Technical Note for

"Multi-stage Intermediation in Display Advertising"

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Abstract

In this technical report, we first characterize an optimal mechanism for intermediaries which are positioned between their captive buyers and a seller who runs a second-price auction when there is no restrictions on the set of mechanisms. This technical report also contains an analysis of the game between intermediaries and a seller, who are organized in a general tree network when agents select mechanisms within the class of strategy-proof (appropriately adapted to our setting) mechanisms where truthful reporting is a dominant strategy. We show that focusing on second-price mechanisms is i) without loss of optimality for the single-stage intermediation, and ii) without loss of optimality within the set of strategy-proof mechanisms for **k**-trees.

Keywords: Intermediary problems, mechanism design, Internet advertising, extensive form games, second-price auction, multi-stage intermediation.

1 Introduction

In this report, we first consider a general mechanism design problem without restriction on the set of mechanisms for an intermediary that participates in a second-price auction on behalf of her multiple buyers whose values are private. Using the result for one intermediary as a building block, we show that focusing on second-price mechanisms for single-stage intermediation with symmetric buyers is without loss of optimality.

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For multi-stage intermediation, we characterize an equilibrium of the game between intermediaries and a seller, who are organized in a general tree network (see Figure 2), when agents select mechanisms within the class of strategy-proof mechanisms (appropriately adapted to our setting). We show that the second-price mechanisms introduced in Balseiro et al. (2017) are optimal for **k**-trees within the class of strategy-proof mechanisms. Therefore, focusing on second-price mechanisms for **k**-trees is without loss of optimality within the set of strategy-proof mechanisms.

The remaining of the report is organized as the following. In Section 2, we first provide relevant definitions that would be used in the following sections. In Section 3, we consider a general mechanism design problem without restrictions on the set of mechanisms for single-stage intermediation. In Section 4, we consider multi-stage intermediation in general trees within the class of strategy-proof mechanisms.

2 Preliminaries

Before stating our results we first provide some useful definitions. Throughout this section, we denote by X a generic random variable with cumulative distribution $G_X(\cdot)$ and continuous positive density $g_X(\cdot) > 0$ over a nonnegative support $\mathcal{X} \subset [0, \infty)$. We also assume that the expected value of X is finite, i.e. $\mathbb{E}[X] < \infty$.[1] Our first definition introduces the virtual value function of X and its inverse.

Definition 1. *The* virtual value function *of random variable X is given by*

$$\phi_X(x) \triangleq x - \frac{1 - G_X(x)}{g_X(x)},$$

for $x \in \mathcal{X}$. Moreover, the inverse of the virtual value function is defined as $\phi_X^{-1}(w) \triangleq \inf\{x \in \mathcal{X} \mid \phi_X(x) \geq w\}$.

Our second definition introduces the *projected virtual value function.* This function is defined for random variables with strictly increasing virtual value functions and it projects an (increasing) virtual value function to nonnegative reals, while extending its domain to $\mathbb{R}$.

Definition 2. *Suppose X is absolutely continuous, and $\phi_X(\cdot)$ is strictly increasing. The* projected

[1] In the following sections, buyers' valuations are assumed to satisfy these requirements.

virtual value function *of random variable* X *is given by*

$$\psi_X(x) = \begin{cases} \sup \mathcal{X} & x \geq \sup \mathcal{X}\,, \\ \phi_X(x) & z_X \leq x < \sup \mathcal{X}\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{1}$$

where the projection point *is given by* $z_X = \phi_X^{-1}(0)$*. If the random variable* X *has an atom at zero and is absolutely continuous elsewhere in its support* $\mathcal{X}$*, then we define* $\psi_X(\cdot)$ *and* z_X *similarly by replacing* $\phi_X(\cdot)$ *in* (1) *with the virtual value* $\phi_{X|X>0}(\cdot)$ *of the strictly positive part of* X*, denoted by* $X|X > 0$.[2]

When the random variable X has finite expected values and strictly increasing virtual values, it is not hard to see that the projection point z_X is well-defined and positive because the projected virtual value function is continuous and $\lim_{t \to \sup \mathcal{X}} \phi_X(t) = \sup \mathcal{X}$ (see Lemma 5 in Balseiro et al. (2017)). Definition 2 extends the projected virtual value function to these cases by defining it through the conditional random variable $X|X > 0$ which satisfies absolute continuity. This guarantees well-definedness of the projected virtual value function since the virtual value function is defined for random variables with distributions that are absolutely continuous. For equilibrium characterization it suffices for an intermediary to restrict attention to the positive part $X|X > 0$ of the downstream reports, because when the downstream report is $X = 0$ the intermediary cannot profit by acquiring the impression from the upstream agent.

3 Single-stage Intermediation

In this section, we consider single-stage intermediation. Specifically, we consider the mechanism design problem of intermediaries positioned between a seller who runs a second-price auction and buyers whose values for the item are private. The value distributions of the buyers are common knowledge and satisfy the conditions in Section 2. We also allow for the possibility that other exogenous agents, i.e., agents other than intermediaries and their downstream buyers, can participate in the mechanism of the seller. As in Balseiro et al. (2017), we model these as random competing bids at the sellers mechanism. The

[2] Note that $\phi_X(x) = \phi_{X|X>0}(x)$ for $x \in \mathcal{X} \setminus \{0\}$. This can be seen by using Definition 1 and noting that the conditional random variable $X|X > 0$ has c.d.f. $G_{X|X>0}(x) = (G_X(x) - G_X(0))/(1 - G_X(0))$, and p.d.f. $g_{X|X>0}(x) = g_X(x)/(1 - G_X(0))$. Thus, focusing on the virtual value of $X|X > 0$ as opposed to X, excludes the atom at zero, without impacting the (projected) virtual values elsewhere.

largest of these bids is denoted by the random variable D, with the c.d.f. $F_D(\cdot)$ and the p.d.f. $f_D(\cdot)$ over the support $\mathcal{D}$. In this setting, each intermediary determines their bids to submit to the auction of the seller on behalf of their buyers, the allocation of the item in case of winning and the payments that would be charged to the buyers.

We first consider a simpler setting with one intermediary and multiple buyers (see Figure 1). Lemma 1 characterizes the optimal mechanism of the intermediary I_ℓ in this setting.

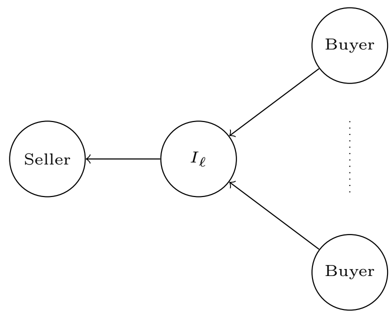

Figure 1: Single intermediary with multiple buyer

Lemma 1. *Consider the network shown in Figure 1 with buyers whose values have increasing virtual value functions. An optimal mechanism for the intermediary I_ℓ is represented by the vector of reserve prices $\mathbf{r}_\ell$ and the vector of reporting functions $\mathbf{Y}_\ell$ where*

$$Y_\ell^i(v) = \psi_{V_i}(v)\,,$$
$$r_\ell^i = z_{V_i}\,,$$

with V_i the random variable that captures the value of buyer i for the impression. The intermediary first ranks buyers according to $\psi_{V_i}(v_i)$ and then reports the maximum $\psi_{V_i}(v_i)$, if greater than zero, to the upstream auction. In case of winning, the intermediary charges a payment to the winning buyer that is equal to the minimum amount which guarantees winning.

In order to capture the simultaneous moves of the intermediaries, we next consider the single stage intermediation with multiple intermediaries. The following proposition formally states that the optimal mechanism in case of a single intermediary in Lemma 1 is still optimal when there are other intermediaries participating in the same upstream mechanism. This can be explained by the fact that the optimal mechanism of the single intermediary is independent of the competing bids in the upstream mechanism.

Proposition 1. *Assume that multiple intermediaries, each with multiple buyers whose values have increasing virtual value functions, participate in a second-price auction run by the seller. Consider the game between intermediaries $I_\ell \in \mathcal{I}$ who choose their mechanisms simultaneously. At a Nash equilibrium of the game among intermediaries, the mechanism of $I_\ell \in \mathcal{I}$ is such that:*

$$
\begin{aligned}
Y_\ell^i(v_i) =& \psi_{V_i}(v_i)\,, \\
r_\ell^i =& z_{V_i}\,,
\end{aligned}
$$

where V_i is the random variable that captures the value distribution of buyer i connected to intermediary I_ℓ.

In addition to characterizing an equilibrium, this proposition also shows that focusing on second-price mechanisms is without loss of optimality in single-stage intermediation with symmetric buyers. Specifically, when all buyers have the same value distribution, i.e., $V_i = V$ for all i, the reporting functions and reserve prices in Proposition 1 are given by $Y_\ell(v) = \psi_V(v)$ and $r_\ell = z_V$, respectively. Therefore, each intermediary first compares the values of her buyers, and reports the projected virtual value evaluated at the maximum report if it is greater than zero to the upstream auction. In case of winning, the payment is determined as the minimum amount which guarantees winning. This mechanism is in the set of second-price mechanisms defined in Balseiro et al. (2017).

4 Multi-stage Intermediation

In this section, we consider multi-stage intermediation in general network structures when the intermediaries are restricted to choose their mechanisms from the set of strategy-proof mechanisms.

4.1 Network Model and Feasible Mechanisms

In comparison to the single-stage intermediation, the network structure and the timing of events are more complex in multi-stage intermediation. Therefore, in this section we start by providing the notation for the network structure and the timing of events. Moreover, we formally define the set of strategy-proof mechanisms which are appropriately adopted to this network structure and the timing of events, and provide examples of the mechanisms in this set.

