intermediary receives from her immediate downstream agents, but also a function of other upstream reports that impact the intermediary's allocation.[2]

To see this more clearly, for intermediary I_ℓ, consider the (unique) path $\mathcal{T}$ in the underlying intermediation tree that connects I_ℓ to the seller. Consider all the reports received by agents in this path (including I_ℓ and the seller), other than the ones submitted by the intermediaries on this path (which are readily a function of the reports in consideration). We refer to these reports as *indirect reports of* I_ℓ, i.e., indirect reports of I_ℓ are given by $\{w_{j,i} \mid \mathcal{U}(I_i) = I_j \in \mathcal{T}, I_i \notin \mathcal{T}\}$, where $w_{j,i}$ denotes the report of agent I_i to its upstream agent I_j. Observe that indirect reports include direct reports $(w_{\ell,c})_{c:I_c \in C_\ell}$ that I_ℓ receives from her immediate

[2] Note that this also is a feature of the set $\mathcal{M}$ discussed in Section 2.

downstream agents, as well as other reports received by agents in $\mathcal{T}$. Note that the latter set of reports also impact the allocation decisions of the intermediary I_ℓ, as a large report in upper tiers potentially diverts the impression away from $\mathcal{T}$ (and hence intermediary I_ℓ), and allocates the impression to a buyer through a different intermediation path. We denote the set of all possible indirect reports of this intermediary by $\mathcal{W}_\ell$, and a particular vector of indirect reports by $\boldsymbol{\omega}_\ell \in \mathcal{W}_\ell$. To make the difference clear, we denote the direct reports intermediary I_ℓ receives by $\mathbf{w}_\ell = (w_{\ell,c})_{c:I_c \in C_\ell} \in \mathbb{R}_+^{C_\ell}$.

Recall that a reporting function of the intermediary maps the direct reports received from immediate lower tier $\mathcal{C}_\ell$ of intermediary I_ℓ to a bid to be submitted to the upstream intermediary's mechanism, i.e., $Y_\ell : \mathbb{R}_+^{C_\ell} \to \mathbb{R}_+$. On the other hand, we allow the contingent payments and allocation to be a function of the remaining indirect reports, i.e., a payment function $\mathbf{X}_\ell : \mathcal{W}_\ell \to \mathbb{R}_+^{C_\ell}$, and an allocation function $\mathbf{Q}_\ell : \mathcal{W}_\ell \to \{0,1\}^{C_\ell}$, respectively, determine the amount charged to downstream agents as well as the allocation based on *all* indirect reports.

It can be seen that any mechanism where (i) contingent upon acquiring the item from upstream, the intermediary bases the allocation decision on direct reports; and (ii) payments are equivalent to the minimum report that guarantees winning belongs to $\mathcal{M}^{\text{SP}}$.[3] Thus, it follows that the mechanisms in $\mathcal{M}$ are a special subclass of the mechanisms discussed here. Moreover, the class of mechanisms considered here is strictly larger in that we no longer impose (i) and (ii). Other important mechanisms such as posted price mechanisms and ranking-based mechanisms also belong to this class.

Observe that the indirect reports admit an alternative recursive definition, which is more convenient for our exposition. Let I_u be the upstream neighbor of intermediary I_ℓ. We let $\boldsymbol{\omega}_{u,-\ell}$ denote the indirect reports of intermediary I_u excluding the direct report she receives from I_ℓ and $\mathcal{W}_{u,-\ell}$ be the set of such indirect reports. With some abuse of notation we can express the indirect reports of intermediary I_ℓ by $\boldsymbol{\omega}_\ell = (\mathbf{w}_\ell, \boldsymbol{\omega}_{u,-\ell}) \in \mathcal{W}_\ell$ where as before $\mathbf{w}_\ell = (w_{\ell,c})_{c:I_c \in C_\ell} \in \mathbb{R}_+^{C_\ell}$ denotes the direct reports and $\boldsymbol{\omega}_{u,-\ell} \in \mathcal{W}_{u,-\ell}$. Alternatively, we let $\boldsymbol{\omega}_\ell = (w_{\ell,c}, \boldsymbol{\omega}_{\ell,-c}) \in \mathcal{W}_\ell$ where $w_{\ell,c} \in \mathbb{R}_+$ is the direct report of downstream agent $I_c \in C_\ell$, and $\boldsymbol{\omega}_{\ell,-c} \in \mathcal{W}_{\ell,-c}$ denotes the indirect reports of the intermediary I_ℓ excluding the report of agent I_c. Thus, indirect reports of intermediaries along the path $\mathcal{T}$ to the seller are closely related.

[3] Specifically, for such mechanisms allocation and payment decisions can always be expressed as a function of the indirect reports along path $\mathcal{T}$.

We illustrate our definition of the direct/indirect reports by considering intermediary $I_{(1,1)}$ in Figure 8. For this intermediary, it can be readily seen that the direct reports are $\mathbf{w}_{(1,1)} = (v_3, v_4)$ where v_3 and v_4 are the reports submitted by Buyer 3 and Buyer 4, respectively. The indirect reports (except the report of $I_{(1,1)}$) for the upstream intermediary, $I_{(2,1)}$, are $\boldsymbol{\omega}_{(2,1),-(1,1)} = (v_1, w_{s,(2,2)})$ where v_1 is the report of Buyer 1, $w_{s,(2,2)}$ is the report of $I_{(2,2)}$. The indirect reports for $I_{(1,1)}$ are given by $\boldsymbol{\omega}_{(1,1)} = (v_3, v_4, v_1, w_{s,(2,2)})$.

We next formally define strategy-proof mechanisms using this notation. We say that a mechanism of I_ℓ is strategy-proof if each downstream agent $I_c \in \mathcal{C}_\ell$ is better off reporting her "type" v over some report v' regardless of the indirect reports $\boldsymbol{\omega}_{\ell,-c}$ of the competitors where $(v', \boldsymbol{\omega}_{\ell,-c}) \in \mathcal{W}_\ell$:

$$vQ_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) \geq vQ_{\ell,c}(v', \boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(v', \boldsymbol{\omega}_{\ell,-c}) \qquad \forall v, v', \boldsymbol{\omega}_{\ell,-c}\,. \tag{18}$$

Note that the mechanism is ex-post w.r.t. the reports of competing bidders in upstream mechanisms. The mechanism should also be individually rational:

$$vQ_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) \geq 0 \qquad \forall v, \boldsymbol{\omega}_{\ell,-c}\,. \tag{19}$$

In case of the seller, the set of direct reports and the set of indirect reports are the same and given by $\mathcal{W}_s = \mathbb{R}^{C_s}$. A mechanism for the seller is given by the duple $(\mathbf{X}_s, \mathbf{Q}_s) \in \mathcal{M}_s$ where the payment function $\mathbf{X}_s : \mathcal{W}_s \to \mathbb{R}_+^{C_s}$ and the allocation function $\mathbf{Q}_s : \mathcal{W}_s \to \{0,1\}^{C_s}$ are as before. The set of feasible mechanisms for the seller is also strategy-proof so $(\mathbf{X}_s, \mathbf{Q}_s) \in \mathcal{M}_s$ satisfy the conditions (18) and (19). Since the seller has no reporting function, we denote $\mathcal{M}_s$ different from $\mathcal{M}^{\text{SP}}$.

We now briefly describe some practical mechanisms that fit within the class of strategy-proof mechanisms.

Second-price auctions. Each intermediary I_ℓ runs a second-price auction with reserve price $r_\ell \geq 0$ and submits $Y_\ell\left(\max_{c \in \mathcal{C}_\ell} w_{\ell,c}\right)$ to the upstream intermediary, whenever the maximum bid $\max_{c \in \mathcal{C}_\ell} w_{\ell,c}$ is above the reserve price r_ℓ. Here $Y_\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing reporting function. Let $I_u = \mathcal{U}(I_\ell)$ be the upstream intermediary of I_ℓ and let $P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})$ be the payment of I_ℓ to I_u in case of winning. When the downstream agent c is the winner, I_ℓ charges her the minimum amount that guarantees winning, which is given by $P_{\ell,c} = \max\left(\max_{c' \in \mathcal{C}_\ell \setminus \{I_c\}} w_{\ell,c'}, r_\ell, Y_\ell^{-1}\left(P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right)\right)$.

Ranking-based mechanisms. Each intermediary I_ℓ transforms downstream reports using a bidder specific increasing function $Y_{\ell,c} : \mathbb{R}_+ \to \mathbb{R}_+$, runs a second-price auction on the transformed bids and reports the maximum transformed bid $\max_{c \in \mathcal{C}_\ell} Y_{\ell,c}\left(w_{\ell,c}\right)$ to the upstream intermediary, whenever the maximum quantity is positive. Let $I_u = \mathcal{U}(I_\ell)$ be the upstream intermediary of I_ℓ and let $P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})$ be the payment of I_ℓ to I_u in case of winning. When the downstream agent c is the winner, I_ℓ charges her the minimum amount that guarantees winning, which is given by $P_{\ell,c} = Y_{\ell,c}^{-1}\left(\max\left(\max_{c' \in \mathcal{C}_\ell \setminus \{I_c\}} Y_{\ell,c'}\left(w_{\ell,c'}\right), P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right)\right)$.

Posted price mechanisms. Each intermediary I_ℓ posts a price $p_\ell \in \mathbb{R}_+$, collects reports from the downstream agents, and allocates the item uniformly at random among the downstream agents with reports above the posted price (whenever the item is won by the intermediary). The intermediary reports upstream a quantity $y_\ell \in \mathbb{R}_+$ whenever some report of a downstream agent is above p_ℓ, and zero otherwise. While the mechanisms described in the previous section are deterministic in nature, i.e., the allocation functions satisfy $\mathbf{Q}_\ell : \mathcal{W}_\ell \to \{0,1\}^{C_\ell}$, it is straightforward to accommodate randomized mechanisms by extending the set of indirect reports to include some random variable, and then having the allocation and payment functions depend on the realization of this random variable. This allows conditions (18) and (19) to hold ex-post with respect to all randomizations in the intermediation network. For example, independent draws from uniform random variables can be used to break ties among buyers.

J.2. Equilibrium Characterization

In our model, buyers are strategic players however assuming that buyers report their values truthfully, we can focus on the game among intermediaries and the seller because truthful reporting is a weakly dominant strategy for buyers. While the game with buyer corresponds to an extensive form incomplete information game, the game among intermediaries and the seller is an extensive form (Stackelberg) game. We first provide a formal definition for this Stackelberg game. Subsequently, we characterize an SPE of the game between the intermediaries and the seller. Similar to the single-stage intermediation, we first identify an optimal mechanism in a simpler setting with only one intermediary. We next extend our characterization to a setting with multiple intermediaries by using the result for one intermediary as a building block.

Definition J.1. The game among intermediaries and the seller is an extensive form game $\Gamma = \langle \mathcal{I} \cup \{I_s\}, \mathcal{H}, \mathcal{S}, \{u_\ell : I_\ell \in \mathcal{I}\} \cup \{u_{n+1}\}, \mathcal{P}\rangle$ where

- The set of players is $\mathcal{I} \cup \{I_s\}$.
- The set of histories is $\mathcal{H} = \cup_{t=1}^{n} \mathcal{H}_t$ where $\mathcal{H}_t$ is the set of all possible tier t histories and is given by:

$$\mathcal{H}_t = \left\{ H_t | H_t = \left\{ \left(\mathbf{X}_{(t',k)}, \mathbf{Q}_{(t',k)}, Y_{(t',k)}\right) \in \mathcal{M}^{\mathrm{SP}} : I_{(t',k)} \in \mathcal{I}, t < t' \right\} \cup \left\{ (\mathbf{X}_{n+1}, \mathbf{Q}_{n+1}) \in \mathcal{M}_s \right\} \right\}.$$

Here, history H_t consists of the mechanisms $\{(\mathbf{X}_{(t',k)}, \mathbf{Q}_{(t',k)}, Y_{(t',k)}) : I_{(t',k)} \in \mathcal{I}, t < t'\}$ chosen by upstream intermediaries $\{I_{(t',k)} \in \mathcal{I}, t < t'\}$ and the seller's mechanism $(\mathbf{X}_{n+1}, \mathbf{Q}_{n+1})$.

- The set of pure strategies for intermediary $I_{(t,j)}$ is $\mathcal{S}_{(t,j)} = \{s | s : \mathcal{H}_t \to \mathcal{M}^{\text{SP}}\}$, and for the seller, I_s, $\mathcal{S}_{n+1} = \mathcal{M}_s$. Then, the set of pure strategy profile is $\mathcal{S} = \prod_{I_{(t,j)} \in \mathcal{I}} \mathcal{S}_{(t,j)} \times \mathcal{S}_{n+1}$.
- The utility functions, $u_\ell : \mathcal{S} \to \mathbb{R}$, for intermediaries $I_\ell \in \mathcal{I}$ in tier t are given by

$$u_\ell(s = \{s_{\ell'} : I_{\ell'} \in \mathcal{I}\} \cup \{s_{n+1}\}) = \mathbb{E}\left[\sum_{c: I_c \in \mathcal{C}(I_\ell)} X_{\ell,c}(\boldsymbol{\omega}_\ell) - X_{u,\ell}(\boldsymbol{\omega}_u)\right]$$

where the upstream agent is $I_u = \mathcal{U}(I_\ell)$, the direct reports of I_ℓ are $\mathbf{w}_\ell$, the indirect reports of I_ℓ are $\boldsymbol{\omega}_\ell = (\mathbf{w}_\ell, \boldsymbol{\omega}_{u,-\ell})$, and the indirect reports of I_u are $\boldsymbol{\omega}_u = (Y_\ell(\mathbf{w}_\ell), \boldsymbol{\omega}_{u,-\ell})$.

The utility function, $u_{n+1} : \mathcal{S} \to \mathbb{R}$, for the seller is given by

$$u_{n+1}(s = \{s_\ell : I_\ell \in \mathcal{I}\} \cup \{s_{n+1}\}) = \mathbb{E}\left[\sum_{c: I_c \in \mathcal{C}(I_s)} X_{s,c}(\mathbf{w}_s)\right],$$

where $\mathbf{w}_s = (Y_c(\mathbf{w}_c))_{c: I_c \in \mathcal{C}(I_s)}$.

- The player function $\mathcal{P} : \mathcal{H} \to \mathcal{I} \cup \{I_s\}$ is $\mathcal{P}(H_t) = \{I_{(t,k)} : I_{(t,k)} \in \mathcal{I}\}$ and $\mathcal{P}(\emptyset) = I_s$.

We next state an important property of mechanisms in $\mathcal{M}^{\text{SP}}$ or $\mathcal{M}_s$. For any feasible allocation function of mechanisms in $\mathcal{M}^{\text{SP}}$ or $\mathcal{M}_s$, payments have a second-price like structure. Let $P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) = \inf\{w \in \mathbb{R} : Q_{\ell,c}(w, \boldsymbol{\omega}_{\ell,-c}) = 1\}$ be the minimum report of downstream agent I_c to her upstream agent I_ℓ that guarantees winning against indirect reports $\boldsymbol{\omega}_{\ell,-c} \in \mathcal{W}_{\ell,-c}$. If $\mathcal{W}_{\ell,-c} = \emptyset$ (e.g., I_ℓ is a seller with a single downstream agent), then we denote by $P_{\ell,c} = \inf\{w \in \mathbb{R} : Q_{\ell,c}(w) = 1\}$ the same amount. This quantity is well-defined because the allocation is non-decreasing by (18) (see Proposition J.1).