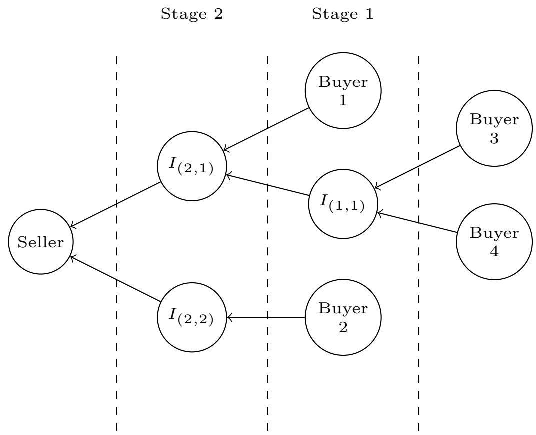

Figure 2: A tree of intermediaries. Here, three intermediaries and four buyers constitute the tree network. The intermediaries $I_{(2,1)}$ and $I_{(2,2)}$ in the upstream tier (tier 2) choose their mechanisms simultaneously. Subsequently, intermediary $I_{(1,1)}$ in tier 1 chooses her mechanism, after observing the mechanisms of $I_{(2,1)}$ and $I_{(2,2)}$. Finally, the values of buyers are realized, and they submit bids to the upstream mechanisms they face.

Network structure and timing of events. A tree network originates from a strategic seller. All leaf nodes in this network correspond to different buyers. Intermediaries are located between buyers and the seller. Each intermediary connects a set of downstream agents (either other intermediaries or buyers) to a single upstream agent which is either the seller or another intermediary. A **k**-tree introduced in Balseiro et al. (2017) is a special case in which all agents in a tier have the same number of downstream agents and buyers' values are identical. Here, the value of buyer $I_\ell \in \mathcal{B}$ is denoted by the random variable V_ℓ. As in Balseiro et al. (2017), we assume that this random variable is absolutely continuous. However, now we allow buyers to have different value distributions. Therefore, V_ℓ is associated with a cumulative distribution function $G_{V_\ell}(\cdot)$, and a probability density function $g_{V_\ell}(\cdot)$ that is strictly positive over $\mathcal{V}_\ell$, the support of V_ℓ. The lower bound of the support is assumed to be at 0, and V_ℓ has finite expected values, i.e., $\mathbb{E}[V_\ell] < \infty$. With some abuse of notation we denote by $\mathcal{C}_\ell = \mathcal{C}(I_\ell)$ the set of downstream agents connecting to I_ℓ and $C_\ell = |\mathcal{C}(I_\ell)|$ the total number of downstream agents.

We model the corresponding game among intermediaries and the seller as an extensive form (Stackelberg) game, where intermediaries move sequentially from upstream to downstream by choosing their mechanisms following the seller. The timing of events is as follows:

1. Each buyer privately draws her type.

2. The seller I_s determines her mechanism.

3. Intermediaries $\{I_{(t,j)}\}_j$ in tier t simultaneously choose their mechanisms, sequentially from tier $t = n$ down to $t = 1$. Specifically, intermediaries in a tier simultaneously choose their mechanisms after observing the mechanisms chosen by upstream tiers.

4. Buyers choose their bidding strategies.

5. If an agent $\{I_{(t,j)}\}_j$ in tier t is:

   (a) a buyer, she bids using her bidding strategy,

   (b) an intermediary, she receives bids from her downstream agents, and submits a report to the upstream tier as determined by her mechanisms,

   sequentially from tier $t = 0$ up to tier $t = n$.

6. The seller, I_s, determines a winner, allocates the impression (if won) and charges payments according to her mechanism.

7. The winning intermediary in tier t, determines a downstream winner, allocates the impression, and charges payments as determined by her mechanism, from $t = n$ down to $t = 1$.

Feasible mechanism. In our problem, we restrict the set of available mechanisms for intermediaries (and the seller) to the set of the strategy-proof mechanisms (appropriately adjusted to our setting) denoted by $\mathcal{M}^{\mathrm{SP}}$. Nisan et al. (2007, p.218) define strategy-proof mechanisms, and discuss that truthful reporting is a dominant strategy under strategy-proof mechanisms (see Nisan et al., 2007, p. 244).

In a setting without intermediation, an agent is not better off reporting any value other than her true "type" when the seller implements a strategy-proof mechanism. When agents implement strategy-proof mechanism in an intermediation network, it is still the case that buyers are better off reporting their true type for the item. Since intermediaries do not inherently value impressions, a priori there is no straightforward interpretation for the "type" of an intermediary and what it means for an intermediary to report her type truthfully. Instead, in intermediation networks, an intermediary's "type" is endogenously determined, at equilibrium, based on the potential revenue she can extract from auctioning the item to a downstream agent. For example, as discussed in Balseiro et al. (2017), adopting Bulow and Roberts (1989) interpretation of the optimal auction as a monopolist's problem in third-degree price discrimination, at equilibrium, we could interpret the marginal revenue of the intermediary as her "type." The mechanisms we consider in this setting naturally extend strategy-proof mechanisms to intermediation networks, allowing one to consider more general mechanisms than the second-price mechanisms introduced in Balseiro et al. (2017).

We proceed by introducing the notation that will be used in the remainder of this report, and formally defining the strategy-proof mechanisms for the intermediaries and the seller.

A mechanism for intermediary I_ℓ is given by the triple $(\mathbf{X}_\ell, \mathbf{Q}_\ell, Y_\ell) \in \mathcal{M}^{\mathrm{SP}}$. As before Y_ℓ denotes the *reporting function* the intermediary uses to map the direct downstream reports to an upstream report. In addition, in the general setting considered in this technical note, we allow the intermediary to choose a *payment function* $\mathbf{X}_\ell$, and an *allocation function* $\mathbf{Q}_\ell$, which possibly do not satisfy the second-price structure introduced in Balseiro et al. (2017). Note that due to the multi-tier intermediation structure, the intermediary can allocate the item downstream if she acquires it from upstream. Thus, her allocation, and possibly the payments, are not only a function of the direct reports the intermediary receives from her immediate downstream agents, but also a function of other upstream reports that impact the intermediary's allocation.[3]

To see this more clearly, for intermediary I_ℓ, consider the (unique) path $\mathcal{T}$ in the underlying intermediation tree that connects I_ℓ to the seller. Consider all the reports received by agents in this path (including I_ℓ and the seller), other than the ones submitted by the intermediaries on this path (which are readily a function of the reports in consideration). We refer to these reports as *indirect reports of* I_ℓ, i.e., indirect reports of I_ℓ are given by $\{w_{j,i} \mid \mathcal{U}(I_i) = I_j \in \mathcal{T}, I_i \notin \mathcal{T}\}$, where $w_{j,i}$ denotes the report of agent I_i to its upstream agent I_j. Observe that indirect reports include direct reports $(w_{\ell,c})_{c:I_c \in C_\ell}$ that I_ℓ receives from her immediate downstream agents, as well as other reports received by agents in $\mathcal{T}$. Note that the latter set of reports also impact the allocation decisions of the intermediary I_ℓ, as a large report in upper tiers potentially diverts the impression away from $\mathcal{T}$ (and hence intermediary I_ℓ), and allocates the impression to a buyer through a different intermediation path. We denote the set of all possible indirect reports of this intermediary by $\mathcal{W}_\ell$, and a particular vector of indirect reports by $\boldsymbol{\omega}_\ell \in \mathcal{W}_\ell$. To make the difference clear, we denote the direct reports intermediary I_ℓ receives by

[3] Note that this also is a feature of the second-price auction mechanisms discussed in Balseiro et al. (2017).

$\mathbf{w}_\ell = (w_{\ell,c})_{c:I_c \in C_\ell} \in \mathbb{R}_+^{C_\ell}$.

Recall that a reporting function of the intermediary maps the direct reports received from immediate lower tier $\mathcal{C}_\ell$ of intermediary I_ℓ to a bid to be submitted to the upstream intermediary's mechanism, i.e., $Y_\ell : \mathbb{R}_+^{C_\ell} \to \mathbb{R}_+$. On the other hand, we allow the contingent payments and allocation to be a function of the remaining indirect reports, i.e., a payment function $\mathbf{X}_\ell : \mathcal{W}_\ell \to \mathbb{R}_+^{C_\ell}$, and an allocation function $\mathbf{Q}_\ell : \mathcal{W}_\ell \to \{0,1\}^{C_\ell}$, respectively, determine the amount charged to downstream agents as well as the allocation based on *all* indirect reports.

It can be seen that any mechanism where (i) contingent upon acquiring the item from upstream, the intermediary bases the allocation decision on direct reports; and (ii) payments are equivalent to the minimum report that guarantees winning belongs to $\mathcal{M}^{\text{sp}}$.[4] Thus, it follows that the second-price mechanisms discussed in Balseiro et al. (2017) are a special subclass of the mechanisms discussed here. Moreover, the class of mechanisms considered here is strictly larger in that we no longer impose (i) and (ii). Other important mechanisms such as posted price mechanisms and ranking-based mechanisms also belong to this class.