Proposition J.1. *Let* $(\mathbf{X}_\ell, \mathbf{Q}_\ell)$ *be allocation and payment functions satisfying* (18)*. By the Envelope Theorem, the allocation and payments can be conveniently written as*

$$\begin{aligned} Q_{\ell,c}(\boldsymbol{\omega}_\ell) =& \mathbf{1}\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}, \\ X_{\ell,c}(\boldsymbol{\omega}_\ell) =& X_{\ell,c}(0, \boldsymbol{\omega}_{\ell,-c}) + P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) \mathbf{1}\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}, \end{aligned}$$

where $\boldsymbol{\omega}_\ell = (w_{\ell,c}, \boldsymbol{\omega}_{\ell,-c}) \in \mathcal{W}_\ell$ *is the vector of all indirect reports of intermediary* I_ℓ *and* $w_{\ell,c}$ *is the direct report of downstream agent* $I_c \in \mathcal{C}(I_\ell)$ *to* I_ℓ*.*

Proof. We first prove that the allocation function $Q_{\ell,c}(\boldsymbol{\omega}_\ell)$ is nondecreasing in $w_{\ell,c}$. Writing (18) for v and v', and v' and v; and then summing the inequalities we obtain:

$$(v - v')Q_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) \geq (v - v')Q_{\ell,c}(v', \boldsymbol{\omega}_{\ell,-c}).$$

When $v > v'$, we get $Q_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) \geq Q_{\ell,c}(v', \boldsymbol{\omega}_{\ell,-c})$, vice versa.

Equation (18) implies that $vQ_{\ell,c}(v,\boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(v,\boldsymbol{\omega}_{\ell,-c}) = \max_w vQ_{\ell,c}(w,\boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(w,\boldsymbol{\omega}_{\ell,-c})$. Applying the Envelope Theorem, we obtain that the payment $X_{\ell,c}(v,\boldsymbol{\omega}_{\ell,-c})$ is given as follows:

$$X_{\ell,c}(\boldsymbol{\omega}_\ell) = w_{\ell,c}Q_{\ell,c}(\boldsymbol{\omega}_\ell) - \int_0^{w_{\ell,c}} Q_{\ell,c}(y,\boldsymbol{\omega}_{\ell,-c})\mathrm{d}y + X_{\ell,c}(0,\boldsymbol{\omega}_{\ell,-c})\,.$$

Recognizing the definition of $P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) = \inf\{w \in \mathbb{R} : Q_{\ell,c}(w,\boldsymbol{\omega}_{\ell,-c}) = 1\}$ and using the monotonicity of $Q_{\ell,c}(\cdot,\boldsymbol{\omega}_{\ell,-c})$, first we can alternatively express the allocation as $Q_{\ell,c}(\boldsymbol{\omega}_\ell) = 1\{w_{\ell,c} \geq P_\ell^c(\boldsymbol{\omega}_{\ell,-c})\}$, and then the payment as:

$$\begin{aligned} X_{\ell,c}(\boldsymbol{\omega}_\ell) &= w_{\ell,c}1\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\} - \int_0^{w_{\ell,c}} 1\{y \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}\mathrm{d}y + X_{\ell,c}(0,\boldsymbol{\omega}_{\ell,-c})\,, \\ &= X_{\ell,c}(0,\boldsymbol{\omega}_{\ell,-c}) + P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})1\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}\,. \end{aligned}$$ Q.E.D.

Here $X_{\ell,c}(0,\boldsymbol{\omega}_{\ell,-c})$ represents the payment of agent I_c when her value is 0 and $P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})$ represents the additional payment conditional on winning. Individual rationality, see (19), further implies that $X_{\ell,c}(0,\boldsymbol{\omega}_{\ell,-c}) \leq 0$, and thus we shall see that in the optimal mechanism this payment is set to 0.

J.2.1. Single-stage Intermediation We start by considering the problem of a single intermediary positioned between multiple buyers and a seller. We will establish that in this setting the optimal mechanism of the intermediary can be explicitly characterized in terms of the virtual value functions of buyers' values and their projections onto nonnegative numbers.

LEMMA J.1. *Suppose that an intermediary, I_ℓ, has an upstream seller, I_s, which implements a strategy-proof mechanism $(\mathbf{X}_s,\mathbf{Q}_s)$ with a set of indirect reports $\mathcal{W}_s = \mathbb{R}_+ \times \mathcal{W}_{s,-\ell}$, where $\mathcal{W}_{s,-\ell}$ is the set of indirect reports of the seller excluding the report of intermediary I_ℓ. Assume that the type of buyer $I_c \in \mathcal{C}(I_\ell)$ is a random variable V_c with strictly increasing virtual value function. Then an optimal mechanism for I_ℓ is given by $(\mathbf{X}_\ell^*,\mathbf{Q}_\ell^*,Y_\ell^*)$, regardless of the distribution of indirect reports in $\mathcal{W}_{s,-\ell}$. The reporting function is*

$$Y_\ell^*(\mathbf{w}_\ell) = \max_{c: I_c \in \mathcal{C}(I_\ell)} \psi_{V_c}(w_{\ell,c})\,,$$

where $\mathbf{w}_\ell = (w_{\ell,c})_{c:I_c\in\mathcal{C}(I_\ell)} \in \mathbb{R}^{C_\ell}$ is a vector of direct reports. The allocation and payment functions are given by

$$\begin{aligned} Q_{\ell,c}^*(\boldsymbol{\omega}_\ell) =& Q_{s,\ell}\left(\psi_{V_c}(w_{\ell,c}),\boldsymbol{\omega}_{s,-\ell}\right)\mathbf{1}\{c = \arg\max_{c':I_{c'}\in\mathcal{C}_\ell} \psi_{V_{c'}}(w_{\ell,c'}) \text{ and } w_{\ell,c} \geq z_{V_c}\}\,, \\ X_{\ell,c}^*(\boldsymbol{\omega}_\ell) =& w_{\ell,c}Q_{\ell,c}^*(\boldsymbol{\omega}_\ell) - \int_0^{w_{\ell,c}} Q_{\ell,c}^*(y,\boldsymbol{\omega}_{\ell,-c})\mathrm{d}y\,, \end{aligned}$$

where $\boldsymbol{\omega}_\ell = (\mathbf{w}_\ell,\boldsymbol{\omega}_{s,-\ell}) \in \mathcal{W}_\ell$ is the vector of indirect reports of I_ℓ and $\boldsymbol{\omega}_{s,-\ell} \in \mathcal{W}_{s,-\ell}$.

Proof. We start our proof by explaining the direct and the indirect reports of the intermediary and the seller.[4] Let $\mathbf{v} = (v_c)_{c:I_c \in \mathcal{C}(I_\ell)}$ be the vector of buyers' values. Since truthful reporting is a dominant strategy, reports of buyers are their values. Therefore, the direct reports of the intermediary are $\mathbf{w}_\ell = \mathbf{v}$. In addition to the set of reports $\mathcal{W}_{s,-\ell}$, the seller also receives reports from intermediary I_ℓ, thus her indirect reports are given by $\boldsymbol{\omega}_s = (w_{s,\ell}, \boldsymbol{\omega}_{s,-\ell}) \in \mathcal{W}_s = \mathbb{R} \times \mathcal{W}_{s,-\ell}$. Note that the set of direct and the set of indirect reports for the seller are the same. The indirect reports of intermediary I_ℓ is her direct reports and the indirect reports of her upstream agent except her own report, so the indirect reports for I_ℓ are $\boldsymbol{\omega}_\ell = (\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) \in \mathcal{W}_\ell = \mathbb{R}^{\mathcal{C}_\ell} \times \mathcal{W}_{s,-\ell}$.