Observe that the indirect reports admit an alternative recursive definition, which is more convenient for our exposition. Let I_u be the upstream neighbor of intermediary I_ℓ. We let $\boldsymbol{\omega}_{u,-\ell}$ denote the indirect reports of intermediary I_u excluding the direct report she receives from I_ℓ and $\mathcal{W}_{u,-\ell}$ be the set of such indirect reports. With some abuse of notation we can express the indirect reports of intermediary I_ℓ by $\boldsymbol{\omega}_\ell = (\mathbf{w}_\ell, \boldsymbol{\omega}_{u,-\ell}) \in \mathcal{W}_\ell$ where as before $\mathbf{w}_\ell = (w_{\ell,c})_{c:I_c \in C_\ell} \in \mathbb{R}_+^{C_\ell}$ denotes the direct reports and $\boldsymbol{\omega}_{u,-\ell} \in \mathcal{W}_{u,-\ell}$. Alternatively, we let $\boldsymbol{\omega}_\ell = (w_{\ell,c}, \boldsymbol{\omega}_{\ell,-c}) \in \mathcal{W}_\ell$ where $w_{\ell,c} \in \mathbb{R}_+$ is the direct report of downstream agent $I_c \in C_\ell$, and $\boldsymbol{\omega}_{\ell,-c} \in \mathcal{W}_{\ell,-c}$ denotes the indirect reports of the intermediary I_ℓ excluding the report of agent I_c. Thus, indirect reports of intermediaries along the path $\mathcal{T}$ to the seller are closely related.

We illustrate our definition of the direct/indirect reports by considering intermediary $I_{(1,1)}$ in Figure 2. For this intermediary, it can be readily seen that the direct reports are $\mathbf{w}_{(1,1)} = (v_3, v_4)$ where v_3 and v_4 are the reports submitted by Buyer 3 and Buyer 4, respectively. The indirect reports (except the report of $I_{(1,1)}$) for the upstream intermediary, $I_{(2,1)}$, are $\boldsymbol{\omega}_{(2,1),-(1,1)} = (v_1, w_{s,(2,2)})$ where v_1 is the report of Buyer 1, $w_{s,(2,2)}$ is the report of $I_{(2,2)}$. The indirect reports for $I_{(1,1)}$ are given by

[4]Specifically, for such mechanisms allocation and payment decisions can always be expressed as a function of the indirect reports along path $\mathcal{T}$.

$\boldsymbol{\omega}_{(1,1)} = (v_3, v_4, v_1, w_{s,(2,2)})$.

We next formally define strategy-proof mechanisms using this notation. We say that a mechanism of I_ℓ is strategy-proof if each downstream agent $I_c \in \mathcal{C}_\ell$ is better off reporting her "type" v over some report v' regardless of the indirect reports $\boldsymbol{\omega}_{\ell,-c}$ of the competitors where $(v', \boldsymbol{\omega}_{\ell,-c}) \in \mathcal{W}_\ell$:

$$vQ_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) \geq vQ_{\ell,c}(v', \boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(v', \boldsymbol{\omega}_{\ell,-c}) \qquad \forall v, v', \boldsymbol{\omega}_{\ell,-c}\,. \tag{2}$$

Note that the mechanism is ex-post w.r.t. the reports of competing bidders in upstream mechanisms. The mechanism should also be individually rational:

$$vQ_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) \geq 0 \qquad \forall v, \boldsymbol{\omega}_{\ell,-c}\,. \tag{3}$$

In case of the seller, the set of direct reports and the set of indirect reports are the same and given by $\mathcal{W}_s = \mathbb{R}^{C_s}$. A mechanism for the seller is given by the duple $(\mathbf{X}_s, \mathbf{Q}_s) \in \mathcal{M}_s$ where the payment function $\mathbf{X}_s : \mathcal{W}_s \to \mathbb{R}_+^{C_s}$ and the allocation function $\mathbf{Q}_s : \mathcal{W}_s \to \{0,1\}^{C_s}$ are as before. The set of feasible mechanisms for the seller is also strategy-proof so $(\mathbf{X}_s, \mathbf{Q}_s) \in \mathcal{M}_s$ satisfy the conditions (2) and (3). Since the seller has no reporting function, we denote $\mathcal{M}_s$ different from $\mathcal{M}^{\mathrm{SP}}$.

We now briefly describe some practical mechanisms that fit within the class of strategy-proof mechanisms.

Second-price auctions. Each intermediary I_ℓ runs a second-price auction with reserve price $r_\ell \geq 0$ and submits $Y_\ell\left(\max_{c\in\mathcal{C}_\ell} w_{\ell,c}\right)$ to the upstream intermediary, whenever the maximum bid $\max_{c\in\mathcal{C}_\ell} w_{\ell,c}$ is above the reserve price r_ℓ. Here $Y_\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing reporting function. Let $I_u = \mathcal{U}(I_\ell)$ be the upstream intermediary of I_ℓ and let $P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})$ be the payment of I_ℓ to I_u in case of winning. When the downstream agent c is the winner, I_ℓ charges her the minimum amount that guarantees winning, which is given by $P_{\ell,c} = \max\left(\max_{c'\in\mathcal{C}_\ell\setminus\{I_c\}} w_{\ell,c'}, r_\ell, Y_\ell^{-1}\left(P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right)\right)$.

Ranking-based mechanisms. Each intermediary I_ℓ transforms downstream reports using a bidder specific increasing function $Y_{\ell,c} : \mathbb{R}_+ \to \mathbb{R}_+$, runs a second-price auction on the transformed bids and reports the maximum transformed bid $\max_{c\in\mathcal{C}_\ell} Y_{\ell,c}\left(w_{\ell,c}\right)$ to the upstream intermediary, whenever the maximum quantity is positive. Let $I_u = \mathcal{U}(I_\ell)$ be the upstream intermediary of I_ℓ

and let $P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})$ be the payment of I_ℓ to I_u in case of winning. When the downstream agent c is the winner, I_ℓ charges her the minimum amount that guarantees winning, which is given by $P_{\ell,c} = Y_{\ell,c}^{-1}\left(\max\left(\max_{c' \in \mathcal{C}_\ell \setminus \{I_c\}} Y_{\ell,c'}\left(w_{\ell,c'}\right), P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right)\right)$.

Posted price mechanisms. Each intermediary I_ℓ posts a price $p_\ell \in \mathbb{R}_+$, collects reports from the downstream agents, and allocates the item uniformly at random among the downstream agents with reports above the posted price (whenever the item is won by the intermediary). The intermediary reports upstream a quantity $y_\ell \in \mathbb{R}_+$ whenever some report of a downstream agent is above p_ℓ, and zero otherwise. While the mechanisms described in the previous section are deterministic in nature, i.e., the allocation functions satisfy $\mathbf{Q}_\ell : \mathcal{W}_\ell \to \{0,1\}^{C_\ell}$, it is straightforward to accommodate randomized mechanisms by extending the set of indirect reports to include some random variable, and then having the allocation and payment functions depend on the realization of this random variable. This allows conditions (2) and (3) to hold ex-post with respect to all randomizations in the intermediation network. For example, independent draws from uniform random variables can be used to break ties among buyers.

4.2 Equilibrium Characterization

Since truthful reporting is a weakly dominant strategy for buyers, we focus on the game among intermediaries and the seller. Thus, we first provide a formal definition for this game. Subsequently, we characterize an SPE of the game between the intermediaries and the seller. Similar to the single-stage intermediation, we first identify an optimal mechanism in a simpler setting with only one intermediary. We next extend our characterization to a setting with multiple intermediaries by using the result for one intermediary as a building block.

Definition 3. *The game among intermediaries and the seller is an extensive form game* $\Gamma = \langle \mathcal{I} \cup \{I_s\}, \mathcal{H}, \mathcal{S}, \{u_\ell : I_\ell \in \mathcal{I}\} \cup \{u_{n+1}\}, \mathcal{P}\rangle$ *where*

- *The set of players is* $\mathcal{I} \cup \{I_s\}$.
- *The set of histories is* $\mathcal{H} = \cup_{t=1}^{n} \mathcal{H}_t$ *where* $\mathcal{H}_t$ *is the set of all possible tier* t *histories and is given by:*

$$\mathcal{H}_t = \left\{ H_t | H_t = \left\{ \left(\mathbf{X}_{(t',k)}, \mathbf{Q}_{(t',k)}, Y_{(t',k)}\right) \in \mathcal{M}^{\mathrm{SP}} : I_{(t',k)} \in \mathcal{I}, t < t' \right\} \cup \{(\mathbf{X}_{n+1}, \mathbf{Q}_{n+1}) \in \mathcal{M}_s\} \right\}.$$

Here, history H_t consists of the mechanisms $\{(\mathbf{X}_{(t',k)}, \mathbf{Q}_{(t',k)}, Y_{(t',k)}) : I_{(t',k)} \in \mathcal{I}, t < t'\}$ chosen by upstream intermediaries $\{I_{(t',k)} \in \mathcal{I}, t < t'\}$ and the seller's mechanism $(\mathbf{X}_{n+1}, \mathbf{Q}_{n+1})$.