The optimal mechanism design problem of the intermediary can now be stated as follows:

$$
\begin{aligned}
\max_{(\mathbf{X}_\ell, \mathbf{Q}_\ell, Y_\ell)} \ & \mathbb{E}_{\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}} \left[\sum_{c: I_c \in \mathcal{C}_\ell} X_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) - X_{s,\ell}(Y_\ell(\mathbf{v}), \boldsymbol{\omega}_{s,-\ell}) \right] \\
\text{s.t. } & vQ_{\ell,c}(v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) - X_{\ell,c}(v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \geq 0\,, && \forall c, v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}\,, \\
& vQ_{\ell,c}(v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) - X_{\ell,c}(v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \\
& \qquad \geq vQ_{\ell,c}(v', \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) - X_{\ell,c}(v', \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell})\,, && \forall c, v, v', \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}\,, \\
& \sum_{c: I_c \in \mathcal{C}_\ell} Q_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) \leq Q_{s,\ell}(Y_\ell(\mathbf{v}), \boldsymbol{\omega}_{s,-\ell})\,, && \forall \mathbf{v}, \boldsymbol{\omega}_{s,-\ell}\,, \\
& Y_\ell(\mathbf{v}) \geq 0\,, && \forall \mathbf{v}\,.
\end{aligned}
$$

Here, the first two constraints are (18) and (19), and they guarantee that a feasible mechanism for this optimization problem is $(\mathbf{X}_\ell, \mathbf{Q}_\ell, Y_\ell) \in \mathcal{M}^{\text{SP}}$. The third constraint implies that the intermediary allocates the impression only if she acquires it from the seller. Finally, the last constraint guarantees that the report of the intermediary is nonnegative. The expectation in the objective is taken w.r.t. the values of the buyers (direct reports), $\mathbf{v}$, and the set of indirect reports of the seller excluding the report of the intermediary, $\boldsymbol{\omega}_{s,-\ell}$.

Following an identical approach to the proof of Proposition J.1, it can be seen that (18) and (19) can be replaced by the following conditions:

$$
X_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) = X_{\ell,c}(0, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) + v_c Q_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) - \int_0^{v_c} Q_{\ell,c}(y, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \mathrm{d}y\,,
$$

$$
Q_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) \in \{0,1\} \text{ and } Q_{\ell,c}(v_c, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \text{ nondecreasing in } v_c\,.
$$

[4] Here, the seller can also be thought of as an upstream intermediary with a strategy-proof mechanisms and a set of indirect reports.

Observe that (19) implies that $X_{\ell,c}(0,\mathbf{v}_{-c},\boldsymbol{\omega}_{s,-\ell}) \le 0$. When the payment $X_{\ell,c}(\mathbf{v},\boldsymbol{\omega}_{s,-\ell})$ is eliminated from the objective function, the objective function is maximized at $X_{\ell,c}(0,\mathbf{v}_{-c},\boldsymbol{\omega}_{s,-\ell}) = 0$ for all $c \in \mathcal{C}(I_\ell)$. Therefore, in the remainder of the proof, we set $X_{\ell,c}(0,\mathbf{v}_{-c},\boldsymbol{\omega}_{s,-\ell}) = 0$.

Reformulated Problem: We can equivalently reformulate the mechanism design problem of the intermediary by replacing the payment in the objective function, and changing the order of integration. Hence, we obtain a new objective function and remove the constraints on the payment function. The resulting problem is:

$$\begin{aligned}
\max_{(\mathbf{Q}_\ell, Y_\ell)} \ & \mathbb{E}_{\mathbf{v},\boldsymbol{\omega}_{s,-\ell}}\left[\sum_{I_c \in \mathcal{C}_\ell} \phi_{v_c}(v_c) Q_{\ell,c}(\mathbf{v},\boldsymbol{\omega}_{s,-\ell}) - X_{s,\ell}(Y_\ell(\mathbf{v}),\boldsymbol{\omega}_{s,-\ell})\right] \\
\text{s.t. } & Q_{\ell,c}(\mathbf{v},\boldsymbol{\omega}_{s,-\ell}) \in \{0,1\} \text{ and } Q_{\ell,c}(v_c,\mathbf{v}_{-c},\boldsymbol{\omega}_{s,-\ell}) \text{ nondecreasing in } v_c\,, && \forall c,\mathbf{v},\boldsymbol{\omega}_{s,-\ell}\,, \\
& \sum_{c: I_c \in \mathcal{C}_\ell} Q_{\ell,c}(\mathbf{v},\boldsymbol{\omega}_{s,-\ell}) \le Q_{s,\ell}(Y_\ell(\mathbf{v}),\boldsymbol{\omega}_{s,-\ell})\,, && \forall \mathbf{v},\boldsymbol{\omega}_{s,-\ell}\,, \\
& Y_\ell(\mathbf{v}) \ge 0\,, && \forall \mathbf{v}\,.
\end{aligned}$$

Point-wise Optimization: Momentarily relaxing the monotonicity constraint on the allocation and maximizing the integrand point-wise over the buyers' values $\mathbf{v}$ and $\boldsymbol{\omega}_{s,-\ell}$, we obtain

$$\begin{aligned}
\max_{(\mathbf{Q}_\ell, Y_\ell)} \ & \sum_{I_c \in \mathcal{C}_\ell} \phi_{V_c}(v_c) Q_{\ell,c} - X_{s,\ell}(Y_\ell,\boldsymbol{\omega}_{s,-\ell}) \\
\text{s.t. } & \sum_{I_c \in \mathcal{C}(I_\ell)} Q_{\ell,c} \le Q_{s,\ell}(Y_\ell,\boldsymbol{\omega}_{s,-\ell})\,, \\
& Y_\ell \ge 0\,.
\end{aligned}$$

Considering the subproblem on $\mathbf{Q}_\ell$ for a given Y_ℓ, it is optimal to allocate to the bidder with the highest virtual value (whenever the highest virtual value is positive) and set allocation to its bound whenever, i.e., $Q_{\ell,c} = Q_{s,\ell}(Y_\ell,\boldsymbol{\omega}_{s,-\ell})\mathbf{1}\{c = \arg\max_{c': I_{c'} \in \mathcal{C}(I_\ell)} \phi_{V_{c'}}(v_{c'}) \text{ and } v_c \ge z_{V_c}\}$. Using this observation, the point-wise optimization problem is written as:

$$\max_{Y_\ell \ge 0} \left\{ \left(\max_{c: I_c \in \mathcal{C}(I_\ell)} \phi_{V_c}(v_c) \right)^+ Q_{s,\ell}(Y_\ell,\boldsymbol{\omega}_{s,-\ell}) - X_{s,\ell}(Y_\ell,\boldsymbol{\omega}_{s,-\ell}) \right\}.$$

Note that $(\mathbf{X}_s, \mathbf{Q}_s)$ is also a strategy-proof mechanism, so $v = \arg\max_{v'} v Q_{s,\ell}(v',\boldsymbol{\omega}_{s,-\ell}) - X_{s,\ell}(v',\boldsymbol{\omega}_{s,-\ell})$. Thus, $Y_\ell = \left(\max_{c': I_{c'} \in \mathcal{C}(I_\ell)} \phi_{V_{c'}}(v_{c'}) \right)^+$ is an optimal solution for the point-wise optimization problem, and thus implying

$$Q_{\ell,c} = Q_{s,\ell}\left(\psi_{V_c}(v_c),\boldsymbol{\omega}_{s,-\ell}\right) \mathbf{1}\{c = \arg\max_{c': I_{c'} \in \mathcal{C}_\ell} \psi_{V_{c'}}(v_{c'}) \text{ and } v_c \ge z_{V_c}\}\,.$$

By using this optimal solution of point-wise optimization, we can construct an optimal solution for the reformulated problem, and hence for the mechanism design problem of the intermediary.

$$Y_\ell^*(\mathbf{v}) = \max_{c': I_{c'} \in \mathcal{C}(I_\ell)} \psi_{V_{c'}}(v_{c'}),$$

$$Q_{\ell,c}^*(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) = Q_{s,\ell}\left(\psi_{V_c}(v_c), \boldsymbol{\omega}_{s,-\ell}\right) \mathbf{1}\{c = \arg\max_{c': I_{c'} \in \mathcal{C}(I_\ell)} \psi_{V_{c'}}(v_{c'}) \text{ and } v_c \geq z_{V_c}\}.$$

These reporting and allocation functions establish an optimal solution for the reformulated mechanism design problem of the intermediary. First, we consider feasibility. In particular, since the value V_c of a buyer is assumed to have a continuous positive density $g_{V_c}(\cdot) > 0$, finite expected value $\mathbb{E}[V_c] < \infty$, and a strictly increasing virtual value function, it follows that the associated projected virtual value function is well defined. Moreover, the projected virtual value function is nondecreasing because the virtual value function is strictly increasing. Since, $Q_{s,\ell}(w_{s,\ell}, \boldsymbol{\omega}_{s,-\ell})$ is also nondecreasing in $w_{s,\ell}$ by Proposition J.1, it follows that $Q_{\ell,c}^*(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell})$ is nondecreasing in v_c. The other constraints in the reformulated problem are clearly satisfied. Second, we consider optimality. Since this solution maximizes the integrand in the objective function of the reformulated problem point-wise, it follows that this solution is also optimal. Finally, point-wise optimality implies that the solution is optimal regardless of the distribution of indirect reports in $\mathcal{W}_{s,-\ell}$. We derive the payment function by using the characterization obtained in Proposition J.1. Q.E.D.

The reporting function of the mechanism derived here, $Y_\ell^*(\cdot)$, does not depend on the distribution of the indirect reports in the mechanism of the seller, and only is a function of the downstream report distribution when the upstream seller also implements a strategy-proof mechanism.

The mechanism $(\mathbf{X}_\ell^*, \mathbf{Q}_\ell^*, Y_\ell^*)$ can be further simplified by considering the mechanism of the seller. The following corollary provides an alternative characterization.

Corollary J.1. *The mechanism characterized in Lemma J.1 can equivalently be expressed as follows:*

$$Y_\ell^*(\mathbf{w}_\ell) = \max_{c: I_c \in \mathcal{C}(I_\ell)} \psi_{V_c}(w_{\ell,c}),$$

$$Q_{\ell,c}^*(\mathbf{w}_\ell, \boldsymbol{\omega}_{s,-\ell}) = \mathbf{1}\left\{w_{\ell,c} \geq P_{\ell,c}(\mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell})\right\},$$

$$X_{\ell,c}^*(\mathbf{w}_\ell, \boldsymbol{\omega}_{s,-\ell}) = P_{\ell,c}(\mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell}) \mathbf{1}\left\{v_c \geq P_{\ell,c}(\mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell})\right\},$$

where the minimum winning report function of buyer $I_c \in \mathcal{C}_\ell$ in the intermediary's mechanism is given by

$$P_{\ell,c}(\mathbf{w}_{-c}, \boldsymbol{\omega}_{s,-\ell}) = \psi_{V_c}^{-1}\left(\max\left(\max_{c': I_{c'} \in \mathcal{C}(I_\ell) \setminus \{I_c\}} \psi_{V_{c'}}(w_{\ell,c'}), P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})\right)\right),$$

where $P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})$ is the minimum winning report function of I_ℓ in the seller's mechanism.

Proof. Since the seller's mechanism is also strategy-proof, by Proposition J.1, it follows that $Q_{s,\ell}(w,\boldsymbol{\omega}_{s,-\ell}) = 1\{w \geq P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})\}$ where $P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell}) = \inf\{w | Q_{s,\ell}(w,\boldsymbol{\omega}_{s,-\ell}) = 1\}$. Using this observation, the allocation function of intermediary I_ℓ is given by

$$Q^*_{\ell,c}(\mathbf{w}_\ell,\boldsymbol{\omega}_{s,-\ell}) = Q_{s,\ell}\left(\max_{c':I_{c'}\in C_\ell} \psi_{V_{c'}}(w_{\ell,c'}),\boldsymbol{\omega}_{s,-\ell}\right)\mathbf{1}\{c = \operatorname*{arg\,max}_{c':I_{c'}\in\mathcal{C}(I_\ell)} \psi_{V_{c'}}(w_{\ell,c'}) \text{ and } w_{\ell,c}\geq z_{V_c}\}$$

$$= \mathbf{1}\{\psi_{V_c}(w_{\ell,c}) \geq P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})\}\mathbf{1}\{c = \operatorname*{arg\,max}_{c':I_{c'}\in\mathcal{C}(I_\ell)} \psi_{V_{c'}}(w_{\ell,c'}) \text{ and } w_{\ell,c}\geq z_{V_c}\}\,.$$

Recognizing that $P_{\ell,c}(\mathbf{w}_{\ell,-c},\boldsymbol{\omega}_{s,-\ell}) = \inf\{w\in\mathbb{R} : Q^*_{\ell,c}(w,\mathbf{w}_{\ell,-c},\boldsymbol{\omega}_{s,-\ell}) = 1\}$, the minimum amount which guarantees the winning of agent I_c is given as

$$P_{\ell,c}(\mathbf{w}_{\ell,-c},\boldsymbol{\omega}_{s,-\ell}) = \psi^{-1}_{V_c}\left(\max\left(\max_{c':I_{c'}\in\mathcal{C}(I_\ell)\setminus\{I_c\}} \psi_{V_{c'}}(w_{\ell,c'}), P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})\right)\right)\,.$$

By this observation, the allocation is $Q^*_{\ell,c}(\mathbf{w}_\ell,\boldsymbol{\omega}_{s,-\ell}) = \mathbf{1}\{w_{\ell,c}\geq P_{\ell,c}(\mathbf{w}_{\ell,-c},\boldsymbol{\omega}_{s,-\ell})\}$. Q.E.D.

J.2.2. Multi-stage Intermediation We next introduce the *anticipated reports* and state our equilibrium characterization. Note that the definition of anticipated reports provided in Section 3 is a special case for **k**-trees. Here, we provide a general definition.

DEFINITION J.2. Let I_ℓ be an agent connected to an upstream agent I_u, i.e., $I_u = \mathcal{U}(I_\ell)$. If I_ℓ is a buyer, her anticipated report is $W_{u,\ell} = V_\ell$. If I_ℓ is an intermediary, her anticipated report is

$$W_{u,\ell} = \max_{c:I_c\in\mathcal{C}(I_\ell)} \psi_{W_{\ell,c}}(W_{\ell,c})\,.$$

It can be seen that the anticipated report $W_{u,\ell}$ coincides with the report of agent I_ℓ to her upstream agent I_u if all intermediaries use the (maximum of) projected virtual value functions of the downstream bids to determine their bids to the upstream mechanism and buyers report their values truthfully.

The next assumption imposes that the anticipated reports of all agents along the tree have strictly increasing virtual values.

ASSUMPTION J.1. *The anticipated reports $W_{u,\ell}$ for every agent $I_\ell\in\mathcal{B}\cup\mathcal{I}$ where $I_u = \mathcal{U}(I_\ell)$ are well-defined and have finite expected values. Moreover, for all intermediaries, i.e., $I_\ell\in\mathcal{I}$, the anticipated reports have strictly increasing virtual values, i.e., $\phi_{W_{u,\ell}}(\cdot)$ (or $\phi_{W_{u,\ell}|W_{u,\ell}>0}(\cdot)$ if $W_{u,\ell}$ has an atom at zero) is strictly increasing.*

When Assumption J.1 is satisfied, an SPE of the game between intermediaries and the seller can be characterized by applying backward induction starting from the last tier of a tree network. As before, due to the incentive compatible nature of the mechanisms, an intermediary along the network is not influenced by the choice of upstream mechanisms, and in turn her mechanism does not influence downstream mechanisms. Hence, each intermediary focuses on optimizing her profits based on the anticipated reports of the downstream agents, which coincide with the reports induced by the fixed (optimal) mechanisms along the equilibrium path.