- *The set of pure strategies for intermediary $I_{(t,j)}$ is $\mathcal{S}_{(t,j)} = \{s | s : \mathcal{H}_t \to \mathcal{M}^{\mathrm{SP}}\}$, and for the seller, I_s, $\mathcal{S}_{n+1} = \mathcal{M}_s$. Then, the set of pure strategy profile is $\mathcal{S} = \prod_{I_{(t,j)} \in \mathcal{I}} \mathcal{S}_{(t,j)} \times \mathcal{S}_{n+1}$.*

- *The utility functions, $u_\ell : \mathcal{S} \to \mathbb{R}$, for intermediaries $I_\ell \in \mathcal{I}$ in tier t are given by*

$$u_\ell(s = \{s_{\ell'} : I_{\ell'} \in \mathcal{I}\} \cup \{s_{n+1}\}) = \mathbb{E}\left[\sum_{c: I_c \in \mathcal{C}(I_\ell)} X_{\ell,c}(\boldsymbol{\omega}_\ell) - X_{u,\ell}(\boldsymbol{\omega}_u)\right]$$

where the upstream agent is $I_u = \mathcal{U}(I_\ell)$, the direct reports of I_ℓ are $\mathbf{w}_\ell$, the indirect reports of I_ℓ are $\boldsymbol{\omega}_\ell = (\mathbf{w}_\ell, \boldsymbol{\omega}_{u,-\ell})$, and the indirect reports of I_u are $\boldsymbol{\omega}_u = (Y_\ell(\mathbf{w}_\ell), \boldsymbol{\omega}_{u,-\ell})$.

The utility function, $u_{n+1} : \mathcal{S} \to \mathbb{R}$, for the seller is given by

$$u_{n+1}(s = \{s_\ell : I_\ell \in \mathcal{I}\} \cup \{s_{n+1}\}) = \mathbb{E}\left[\sum_{c: I_c \in \mathcal{C}(I_s)} X_{s,c}(\mathbf{w}_s)\right],$$

where $\mathbf{w}_s = (Y_c(\mathbf{w}_c))_{c: I_c \in \mathcal{C}(I_s)}$.

- *The player function $\mathcal{P} : \mathcal{H} \to \mathcal{I} \cup \{I_s\}$ is $\mathcal{P}(H_t) = \{I_{(t,k)} : I_{(t,k)} \in \mathcal{I}\}$ and $\mathcal{P}(\emptyset) = I_s$.*

We next state an important property of mechanisms in $\mathcal{M}^{\mathrm{SP}}$ or $\mathcal{M}_s$. For any feasible allocation function of mechanisms in $\mathcal{M}^{\mathrm{SP}}$ or $\mathcal{M}_s$, payments have a second-price like structure. Let $P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) = \inf\{w \in \mathbb{R} : Q_{\ell,c}(w, \boldsymbol{\omega}_{\ell,-c}) = 1\}$ be the minimum report of downstream agent I_c to her upstream agent I_ℓ that guarantees winning against indirect reports $\boldsymbol{\omega}_{\ell,-c} \in \mathcal{W}_{\ell,-c}$. If $\mathcal{W}_{\ell,-c} = \emptyset$ (e.g., I_ℓ is a seller with a single downstream agent), then we denote by $P_{\ell,c} = \inf\{w \in \mathbb{R} : Q_{\ell,c}(w) = 1\}$ the same amount. This quantity is well-defined because the allocation is non-decreasing by (2) (see Proposition 2).

Proposition 2. *Let $(\mathbf{X}_\ell, \mathbf{Q}_\ell)$ be allocation and payment functions satisfying (2). By the Envelope Theorem, the allocation and payments can be conveniently written as*

$$\begin{aligned} Q_{\ell,c}(\boldsymbol{\omega}_\ell) =& \mathbf{1}\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}, \\ X_{\ell,c}(\boldsymbol{\omega}_\ell) =& X_{\ell,c}(0, \boldsymbol{\omega}_{\ell,-c}) + P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) \mathbf{1}\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}, \end{aligned}$$

where $\boldsymbol{\omega}_\ell = (w_{\ell,c}, \boldsymbol{\omega}_{\ell,-c}) \in \mathcal{W}_\ell$ is the vector of all indirect reports of intermediary I_ℓ and $w_{\ell,c}$ is the direct report of downstream agent $I_c \in \mathcal{C}(I_\ell)$ to I_ℓ.

Here $X_{\ell,c}(0, \boldsymbol{\omega}_{\ell,-c})$ represents the payment of agent I_c when her value is 0 and $P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})$ represents the additional payment conditional on winning. Individual rationality, see (3), further implies that $X_{\ell,c}(0, \boldsymbol{\omega}_{\ell,-c}) \leq 0$, and thus we shall see that in the optimal mechanism this payment is set to 0.

4.2.1 Single Stage Intermediation

We start by considering the problem of a single intermediary positioned between multiple buyers and a seller. We will establish that in this setting the optimal mechanism of the intermediary can be explicitly characterized in terms of the virtual value functions of buyers' values and their projections onto nonnegative numbers.

Lemma 2. *Suppose that an intermediary, I_ℓ, has an upstream seller, I_s, which implements a strategy-proof mechanism $(\mathbf{X}_s, \mathbf{Q}_s)$ with a set of indirect reports $\mathcal{W}_s = \mathbb{R}_+ \times \mathcal{W}_{s,-\ell}$, where $\mathcal{W}_{s,-\ell}$ is the set of indirect reports of the seller excluding the report of intermediary I_ℓ. Assume that the type of buyer $I_c \in \mathcal{C}(I_\ell)$ is a random variable V_c with strictly increasing virtual value function. Then an optimal mechanism for I_ℓ is given by $(\mathbf{X}^*_\ell, \mathbf{Q}^*_\ell, Y^*_\ell)$, regardless of the distribution of indirect reports in $\mathcal{W}_{s,-\ell}$. The reporting function is*

$$Y^*_\ell(\mathbf{w}_\ell) = \max_{c:I_c \in \mathcal{C}(I_\ell)} \psi_{V_c}(w_{\ell,c}) \,,$$

where $\mathbf{w}_\ell = (w_{\ell,c})_{c:I_c \in \mathcal{C}(I_\ell)} \in \mathbb{R}^{C_\ell}$ is a vector of direct reports. The allocation and payment functions are given by

$$\begin{aligned} Q^*_{\ell,c}(\boldsymbol{\omega}_\ell) =& Q_{s,\ell}\left(\psi_{V_c}(w_{\ell,c}), \boldsymbol{\omega}_{s,-\ell}\right) \mathbf{1}\{c = \operatorname*{argmax}_{c':I_{c'} \in \mathcal{C}_\ell} \psi_{V_{c'}}(w_{\ell,c'}) \text{ and } w_{\ell,c} \geq z_{V_c}\} \,, \\ X^*_{\ell,c}(\boldsymbol{\omega}_\ell) =& w_{\ell,c} Q^*_{\ell,c}(\boldsymbol{\omega}_\ell) - \int_0^{w_{\ell,c}} Q^*_{\ell,c}(y, \boldsymbol{\omega}_{\ell,-c}) \mathrm{d}y \,, \end{aligned}$$

where $\boldsymbol{\omega}_\ell = (\mathbf{w}_\ell, \boldsymbol{\omega}_{s,-\ell}) \in \mathcal{W}_\ell$ is the vector of indirect reports of I_ℓ and $\boldsymbol{\omega}_{s,-\ell} \in \mathcal{W}_{s,-\ell}$.

The reporting function of the mechanism derived here, $Y^*_\ell(\cdot)$, does not depend on the distribution of the indirect reports in the mechanism of the seller, and only is a function of the downstream report distribution when the upstream seller also implements a strategy-proof mechanism.

The mechanism $(\mathbf{X}_\ell^*, \mathbf{Q}_\ell^*, Y_\ell^*)$ can be further simplified by considering the mechanism of the seller. The following corollary provides an alternative characterization.

Corollary 1. *The mechanism characterized in Lemma 2 can equivalently be expressed as follows:*

$$
\begin{aligned}
Y_\ell^*(\mathbf{w}_\ell) &= \max_{c: I_c \in \mathcal{C}(I_\ell)} \psi_{V_c}(w_{\ell,c}) \,, \\
Q_{\ell,c}^*(\mathbf{w}_\ell, \boldsymbol{\omega}_{s,-\ell}) &= \mathbf{1}\left\{ w_{\ell,c} \geq P_{\ell,c}(\mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell}) \right\} \,, \\
X_{\ell,c}^*(\mathbf{w}_\ell, \boldsymbol{\omega}_{s,-\ell}) &= P_{\ell,c}(\mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell}) \mathbf{1}\left\{ v_c \geq P_{\ell,c}(\mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell}) \right\} \,,
\end{aligned}
$$

where the minimum winning report function of buyer $I_c \in \mathcal{C}_\ell$ in the intermediary's mechanism is given by

$$
P_{\ell,c}(\mathbf{w}_{-c}, \boldsymbol{\omega}_{s,-\ell}) = \psi_{V_c}^{-1}\left(\max\left(\max_{c': I_{c'} \in \mathcal{C}(I_\ell) \setminus \{I_c\}} \psi_{V_{c'}}(w_{\ell,c'}), P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell}) \right) \right) \,,
$$

where $P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})$ is the minimum winning report function of I_ℓ in the seller's mechanism.

4.2.2 Multi-stage Intermediation

We next introduce the *anticipated reports* and state our equilibrium characterization. Note that the definition of anticipated reports provided in Balseiro et al. (2017) is a special case for **k**-trees. Here, we provide a general definition.

Definition 4. *Let I_ℓ be an agent connected to an upstream agent I_u, i.e., $I_u = \mathcal{U}(I_\ell)$. If I_ℓ is a buyer, her anticipated report is $W_{u,\ell} = V_\ell$. If I_ℓ is an intermediary, her anticipated report is*

$$
W_{u,\ell} = \max_{c: I_c \in \mathcal{C}(I_\ell)} \psi_{W_{\ell,c}}(W_{\ell,c}) \,.
$$

It can be seen that the anticipated report $W_{u,\ell}$ coincides with the report of agent I_ℓ to her upstream agent I_u if all intermediaries use the (maximum of) projected virtual value functions of the downstream bids to determine their bids to the upstream mechanism and buyers report their values truthfully.