Using H_ℓ to denote the history of upstream mechanisms intermediary I_ℓ observes when she chooses her mechanism, the following theorem formally characterizes an SPE for tree networks under Assumption J.1. Recall that all intermediaries in the same tier choose their mechanisms simultaneously. Thus, the history H_ℓ at which intermediary I_ℓ chooses her mechanism consists of mechanisms of intermediaries in all upper tiers (including intermediaries that do not lie on the path between I_ℓ and the seller), and this history is also common to other intermediaries in the same tier.

THEOREM J.1. *Suppose that Assumption J.1 holds. Let s^* be a strategy profile such that for history $H_\ell \in \mathcal{H}_\ell$ the mechanism of intermediary I_ℓ is $s^*_\ell(H_\ell) = (\mathbf{X}^*_\ell, \mathbf{Q}^*_\ell, Y^*_\ell)$. The reporting function is given by*

$$Y^*_\ell(\mathbf{w}_\ell) = \max_{c: I_c \in \mathcal{C}(I_\ell)} \psi_{W_{\ell,c}}(w_{\ell,c}),$$

where $W_{\ell,c}$ is the anticipated report of I_c, and $\mathbf{w}_\ell = (w_{\ell,c})_{c:I_c\in\mathcal{C}(I_\ell)}$ is the vector of direct reports of intermediary I_ℓ. The allocation and payment functions are given by

$$\begin{aligned} Q^*_{\ell,c}(\boldsymbol{\omega}_\ell) =& Q^*_{u,\ell}\left(\psi_{W_{\ell,c}}(w_{\ell,c}), \boldsymbol{\omega}_{u,-\ell}\right) \mathbf{1}\{c = \operatorname*{arg\,max}_{c': I_{c'} \in \mathcal{C}(I_\ell)} \psi_{W_{\ell,c'}}(w_{\ell,c'}) \text{ and } w_{\ell,c} \geq z_{W_{\ell,c}}\}, \\ X^*_{\ell,c}(\boldsymbol{\omega}_\ell) =& w_{\ell,c} Q^*_{\ell,c}(\boldsymbol{\omega}_\ell) - \int_0^{w_{\ell,c}} Q^*_{\ell,c}(t, \boldsymbol{\omega}_{\ell,-c}) \mathrm{d}t, \end{aligned}$$

where $I_u = \mathcal{U}(I_\ell)$ and $\boldsymbol{\omega}_\ell = (\mathbf{w}_\ell, \boldsymbol{\omega}_{u,-\ell}) \in \mathcal{W}_\ell$ is the vector of indirect reports. For the seller I_s, the reporting function is omitted, and $s^*_s(\emptyset) = (\mathbf{X}^*_s, \mathbf{Q}^*_s)$ where

$$\begin{aligned} Q^*_{s,c}(\boldsymbol{\omega}_s) =& \mathbf{1}\{c = \operatorname*{arg\,max}_{c': I_{c'} \in \mathcal{C}(I_s)} \psi_{W_{s,c'}}(w_{s,c'}) \text{ and } w_{s,c} \geq z_{W_{s,c}})\}, \\ X^*_{s,c}(\boldsymbol{\omega}_s) =& w_{s,c} Q^*_{s,c}(\boldsymbol{\omega}_s) - \int_0^{w_{s,c}} Q^*_{s,c}(t, \boldsymbol{\omega}_{s,-c}) \mathrm{d}t, \end{aligned}$$

where $\boldsymbol{\omega}_s$ are indirect reports for the seller.

Then s^* constitutes an SPE of the game among intermediaries.

Proof. If an intermediary faces an upstream strategy-proof mechanism, her optimal mechanism is characterized by Lemma J.1. Using this result, we can characterize the equilibrium strategies of intermediaries starting from the last tier of a tree network via backward induction.

Base Case. Let H_1 be a history observed by intermediaries $I_{(1,j)} \in \mathcal{I}$. By Assumption J.1, the type of the buyers connected to $I_{(1,j)}$ has strictly increasing virtual value functions. Then, an optimal mechanism is given by Lemma J.1, that is also given in the statement of the theorem.

Inductive Case. Consider any tier t and assume that all lower tiers adopt the equilibrium strategy given in the statement of the theorem. Therefore, an intermediary in tier t, $I_{(t,j)}$ observes that her downstream reports are characterized by Definition J.2, and we also know that the anticipated reports have strictly increasing virtual value functions by Assumption J.1. By these observations, we can invoke Lemma J.1 to characterize the equilibrium strategy of the intermediary. However, for intermediary $I_{(t,j)}$ the anticipated reports coming from the downstream agents potentially have an atom at zero but are continuous elsewhere due to the positive reserve prices. As discussed in our paper, when the value of the anticipated report is zero an intermediary cannot profit from this report. Thus focusing on the strictly positive part is without loss of optimality and it follows that the mechanisms that maximize the expected profit of the intermediaries are given as in the statement of the theorem.

Seller's Mechanism. By backward induction, it follows that the reports received by the seller are as defined in Definition J.2, and thus satisfy Assumption J.1. Recognizing the seminal work of Myerson (1981), the optimal mechanism for the seller can be found. First, note that the set of interim incentive compatible and interim individually rational mechanisms is a larger class than the set of strategy-proof mechanisms. By relaxing strategy-proofness and considering instead the weaker notion of interim IC and IR, we obtain the optimal mechanism. Second, the optimal interim IC, IR mechanism from Myerson (1981) is a strategy-proof mechanism. Furthermore, it coincides with the mechanism given in the statement of the theorem. Q.E.D.

This theorem suggests that in an SPE for tree networks, an intermediary first determines the virtual values of downstream reports, then ranks bids according to their virtual values, and submits the maximum virtual value upstream whenever this quantity is positive. The seller also ranks bids according to their virtual values, and allocated the impression whenever the maximum virtual value is positive and larger than cost of acquiring the impression. Note that the equilibrium mechanisms provided in Theorem J.1 are independent of histories, i.e., $(\mathbf{X}_\ell^*, \mathbf{Q}_\ell^*, Y_\ell^*)$ is independent of H_ℓ. Therefore, the strategy profile s^* can be represented by the set of mechanisms $\{(\mathbf{X}_\ell^*, \mathbf{Q}_\ell^*, Y_\ell^*)\}_{I_\ell \in \mathcal{I}}$ for intermediaries and $(\mathbf{X}_s^*, \mathbf{Q}_s^*)$ for the seller.

Following Theorem J.1 and Proposition J.1, we can express the payment of an intermediary as function of the payment of his upstream agent in case of winning the impression. The following corollary formalizes this result.