The next assumption imposes that the anticipated reports of all agents along the tree have strictly increasing virtual values.

Assumption 1. *The anticipated reports $W_{u,\ell}$ for every agent $I_\ell \in \mathcal{B} \cup \mathcal{I}$ where $I_u = \mathcal{U}(I_\ell)$ are well-defined and have finite expected values. Moreover, for all intermediaries, i.e., $I_\ell \in \mathcal{I}$, the anticipated reports have strictly increasing virtual values, i.e., $\phi_{W_{u,\ell}}(\cdot)$ (or $\phi_{W_{u,\ell}|W_{u,\ell}>0}(\cdot)$ if $W_{u,\ell}$ has an atom at zero) is strictly increasing.*

When Assumption 1 is satisfied, an SPE of the game between intermediaries and the seller can be characterized by applying backward induction starting from the last tier of a tree network. As before, due to the incentive compatible nature of the mechanisms, an intermediary along the network is not influenced by the choice of upstream mechanisms, and in turn her mechanism does not influence downstream mechanisms. Hence, each intermediary focuses on optimizing her profits based on the anticipated reports of the downstream agents, which coincide with the reports induced by the fixed (optimal) mechanisms along the equilibrium path.

Using H_ℓ to denote the history of upstream mechanisms intermediary I_ℓ observes when she chooses her mechanism, the following theorem formally characterizes an SPE for tree networks under Assumption 1. Recall that all intermediaries in the same tier choose their mechanisms simultaneously. Thus, the history H_ℓ at which intermediary I_ℓ chooses her mechanism consists of mechanisms of intermediaries in all upper tiers (including intermediaries that do not lie on the path between I_ℓ and the seller), and this history is also common to other intermediaries in the same tier.

Theorem 1. *Suppose that Assumption 1 holds. Let s^* be a strategy profile such that for history $H_\ell \in \mathcal{H}_\ell$ the mechanism of intermediary I_ℓ is $s^*_\ell(H_\ell) = (\mathbf{X}^*_\ell, \mathbf{Q}^*_\ell, Y^*_\ell)$. The reporting function is given by*

$$Y^*_\ell(\mathbf{w}_\ell) = \max_{c: I_c \in \mathcal{C}(I_\ell)} \psi_{W_{\ell,c}}(w_{\ell,c}) \,,$$

where $W_{\ell,c}$ is the anticipated report of I_c, and $\mathbf{w}_\ell = (w_{\ell,c})_{c:I_c \in \mathcal{C}(I_\ell)}$ is the vector of direct reports of intermediary I_ℓ. The allocation and payment functions are given by

$$\begin{aligned}
Q^*_{\ell,c}(\boldsymbol{\omega}_\ell) =& Q^*_{u,\ell}\left(\psi_{W_{\ell,c}}(w_{\ell,c}), \boldsymbol{\omega}_{u,-\ell}\right) \mathbf{1}\{c = \operatorname*{argmax}_{c': I_{c'} \in \mathcal{C}(I_\ell)} \psi_{W_{\ell,c'}}(w_{\ell,c'}) \text{ and } w_{\ell,c} \geq z_{W_{\ell,c}}\} \,, \\
X^*_{\ell,c}(\boldsymbol{\omega}_\ell) =& w_{\ell,c} Q^*_{\ell,c}(\boldsymbol{\omega}_\ell) - \int_0^{w_{\ell,c}} Q^*_{\ell,c}(t, \boldsymbol{\omega}_{\ell,-c}) \mathrm{d}t \,,
\end{aligned}$$

where $I_u = \mathcal{U}(I_\ell)$ and $\boldsymbol{\omega}_\ell = (\mathbf{w}_\ell, \boldsymbol{\omega}_{u,-\ell}) \in \mathcal{W}_\ell$ is the vector of indirect reports. For the seller I_s, the *reporting function is omitted, and $s_s^*(\emptyset) = (\mathbf{X}_s^*, \mathbf{Q}_s^*)$ where*

$$
\begin{aligned}
Q_{s,c}^*(\boldsymbol{\omega}_s) =& \mathbf{1}\{c = \underset{c': I_{c'} \in \mathcal{C}(I_s)}{\operatorname{argmax}} \ \psi_{W_{s,c'}}(w_{s,c'}) \ \textit{and} \ w_{s,c} \geq z_{W_{s,c}})\}\,, \\
X_{s,c}^*(\boldsymbol{\omega}_s) =& w_{s,c} Q_{s,c}^*(\boldsymbol{\omega}_s) - \int_0^{w_{s,c}} Q_{s,c}^*(t, \boldsymbol{\omega}_{s,-c}) \mathrm{d}t\,,
\end{aligned}
$$

where $\boldsymbol{\omega}_s$ are indirect reports for the seller.

Then s^* constitutes an SPE of the game among intermediaries.

This theorem suggests that in an SPE for tree networks, an intermediary first determines the virtual values of downstream reports, then ranks bids according to their virtual values, and submits the maximum virtual value upstream whenever this quantity is positive. The seller also ranks bids according to their virtual values, and allocated the impression whenever the maximum virtual value is positive and larger than cost of acquiring the impression. Note that the equilibrium mechanisms provided in Theorem 1 are independent of histories, i.e., $(\mathbf{X}_\ell^*, \mathbf{Q}_\ell^*, Y_\ell^*)$ is independent of H_ℓ. Therefore, the strategy profile s^* can be represented by the set of mechanisms $\{(\mathbf{X}_\ell^*, \mathbf{Q}_\ell^*, Y_\ell^*)\}_{I_\ell \in \mathcal{I}}$ for intermediaries and $(\mathbf{X}_s^*, \mathbf{Q}_s^*)$ for the seller.

Finally, we discuss that for a $\mathbf{k}$-tree, the equilibrium characterization provided in Theorem 1 coincides with the one in Theorem 1 in Balseiro et al. (2017).

Corollary 2. *Consider the game in Definition 3 for a $\boldsymbol{k}$-tree where buyers' values are independent and identically distributed. Suppose that Assumption 1 holds. Then, mechanisms $\{(\mathbf{X}_\ell^*, \mathbf{Q}_\ell^*, Y_\ell^*)\}_{I_\ell \in \mathcal{I}}$ (for intermediaries) and $(\mathbf{X}_s^*, \mathbf{Q}_s^*)$ (for the seller) which constitute the SPE in Theorem 1 are equivalent to the mechanisms provided in Theorem 1 in Balseiro et al. (2017).*

This corollary implies that the second-price mechanisms introduced in Balseiro et al. (2017) are optimal within the larger class of strategy-proof mechanisms. Therefore, focusing on second-price mechanisms for $\mathbf{k}$-trees is without loss of optimality within the larger set of strategy-proof mechanisms.

References

Balseiro, S., Candogan, O. and Gurkan, H. (2017), 'Multi-stage intermediation in display advertising'. Working paper.

Bulow, J. and Roberts, J. (1989), 'The simple economics of optimal auctions', *Journal of Political Economy* **97**(5), pp. 1060–1090.

Myerson, R. (1981), 'Optimal auction design', *Mathematics of Operations Research* **6**(1), 58–73.

Nisan, N., Roughgarden, T., Tardos, E. and Vazirani, V. V. (2007), *Algorithmic game theory*, Vol. 1, Cambridge University Press Cambridge.

A Appendix for Section 3

A.1 Proof of Lemma 1

Proof. We prove this result in three main steps. In order to prove this lemma, we solve a general mechanism design problem with no restrictions on the set of mechanisms for an intermediary that participates in a second-price auction on behalf of her multiple buyers.

Step 1: Formulating the mechanism design problem. We introduce the notation used for the general mechanism design problem formulation. Let N denote the set of buyers and $V = (V_i)_{i \in N}$ be the random vector denoting the values of the buyers. With some abuse of notation we denote by N the number of buyers. We next define the functions which are used to represent a general mechanism for the intermediary. The mechanism of the intermediary consists of a *reporting function* $Y(v) : \mathbb{R}^N \to \mathbb{R}_+$ that maps the reports of the buyers to a bid in the auction of the seller; a *payment function* $X(v) : \mathbb{R}^N \to \mathbb{R}_+^N$, which determines the amount charged to the buyers; and an *allocation function* $Q(v) : \mathbb{R}^N \to [0,1]^N$, which determines the winner of the auction. The intermediary can allocate the impression to the buyer only if she wins at the auction of the seller, so the allocation probability is bounded by the probability of winning the impression for the intermediary. By the Revelation Principle we restrict attention, without loss of generality, to direct mechanisms in which the buyer reports her type truthfully to the intermediary.

Additionally, we define the interim allocation and payments by

$$q_i(v_i) = \int_{V_{-i}} Q_i(v_i, v_{-i}) g_{V_{-i}}(v_{-i}) d_{v_{-i}} \,,$$

$$x_i(v_i) = \int_{V_{-i}} X_i(v_i, v_{-i}) g_{V_{-i}}(v_{-i}) d_{v_{-i}} \,,$$

where $g_{V_i}(\cdot)$ denotes the p.d.f. of V_i and $g_{V_{-i}}(\cdot)$ denote the p.d.f. of the induced joint probability distribution for $\{V_j\}_{j \neq i}$. Here $q_i(v_i)$ and $x_i(v_i)$ respectively denote the expected allocation probability and payment of buyer i when she reports her value as v_i. Finally we denote by

$$u(v_i, v_i') = v_i q_i(v_i') - x_i(v_i') \,,$$

the expected payoff of buyer i when her true value is v_i and she reports v_i' to the seller.

The optimal mechanism design problem of the intermediary can now be stated in terms of the payment, allocation functions defined above as follows:

$$\max_{X,Q,Y} \sum_{i\in N} \mathbb{E}_{V_i}\left[x_i(V_i)\right] - \mathbb{E}_V\left[\int_0^{Y(V)} z f_D(z)\mathrm{d}z\right] \tag{Profit of Intermediary}$$

$$\text{st. } u_i(v_i, v_i) \geq 0\,, \forall i, v_i \tag{Individual Rationality}$$

$$u_i(v_i, v_i) \geq u_i(v_i, v_i')\,, \forall i, v_i, v_i' \tag{Incentive Compatibility}$$

$$0 \leq \sum_{i\in N} Q_i(v) \leq F_D(Y(v))\,, \forall v \tag{Feasible Allocation}$$

$$0 \leq Y(v)\,, \forall v\,.$$

Here the first two constraints are the standard IR, IC constraints. The third constrain ensures that the total allocation probability for a realization V of the reports is less than the probability that intermediary acquires the impression from upstream by report $Y(V)$ to the exchange.

Following an identical approach to Myerson (1981), it can be seen that incentive compatibility and individual rationality constraints can be replaced by the following conditions: $u_i(v_i) \triangleq u_i(v_i, v_i) = u_i(0) + \int_0^{v_i} q_i(t)\mathrm{d}t$ and $q_i(\cdot)$ nondecreasing. Note that this result immediately implies that the expected payment of buyer i when she reports v_i is expressed as follows:

$$x_i(v_i) = x_i(0) - \int_0^{v_i} q_i(t)\mathrm{d}t + v_i q_i(v_i)\,. \tag{4}$$

Observe that the IR constraint implies that $x_i(0) \leq 0$. When the payment $x_i(\cdot)$ is eliminated from the objective function in the optimal mechanism design problem of the intermediary by (4), the objective function is maximized at $x_i(0) = 0$ for all $i \in N$. Therefore, in the remainder of the proof, we set $x_i(0) = 0$.

Step 2: Point-wise optimization. Eliminating the payments from the objective by using (4) and integrating by parts, the optimal mechanism design problem can equivalently be stated as follows:

$$\begin{aligned}
&\max_{Q,Y} \mathbb{E}_V \left\{ \sum_{i \in N} \phi_{V_i}(V_i) Q_i(V) - \int_0^{Y(V)} z f_D(z) \mathrm{d}z \right\} \\
&\text{st. } 0 \leq \sum_{i \in N} Q_i(v) \leq F_D(Y(v))\,, \forall v \\
&\quad\;\; 0 \leq Y(v)\,, \forall v \\
&\quad\;\; q_i(\cdot) \text{ nondecreasing.}
\end{aligned} \tag{OMDP}$$

Momentarily relaxing the constraint that $q_i(\cdot)$ is nondecreasing and maximizing the integrand point-wise over V, we get

$$\begin{aligned}
&\max_{Q,Y} \sum_{i \in N} \phi_{V_i}(v_i) Q_i - \int_0^{Y} z f_D(z) \mathrm{d}z \\
&\text{st. } 0 \leq \sum_{i \in N} Q_i \leq F_D(Y)\,, 0 \leq Y
\end{aligned}$$

Define the subproblem for a given Y as

$$\begin{aligned}
T(Y) = &\max_{Q} \sum_{i \in N} \phi_{V_i}(v_i) Q_i \\
&\text{st. } 0 \leq \sum_{i \in N} Q_i \leq F_D(Y).
\end{aligned}$$

For this subproblem, the optimal solution can readily be given as follows:

$$Q_i = \begin{cases} F_D(Y) & \text{if } i = \operatorname*{argmax}_{j \in N} \phi_{V_j}(v_j) \text{ and } \phi_{V_i}(v_i) \geq 0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Thus, $T(Y) = F_D\left(Y\right) \max_{j \in N} \phi^+_{V_j}(v_j)$ and hence the point-wise optimization problem becomes,

$$\max_{Y \geq 0} F_D(Y) \max_{j \in N} \phi^+_{V_j}(v_j) - \int_0^{Y} z f_D(z) \mathrm{d}z$$

The first-order condition with respect to Y can be expressed as follows:

$$f_D(Y)\left(\max_{j\in N}\phi^+_{V_j}(v_j) - Y\right) = 0\,.$$

The objective function of the point-wise maximization problem is unimodal due to the strictly increasing property of virtual value functions so the first-order condition is sufficient for optimality. Summarizing the optimal solution of the point-wise optimization problem for any given $\{v_j\}_{j\in N}$ is such that $Y(v) = \max_{j\in N}\phi^+_{V_j}(v_j)$, and the corresponding allocation function is

$$Q_i(v) = \begin{cases} F_D\left(\max_{j\in N}\phi^+_{V_j}(v_j)\right)\,, & \text{if } i = \operatorname*{argmax}_{j\in N}\phi_{V_j}(v_j) \text{ and } \phi_{V_i}(v_i) \geq 0\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{6}$$

Next, we verify that these allocation and reporting functions also satisfy feasibility in (OMDP). Observe that the first two constraints are trivially satisfied by construction (through the point-wise optimization problem). Thus, for the feasibility it suffices to check that the induced $q_i(\cdot)$ is nondecreasing. On the other hand, by the strictly increasing property of virtual value functions and (6), $Q_i(v_i, v_{-i})$ is nondecreasing in v_i for every v_{-i}. Observe that this immediately implies that the constructed solution is feasible in (OMDP). Moreover, since it maximizes the integrand point-wise, it follows that this solution is also optimal.

For the payment we set

$$X_i(v) = v_i Q_i(v) - \int_0^{v_i} Q_i(t, v_{-i})\ \mathrm{d}t\,.$$

Because the interim allocation and payments are given by $q_i(v_i) = \mathbb{E}[Q_i(v_i, V_{-i})]$ and $x_i(v_i) = \mathbb{E}[X_i(v_i, V_{-i})]$, it follows from Tonelli's Theorem that the latter payment function satisfies (4) as required.

Step 3: Implementation. We next construct an ex-post incentive compatible mechanism which has the same expected profit as the optimal interim mechanism (after taking the expectation on the realization of the highest exogenous bid d). To this end, we define the following allocation rule:

$$Q_i(v,d) = 1\left\{\phi_{V_i}(v_i) \geq d, i = \operatorname*{argmax}_{j\in N}\phi_{V_j}(v_j), \phi_{V_i}(v_i) \geq 0\right\}\,, \tag{7}$$

and payment rule

$$X_i(v,d) = v_i Q_i(v,d) - \int_0^{v_i} Q_i(t, v_{-i}, d)\, \mathrm{d}t\,. \tag{8}$$

Observe that taking expectations over the exogenous bid d leads to identical profit and mechanism as before since $Q_i(v) = \mathbb{E}_D[Q_i(v,D)]$ and $X_i(v) = \mathbb{E}_D[X_i(v,D)]$. We next write the payment rule explicitly by letting $P_i = \phi_{V_i}^{-1}\left(\max\left\{0, d, \max_{j\in N\setminus i} \phi_{V_j}(v_j)\right\}\right)$ be the payment of buyer i when she wins the impression (i.e., she wins both at the intermediary's and seller's auction). In this notation we can write the allocation rule as $Q_i(v,d) = 1_{\{v\geq P_i\}}$ and the payment rule as

$$\begin{aligned} X_i(v,d) &= v_i 1_{\{v\geq P_i\}} - \int_0^{v_i} 1_{\{t\geq P_i\}}\, \mathrm{d}t = v_i 1_{\{v\geq P_i\}} - (v_i - P_i) 1_{\{v\geq P_i\}} \\ &= P_i 1_{\{v\geq P_i\}}\,. \end{aligned} \tag{9}$$

Using this explicit characterization of $X_i(\cdot,\cdot)$, this mechanism can be expressed as follows: The intermediary first computes $\phi_{V_i}(\cdot)$ for each buyer, and identifies the buyer $i = \operatorname{argmax}_{t\in N} \phi_{V_t}(v_t)$. She submits $\phi_{V_i}(v_i)$ to the seller if this bid is nonnegative. In case of winning, (i.e., $\phi_{V_i}(v_i) \geq d$) the impression is then assigned to buyer i. Note that this reporting function and allocation is consistent with (7) and can alternatively be expressed in terms of reporting functions and reserve prices:

$$\begin{aligned} Y_\ell^i(v_i) =& \psi_{V_i}(v_i)\,, \\ r_\ell^i =& z_{V_i} \end{aligned}$$

where V_i denotes the value distribution of the buyer i. That is, the allocation in (7) can be supported by a mechanism in second-price mechanisms. From the previous discussion we see that a payment of amount P_i is charged to buyer i only if she obtains the impression. Note that this payment is supported by a mechanism in a second-price mechanism since P_i is exactly the minimum amount buyer i should bid to win the impression.