Corollary J.2. *Suppose that Assumption J.1 holds, and the seller and intermediaries follow the equilibrium strategies provided in Theorem J.1. Furthermore, assume that intermediary I_c wins the impression from his upstream agent I_ℓ. Let $\hat{P}_{u,\ell}$ denote the payment of I_ℓ to his upstream agent I_u. Then $\hat{P}_{\ell,c}$ the payment of I_c to I_ℓ is given as follows:*

$$\hat{P}_{\ell,c} = \psi^{-1}_{W_{\ell,c}}\left(\max\left(\hat{P}_{u,\ell}, \max_{c':I_{c'}\in\mathcal{C}(I_\ell)\setminus\{I_c\}} \psi_{W_{\ell,c'}}(w_{\ell,c'})\right)\right).$$

Proof. Note that the projection point $z_{W_{u,\ell}}$ is larger than 0 by definition because virtual value functions are below the 45 degree line. This observation implies that $X^*_{u,\ell}(0,\boldsymbol{\omega}_{u,-\ell}) = 0$ for all $\boldsymbol{\omega}_{u,-\ell}$ since $Q^*_{u,\ell}(0,\boldsymbol{\omega}_{u,-\ell}) = 0$ for all $\boldsymbol{\omega}_{u,-\ell}$. Following Proposition J.1, the payment and allocation function of I_u for I_ℓ are given as follows:

$$Q^*_{u,\ell}(\boldsymbol{\omega}_u) = \mathbf{1}\left\{w_{u,\ell} \geq P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right\},$$
$$X^*_{u,\ell}(\boldsymbol{\omega}_u) = P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\mathbf{1}\left\{w_{u,\ell} \geq P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right\}.$$

Replacing $Q^*_{u,\ell}(\boldsymbol{\omega}_u)$ with $\mathbf{1}\{w_{u,\ell} \geq P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\}$ inside $Q^*_{\ell,c}(\boldsymbol{\omega}_\ell)$ and merging indicator functions, we can rewrite the allocation and the payment functions of I_ℓ for I_c provided in Theorem J.1 as follows:

$$Q^*_{\ell,c}(\boldsymbol{\omega}_\ell) = \mathbf{1}\left\{w_{\ell,c} \geq \psi^{-1}_{W_{\ell,c}}\left(\max\left(P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell}), \max_{c':I_{c'}\in\mathcal{C}(I_\ell)\setminus\{I_c\}} \psi_{W_{\ell,c'}}(w_{\ell,c'})\right)\right)\right\}$$
$$X^*_{\ell,c}(\boldsymbol{\omega}_\ell) = \psi^{-1}_{W_{\ell,c}}\left(\max\left(P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell}), \max_{c':I_{c'}\in\mathcal{C}(I_\ell)\setminus\{I_c\}} \psi_{W_{\ell,c'}}(w_{\ell,c'})\right)\right) Q^*_{\ell,c}(\boldsymbol{\omega}_\ell).$$

Here, we can drop the projection point $z_{W_{\ell,c}} = \psi^{-1}_{W_{\ell,c}}(0)$ inside the indicator function in $Q^*_{\ell,c}(\boldsymbol{\omega}_\ell)$ because the monotonicity of $\psi_{W_{\ell,c}}(\cdot)$ (which is assumed in Assumption J.1) and $P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell}) \geq 0$ implies $\psi^{-1}_{W_{\ell,c}}(P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})) \geq \psi^{-1}_{W_{\ell,c}}(0)$.

Assume that the report realizations $\boldsymbol{\omega}$ over the network are such that I_ℓ and I_c win the impression from their upstream mechanisms. Let $\hat{P}_{u,\ell} = P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})$. By the definition of $X^*_{u,\ell}(\boldsymbol{\omega}_u)$, intermediary I_ℓ pays $\hat{P}_{u,\ell}$ to I_u. Using the expression derived for $X^*_{\ell,c}(\boldsymbol{\omega}_\ell)$, it follows that $\hat{P}_{\ell,c}$, the payment of intermediary I_c to I_ℓ, is given by

$$\hat{P}_{\ell,c} = \psi^{-1}_{W_{\ell,c}}\left(\max\left(\hat{P}_{u,\ell}, \max_{c':I_{c'}\in\mathcal{C}(I_\ell)\setminus\{I_c\}} \psi_{W_{\ell,c'}}(w_{\ell,c'})\right)\right). \qquad \text{Q.E.D.}$$

Finally, we discuss that for a **k**-tree, the equilibrium characterization provided in Theorem J.1 coincides with the one in Theorem 1.

Corollary J.3. *Consider the game in Definition J.1 for a* $\boldsymbol{k}$*-tree where buyers' values are independent and identically distributed. Suppose that Assumption J.1 holds. Then, mechanisms* $\{(\mathbf{X}^*_\ell, \mathbf{Q}^*_\ell, Y^*_\ell)\}_{I_\ell \in \mathcal{I}}$ *(for intermediaries) and* $(\mathbf{X}^*_s, \mathbf{Q}^*_s)$ *(for the seller) which constitute the SPE in Theorem J.1 are equivalent to the mechanisms provided in Theorem 1.*

Proof. First, we consider the mechanisms of intermediaries. By symmetry of a $\mathbf{k}$-tree, the anticipated reports of the agents in the same tier follow the same distribution, and hence they have the same virtual value function. Therefore, comparing the virtual values is equivalent to directly comparing bids, and the reporting function can be written as:

$$Y^*_\ell(\mathbf{w}_\ell) = \max_{I_c \in \mathcal{C}(I_\ell)} \psi_{W_{\ell,c}}(w_{\ell,c}) = \psi_{W_{\ell,c}}\left(\max_{I_c \in \mathcal{C}(I_\ell)} w_{\ell,c}\right).$$

Next, we consider payments and allocations. By Corollary J.1, the equilibrium mechanisms can alternatively be expressed as follows:

$$Q^*_{\ell,c}(\boldsymbol{\omega}_\ell) = \mathbf{1}\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\},$$
$$X^*_{\ell,c}(\boldsymbol{\omega}_\ell) = P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\mathbf{1}\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}.$$

where $P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) = \psi^{-1}_{W_{\ell,c}}\left(\max\left(\max_{I_{c'} \in \mathcal{C}_\ell \setminus \{I_c\}} \psi_{W_{\ell,c'}}(w_{\ell,c'}), P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right)\right)$ and $P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})$ is the payment of I_ℓ to her upstream intermediary, $I_u \in \mathcal{U}(I_\ell)$.[5] Using that the anticipated reports are identically distributed for all agents in the same tier we obtain

$$P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) = \max\left(\max_{I_{c'} \in \mathcal{C}_\ell \setminus \{I_c\}} w_{\ell,c'}, \psi^{-1}_{W_{\ell,c}}\left(P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right)\right).$$

Note that this payment is the same as the payment with the reporting function and the reserve price of Theorem 1.

Also note that the intermediary allocates the impression to the downstream agent $I_c \in \mathcal{C}(I_\ell)$ when $w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})$ which corresponds to the case that intermediary I_ℓ acquires the impression from her upstream agent and an agent I_c is the winner. Then, it follows that the mechanism of intermediary I_ℓ is the same as the mechanism provided in Theorem 1.

[5] In Corollary J.1, we have $P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})$ instead of $P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})$ because the intermediary connects to the seller. Here, the intermediary could connect either to another intermediary or the seller. However, the result would still follow because the upstream mechanism in either case would be a strategy-proof mechanism.

Second, we consider the seller. Using the fact that the virtual values are same for agents in the same tier, the seller's mechanism is given by

$$Q^*_{s,c}(\boldsymbol{\omega}_s) = \mathbf{1}\left\{ w_{s,c} \geq \max\left(\max_{c': I_{c'} \in \mathcal{C}_s \setminus \{I_c\}} w_{s,c'}, z_{W_{s,c}} \right) \right\},$$

$$X^*_{s,c}(\boldsymbol{\omega}_s) = \begin{cases} \max\left(\max_{c': I_{c'} \in \mathcal{C}_s \setminus \{I_c\}} w_{s,c'}, z_{W_{s,c}} \right) & \text{if } w_{s,c} \geq \max\left(\max_{c': I_{c'} \in \mathcal{C}_s \setminus \{I_c\}} w_{s,c'}, z_{W_{s,c}} \right), \\ 0 & \text{otherwise.} \end{cases}$$

This mechanism is simply a second-price auction with an optimal reserve price of $z_{W_{s,c}}$ for downstream agent $I_c \in \mathcal{C}(I_s)$, which is the same as the seller's mechanism in Theorem 1. Q.E.D.

This corollary implies that the class $\mathcal{M}$ is optimal within the larger class of strategy-proof mechanisms. Therefore, focusing on second-price mechanisms for **k**-trees is without loss of optimality within the larger set of strategy-proof mechanisms.