This mechanism also given in Balseiro et al. (2017), achieves the allocation and the payments in (7) and (9), and hence is optimal. $\square$

A.2 Proof of Proposition 1

Proof. For the proof of this proposition, we invoke Lemma 1 to characterize the best response functions of intermediaries. The best response of intermediary I_ℓ, assuming other intermediaries use mechanisms given in the statement of the proposition as follows, can be obtained by solving the following optimization problem.

$$\max_{(\mathbf{r}_\ell, \mathbf{Y}_\ell) \in \mathcal{M}_\ell} \sum_{I_i \in \mathcal{C}(I_\ell)} \mathbb{E}_{V,T} \left\{ \left[\left(Y_\ell^i\right)^{-1} \left(\max \left(Y_\ell^i(r_\ell^i), \max_{j \neq i} Y_\ell^j(V_j), T \right) \right) - T \right] 1_{E_i} \right\} ,$$

where $E_i = \left\{ V_i \geq \left(Y_\ell^i\right)^{-1} \left(\max(Y_\ell^i(r_\ell^i), \max_{j \neq i} Y_\ell^j(V_j), T) \right) \right\}$ represents the event that the buyer i wins the impression and $T = \max\left(D, \max_{I_k \in \mathcal{C}(I_c) \setminus \{I_\ell\}} Y_c^k(V_k) \right)$ represents the competing bid at the auction of the seller. It can be seen that this maximization problem is equivalent to the profit maximization problem of a single intermediary participating in the seller's mechanism with competing bid T. Thus, Lemma 1 can be applied for characterizing the optimal mechanism of I_ℓ, which is independent of the distribution of T and hence the mechanism of the remaining intermediaries. In particular, Lemma 1 implies that the best response for I_ℓ is

$$\begin{aligned} Y_\ell^i(v_i) =& \psi_{V_i}(v_i) , \\ r_\ell^i =& z_{V_i} , \end{aligned}$$

where V_i denotes the random variable of buyer i connected to I_ℓ. Since we consider an arbitrary intermediary, it follows that the strategy profile given in Proposition 1 constitutes a Nash equilibrium of the game among intermediaries. $\square$

B Appendix for Section 4

B.1 Proof of Proposition 2

Proof. We first prove that the allocation function $Q_{\ell,c}(\boldsymbol{\omega}_\ell)$ is nondecreasing in $w_{\ell,c}$. Writing (2) for v and v', and v' and v; and then summing the inequalities we obtain:

$$(v - v')Q_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) \geq (v - v')Q_{\ell,c}(v', \boldsymbol{\omega}_{\ell,-c}) .$$

When $v > v'$, we get $Q_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) \geq Q_{\ell,c}(v', \boldsymbol{\omega}_{\ell,-c})$, vice versa.

Equation (2) implies that $vQ_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c}) = \max_w vQ_{\ell,c}(w, \boldsymbol{\omega}_{\ell,-c}) - X_{\ell,c}(w, \boldsymbol{\omega}_{\ell,-c})$. Applying the Envelope Theorem, we obtain that the payment $X_{\ell,c}(v, \boldsymbol{\omega}_{\ell,-c})$ is given as follows:

$$X_{\ell,c}(\boldsymbol{\omega}_\ell) = w_{\ell,c}Q_{\ell,c}(\boldsymbol{\omega}_\ell) - \int_0^{w_{\ell,c}} Q_{\ell,c}(y, \boldsymbol{\omega}_{\ell,-c})\mathrm{d}y + X_{\ell,c}(0, \boldsymbol{\omega}_{\ell,-c}) .$$

Recognizing the definition of $P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) = \inf \{w \in \mathbb{R} : Q_{\ell,c}(w, \boldsymbol{\omega}_{\ell,-c}) = 1\}$ and using the monotonicity of $Q_{\ell,c}(\cdot, \boldsymbol{\omega}_{\ell,-c})$, first we can alternatively express the allocation as $Q_{\ell,c}(\boldsymbol{\omega}_\ell) = 1_{\{w_{\ell,c} \geq P_\ell^c(\boldsymbol{\omega}_{\ell,-c})\}}$, and then the payment as:

$$\begin{aligned} X_{\ell,c}(\boldsymbol{\omega}_\ell) &= w_{\ell,c}1_{\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}} - \int_0^{w_{\ell,c}} 1_{\{y \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}}\mathrm{d}y + X_{\ell,c}(0, \boldsymbol{\omega}_{\ell,-c}) , \\ &= X_{\ell,c}(0, \boldsymbol{\omega}_{\ell,-c}) + P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})1_{\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\}} . \end{aligned}$$

□

B.2 Proof of Lemma 2

Proof. We start our proof by explaining the direct and the indirect reports of the intermediary and the seller.[5] Let $\mathbf{v} = (v_c)_{c:I_c \in \mathcal{C}(I_\ell)}$ be the vector of buyers' values. Since truthful reporting is a dominant strategy, reports of buyers are their values. Therefore, the direct reports of the intermediary are $\mathbf{w}_\ell = \mathbf{v}$. In addition to the set of reports $\mathcal{W}_{s,-\ell}$, the seller also receives reports from intermediary I_ℓ, thus her indirect reports are given by $\boldsymbol{\omega}_s = (w_{s,\ell}, \boldsymbol{\omega}_{s,-\ell}) \in \mathcal{W}_s = \mathbb{R} \times \mathcal{W}_{s,-\ell}$. Note that the set of direct and the set of indirect reports for the seller are the same. The indirect reports of intermediary I_ℓ is her direct reports and the indirect reports of her upstream agent except her own report, so the indirect reports

[5]Here, the seller can also be thought of as an upstream intermediary with a strategy-proof mechanisms and a set of indirect reports.

for I_ℓ are $\boldsymbol{\omega}_\ell = (\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) \in \mathcal{W}_\ell = \mathbb{R}^{\mathcal{C}_\ell} \times \mathcal{W}_{s,-\ell}$.

The optimal mechanism design problem of the intermediary can now be stated as follows:

$$
\begin{aligned}
&\max_{(\mathbf{X}_\ell, \mathbf{Q}_\ell, Y_\ell)} \mathbb{E}_{\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}} \left[\sum_{c: I_c \in \mathcal{C}_\ell} X_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) - X_{s,\ell}(Y_\ell(\mathbf{v}), \boldsymbol{\omega}_{s,-\ell}) \right] \\
&\quad \text{s.t. } vQ_{\ell,c}(v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) - X_{\ell,c}(v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \geq 0 \,, && \forall c, v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell} \,, \\
&\qquad vQ_{\ell,c}(v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) - X_{\ell,c}(v, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \\
&\qquad\qquad \geq vQ_{\ell,c}(v', \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) - X_{\ell,c}(v', \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \,, && \forall c, v, v', \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell} \,, \\
&\qquad \sum_{c: I_c \in \mathcal{C}_\ell} Q_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) \leq Q_{s,\ell}(Y_\ell(\mathbf{v}), \boldsymbol{\omega}_{s,-\ell}) \,, && \forall \mathbf{v}, \boldsymbol{\omega}_{s,-\ell} \,, \\
&\qquad Y_\ell(\mathbf{v}) \geq 0 \,, && \forall \mathbf{v} \,.
\end{aligned}
$$

Here, the first two constraints are (2) and (3), and they guarantee that a feasible mechanism for this optimization problem is $(\mathbf{X}_\ell, \mathbf{Q}_\ell, Y_\ell) \in \mathcal{M}^{\text{SP}}$. The third constraint implies that the intermediary allocates the impression only if she acquires it from the seller. Finally, the last constraint guarantees that the report of the intermediary is nonnegative. The expectation in the objective is taken w.r.t. the values of the buyers (direct reports), $\mathbf{v}$, and the set of indirect reports of the seller excluding the report of the intermediary, $\boldsymbol{\omega}_{s,-\ell}$.

Following an identical approach to the proof of Proposition 2, it can be seen that (2) and (3) can be replaced by the following conditions:

$$
\begin{aligned}
X_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) = X_{\ell,c}(0, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) + v_c Q_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) - \int_0^{v_c} Q_{\ell,c}(y, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \mathrm{d}y \,, \\
Q_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) \in \{0,1\} \text{ and } Q_{\ell,c}(v_c, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \text{ nondecreasing in } v_c \,.
\end{aligned}
$$

Observe that (3) implies that $X_{\ell,c}(0, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \leq 0$. When the payment $X_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell})$ is eliminated from the objective function, the objective function is maximized at $X_{\ell,c}(0, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) = 0$ for all $c \in \mathcal{C}(I_\ell)$. Therefore, in the remainder of the proof, we set $X_{\ell,c}(0, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) = 0$.

Reformulated Problem: We can equivalently reformulate the mechanism design problem of the intermediary by replacing the payment in the objective function, and changing the order of integration.