J.3. Numerical Analysis

In this section, we provide a numerical study conducted to compute the ratio of expected profits of an upstream intermediary and a downstream intermediary in two-tier general tree networks by Monte Carlo simulation. We randomly generate tree networks by drawing random numbers from a Poisson distribution to determine the number downstream connections of each agent in tiers 3, 2 and 1 (e.g., see Figure 8). For a given network, we compute the sample profits that would be obtained when advertisers' values are drawn from GPDs, and agents use the equilibrium strategies in Theorem J.1.

Random Network Generation. We first draw the value of X_3 from a Poisson distribution with mean λ, and set the seller's number of downstream connections to $\max(1, X_3)$. We project the value of X_3 at 1 in order to guarantee that there exists at least one intermediary in tier 2. For each agent in tier 2, we draw the values of $\{X_{2,i}\}_{i=1}^{\max(1,X_3)}$ from the same Poisson distribution to determine their number of downstream connections. Similarly to the seller's number of downstream connections, we set the number of downstream connections of the first intermediary in tier 2 to $\max(1, X_{2,1})$ in order to guarantee that there exists at least one intermediary in tier 1, and we set the number of downstream connections of the remaining agents in tier 2 to $X_{2,i}$. We denote the total number of agents in tier 1 by $\bar{X}_2 = \max(1, X_{2,1}) + \sum_{j \neq 1} X_{2,j}$. Finally, we draw the values of $\{X_{1,i}\}_{i=1}^{\bar{X}_2}$ and we set the number of downstream connections of the first intermediary in tier 1 to $\max(1, X_{1,1})$. The agents in tier 0 and the agents (except $I_{(2,1)}$ and $I_{(1,1)}$) in tier 2 and tier 1 with no downstream connections are buyers.

An example of a randomly generated network is illustrated in Figure 8. In this example, the seller has two downstream connections, i.e, $X_3 = 2$. Therefore, there are two agents in tier 2. The first agent in tier 2 has two downstream connections, $X_{2,1} = 2$, and the second agent has one, $X_{2,2} = 1$. Because these agents have downstream connections, they are intermediaries. In tier 3, there are three agents. For these three agents, we draw $\{X_{1,i}\}_{i=1}^3$ whose values are $X_{1,1} = 2$, $X_{1,2} = X_{1,3} = 0$. Observe that drawing zero for $X_{1,2}$, $X_{1,3}$ implies that the second and third agents in that tier are buyers. Finally, we add two buyers to tier 0 because the intermediary in tier 1 has two downstream connections.

Simulation Steps. In our simulation, we first randomly generate a network as explained above. For each network, we simulate the following sequence of events where agents use the equilibrium strategies provided in the previous section. In particular, buyers report their values truthfully, and intermediaries and the seller use the mechanisms provided in Theorem J.1.

1. Buyers' values are drawn.
2. If an agent $\{I_{(t,j)}\}_j$ in tier t is
   (a) a buyer, she bids her value truthfully,
   (b) an intermediary, she receives bids from her downstream agents, and submits a report to the upstream tier,

   sequentially from tier $t = 0$ up to tier $t = 2$.
3. The seller determines a winner, allocates the impression (if won), and charges payments.
4. If the winning agent in tier t is
   (a) an intermediary, then she determines a downstream winner, allocates the impression, and charges payments as determined by her mechanism, from $t = 2$ down to $t = 1$,
   (b) a buyer, she acquires the impression and pays the amount charged by her upstream agent.

We draw buyers' values from a GPD with parameters $(\xi, \sigma = 1, \mu = 0)$ whose shape parameter ξ takes values from the set $\{-1, 0, 0.5, 0.7\}$. The GPD with $(\xi = -1, \sigma = 1, \mu = 0)$ corresponds to a standard uniform distribution with support $[0, 1]$ and the GPD with $(\xi = 0, \sigma = 1, \mu = 0)$ corresponds to an exponential distribution with mean $\sigma = 1$. When the shape parameter $\xi > 0$, GPDs correspond to shifted Pareto distributions with support $[0, \infty)$. When the support is unbounded, we draw values from truncated distributions. Specifically, we truncate the exponential distribution with mean $\sigma = 1$ to the interval $[0, 20]$, and the shifted Pareto distributions to $[0, 500]$.

Results. We compute the expected profits of $I_{(2,1)}$ and $I_{(1,1)}$ over randomly generated networks for different shape parameter values. For each value of the shape parameter ξ, we generated 100 networks and for each network, the sample size is 3000. The summary of the results are provided in Table 1.

Table 1 The ratio of the expected profits of $I_{(2,1)}$ and $I_{(1,1)}$. Note that the upstream intermediary $I_{(2,1)}$ profits more than the downstream intermediary $I_{(1,1)}$ when the ratio is higher than one.

λ \ ξ	-1	0	0.5	0.7
1	2.8397	1.3252	0.9510	0.5303
2	4.2290	2.1390	1.2315	0.6982
3	5.7571	3.0483	1.7132	0.8782

As the shape parameter ξ increases (i.e., as the tail of the value distribution gets heavier), the ratio of expected profits decreases and eventually the downstream intermediary makes more profit, i.e., the ratio of the (sample average) profits of an upstream intermediary to that of its downstream intermediary becomes smaller than one. In networks with a larger number of expected downstream connections (i.e., when λ is larger), a larger shape parameter is required for downstream intermediaries to be more profitable (similarly to the sufficient condition required in Proposition 2). For example, the shape parameter $\xi = 0.5$ is large enough when $\lambda = 1$; however, a larger shape parameter such as $\xi = 0.7$ (i.e., a more heavy-tailed distribution) is needed when $\lambda = 3$ in order for the downstream intermediary to make more profit.

We also consider the case where we generate random graphs by using different λ values for different tiers.

Table 2 The ratio of the expected profits of $I_{(2,1)}$ and $I_{(1,1)}$

λ \ ξ	-1	0	0.5	0.7
$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$	3.9390	2.1341	1.1389	0.6471
$\lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_3 = 1$	4.9936	2.4962	1.5189	1.1317
$\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 2$	7.0868	3.8863	2.5818	1.4037
$\lambda_1 = 3,\ \lambda_2 = 1,\ \lambda_3 = 2$	2.8897	1.4807	0.7124	0.4049
$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 3$	2.5782	1.7290	0.6808	0.4255
$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 1$	6.0734	3.3635	1.4922	0.9776

Although we allow different tiers to have different size distributions, our main insight into the impact of the advertisers' value distribution and the ratio of intermediaries' expected profits holds as illustrated in Table 2. In particular, an increase in the shape parameter ξ decreases the ratio of expected profits.

References

Bresnahan TF, Reiss PC (1985) Dealer and manufacturer margins. *The RAND Journal of Economics* 16(2):253–268.

Feldman J, Mirrokni V, Muthukrishnan S, Pai MM (2010) Auctions with intermediaries: Extended abstract. *Proceedings of the 11th ACM Conference on Electronic Commerce*, 23–32, EC '10 (New York: ACM).

Laffont JJ, Maskin E (1980) Optimal reservation price in the vickrey auction. *Economics Letters* 6(4):309 – 313.

Lariviere MA, Porteus EL (2001) Selling to the newsvendor: An analysis of price-only contracts. *Manufacturing & Service Operations Management* 3(4):293–305.

Myerson R (1981) Optimal auction design. *Mathematics of Operations Research* 6(1):58–73.

Nisan N, Roughgarden T, Tardos E, Vazirani VV (2007) *Algorithmic game theory*, volume 1 (Cambridge University Press Cambridge).

Osborne MJ, Rubinstein A (1994) *A course in game theory* (MIT press).

Rinne H (2014) *The Hazard rate : Theory and inference (with supplementary MATLAB-Programs)* (Justus-Liebig-Universitt), URL `http://geb.uni-giessen.de/geb/volltexte/2014/10793`.

Ross SM (1996) *Stochastic Processes* (John Wiley and Sons).