Hence, we obtain a new objective function and remove the constraints on the payment function. The resulting problem is:

$$
\begin{aligned}
&\max_{(\mathbf{Q}_\ell, Y_\ell)} \mathbb{E}_{\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}} \left[\sum_{I_c \in \mathcal{C}_\ell} \phi_{v_c}(v_c) Q_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) - X_{s,\ell}(Y_\ell(\mathbf{v}), \boldsymbol{\omega}_{s,-\ell}) \right] \\
&\text{s.t. } Q_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) \in \{0,1\} \text{ and } Q_{\ell,c}(v_c, \mathbf{v}_{-c}, \boldsymbol{\omega}_{s,-\ell}) \text{ nondecreasing in } v_c \,, \qquad \forall c, \mathbf{v}, \boldsymbol{\omega}_{s,-\ell} \,, \\
&\qquad \sum_{c: I_c \in \mathcal{C}_\ell} Q_{\ell,c}(\mathbf{v}, \boldsymbol{\omega}_{s,-\ell}) \le Q_{s,\ell}(Y_\ell(\mathbf{v}), \boldsymbol{\omega}_{s,-\ell}) \,, \qquad \forall \mathbf{v}, \boldsymbol{\omega}_{s,-\ell} \,, \\
&\qquad Y_\ell(\mathbf{v}) \ge 0 \,, \qquad \forall \mathbf{v} \,.
\end{aligned}
$$

Point-wise Optimization: Momentarily relaxing the monotonicity constraint on the allocation and maximizing the integrand point-wise over the buyers' values $\mathbf{v}$ and $\boldsymbol{\omega}_{s,-\ell}$, we obtain

$$
\begin{aligned}
&\max_{(\mathbf{Q}_\ell, Y_\ell)} \sum_{I_c \in \mathcal{C}_\ell} \phi_{V_c}(v_c) Q_{\ell,c} - X_{s,\ell}(Y_\ell, \boldsymbol{\omega}_{s,-\ell}) \\
&\text{s.t.} \sum_{I_c \in \mathcal{C}(I_\ell)} Q_{\ell,c} \le Q_{s,\ell}(Y_\ell, \boldsymbol{\omega}_{s,-\ell}) \,, \\
&\qquad Y_\ell \ge 0 \,.
\end{aligned}
$$

Considering the subproblem on $\mathbf{Q}_\ell$ for a given Y_ℓ, it is optimal to allocate to the bidder with the highest virtual value (whenever the highest virtual value is positive) and set allocation to its bound whenever, i.e., $Q_{\ell,c} = Q_{s,\ell}(Y_\ell, \boldsymbol{\omega}_{s,-\ell}) \mathbf{1}\{c = \operatorname*{argmax}_{c': I_{c'} \in \mathcal{C}(I_\ell)} \phi_{V_{c'}}(v_{c'}) \text{ and } v_c \ge z_{V_c}\}$. Using this observation, the point-wise optimization problem is written as:

$$
\max_{Y_\ell \ge 0} \left\{ \left(\max_{c: I_c \in \mathcal{C}(I_\ell)} \phi_{V_c}(v_c) \right)^+ Q_{s,\ell}(Y_\ell, \boldsymbol{\omega}_{s,-\ell}) - X_{s,\ell}(Y_\ell, \boldsymbol{\omega}_{s,-\ell}) \right\} .
$$

Note that $(\mathbf{X}_s, \mathbf{Q}_s)$ is also a strategy-proof mechanism, so $v = \operatorname*{argmax}_{v'} v Q_{s,\ell}(v', \boldsymbol{\omega}_{s,-\ell}) - X_{s,\ell}(v', \boldsymbol{\omega}_{s,-\ell})$. Thus, $Y_\ell = \left(\max_{c': I_{c'} \in \mathcal{C}(I_\ell)} \phi_{V_{c'}}(v_{c'}) \right)^+$ is an optimal solution for the point-wise optimization problem, and thus implying

$$
Q_{\ell,c} = Q_{s,\ell}\left(\psi_{V_c}(v_c), \boldsymbol{\omega}_{s,-\ell}\right) \mathbf{1}\{c = \operatorname*{argmax}_{c': I_{c'} \in \mathcal{C}_\ell} \psi_{V_{c'}}(v_{c'}) \text{ and } v_c \ge z_{V_c}\} \,.
$$

By using this optimal solution of point-wise optimization, we can construct an optimal solution for the reformulated problem, and hence for the mechanism design problem of the intermediary.

$$Y^*_\ell(\mathbf{v}) = \max_{c':I_{c'}\in\mathcal{C}(I_\ell)} \psi_{V_{c'}}(v_{c'})\,,$$

$$Q^*_{\ell,c}(\mathbf{v},\boldsymbol{\omega}_{s,-\ell}) = Q_{s,\ell}\left(\psi_{V_c}(v_c),\boldsymbol{\omega}_{s,-\ell}\right)\mathbf{1}\{c = \operatorname*{argmax}_{c':I_{c'}\in\mathcal{C}(I_\ell)} \psi_{V_{c'}}(v_{c'}) \text{ and } v_c \geq z_{V_c}\}\,.$$

These reporting and allocation functions establish an optimal solution for the reformulated mechanism design problem of the intermediary. First, we consider feasibility. In particular, since the value V_c of a buyer is assumed to have a continuous positive density $g_{V_c}(\cdot) > 0$, finite expected value $\mathbb{E}[V_c] < \infty$, and a strictly increasing virtual value function, it follows that the associated projected virtual value function is well defined. Moreover, the projected virtual value function is nondecreasing because the virtual value function is strictly increasing. Since, $Q_{s,\ell}(w_{s,\ell},\boldsymbol{\omega}_{s,-\ell})$ is also nondecreasing in $w_{s,\ell}$ by Proposition 2, it follows that $Q^*_{\ell,c}(\mathbf{v},\boldsymbol{\omega}_{s,-\ell})$ is nondecreasing in v_c. The other constraints in the reformulated problem are clearly satisfied. Second, we consider optimality. Since this solution maximizes the integrand in the objective function of the reformulated problem point-wise, it follows that this solution is also optimal. Finally, point-wise optimality implies that the solution is optimal regardless of the distribution of indirect reports in $\mathcal{W}_{s,-\ell}$. We derive the payment function by using the characterization obtained in Proposition 2. □

B.3 Proof of Corollary 1

Proof. Since the seller's mechanism is also strategy-proof, by Proposition 2, it follows that $Q_{s,\ell}(w,\boldsymbol{\omega}_{s,-\ell}) = 1_{\{w\geq P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})\}}$ where $P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell}) = \inf\{w | Q_{s,\ell}(w,\boldsymbol{\omega}_{s,-\ell}) = 1\}$. Using this observation, the allocation function of intermediary I_ℓ is given by

$$\begin{aligned} Q^*_{\ell,c}(\mathbf{w}_\ell,\boldsymbol{\omega}_{s,-\ell}) =& Q_{s,\ell}\left(\max_{c':I_{c'}\in C_\ell} \psi_{V_{c'}}(w_{\ell,c'}),\boldsymbol{\omega}_{s,-\ell}\right)\mathbf{1}\{c = \operatorname*{argmax}_{c':I_{c'}\in\mathcal{C}(I_\ell)} \psi_{V_{c'}}(w_{\ell,c'}) \text{ and } w_{\ell,c} \geq z_{V_c}\} \\ =& \mathbf{1}\{\psi_{V_c}(w_{\ell,c}) \geq P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})\}\mathbf{1}\{c = \operatorname*{argmax}_{c':I_{c'}\in\mathcal{C}(I_\ell)} \psi_{V_{c'}}(w_{\ell,c'}) \text{ and } w_{\ell,c} \geq z_{V_c}\}\,. \end{aligned}$$

Recognizing that $P_{\ell,c}(\mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell}) = \inf\{w \in \mathbb{R} : Q^*_{\ell,c}(w, \mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell}) = 1\}$, the minimum amount which guarantees the winning of agent I_c is given as

$$P_{\ell,c}(\mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell}) = \psi_{V_c}^{-1}\left(\max\left(\max_{c':I_{c'}\in\mathcal{C}(I_\ell)\setminus\{I_c\}} \psi_{V_{c'}}(w_{\ell,c'}), P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})\right)\right).$$

By this observation, the allocation is $Q^*_{\ell,c}(\mathbf{w}_\ell, \boldsymbol{\omega}_{s,-\ell}) = \mathbf{1}\{w_{\ell,c} \geq P_{\ell,c}(\mathbf{w}_{\ell,-c}, \boldsymbol{\omega}_{s,-\ell})\}$. □

B.4 Proof of Theorem 1

Proof. If an intermediary faces an upstream strategy-proof mechanism, her optimal mechanism is characterized by Lemma 2. Using this result, we can characterize the equilibrium strategies of intermediaries starting from the last tier of a tree network via backward induction.

Base Case. Let H_1 be a history observed by intermediaries $I_{(1,j)} \in \mathcal{I}$. By Assumption 1, the type of the buyers connected to $I_{(1,j)}$ has strictly increasing virtual value functions. Then, an optimal mechanism is given by Lemma 2, that is also given in the statement of the theorem.

Inductive Case. Consider any tier t and assume that all lower tiers adopt the equilibrium strategy given in the statement of the theorem. Therefore, an intermediary in tier t, $I_{(t,j)}$ observes that her downstream reports are characterized by Definition 4, and we also know that the anticipated reports have strictly increasing virtual value functions by Assumption 1. By these observations, we can invoke Lemma 2 to characterize the equilibrium strategy of the intermediary. However, for intermediary $I_{(t,j)}$ the anticipated reports coming from the downstream agents potentially have an atom at zero but are continuous elsewhere due to the positive reserve prices. As discussed in Balseiro et al. (2017), when the value of the anticipated report is zero an intermediary cannot profit from this report. Thus focusing on the strictly positive part is without loss of optimality and it follows that the mechanisms that maximize the expected profit of the intermediaries are given as in the statement of the theorem.

Seller's Mechanism. By backward induction, it follows that the reports received by the seller are as defined in Definition 4, and thus satisfy Assumption 1. Recognizing the seminal work of Myerson (1981), the optimal mechanism for the seller can be found. First, note that the set of interim incentive compatible and interim individually rational mechanisms is a larger class than the set of strategy-proof mechanisms. By relaxing strategy-proofness and considering instead the weaker notion of interim IC and IR, we obtain the optimal mechanism. Second, the optimal interim IC, IR mechanism from Myerson (1981) is a strategy-proof mechanism. Furthermore, it coincides with the mechanism given in the statement of the theorem. □

B.5 Proof of Corollary 2

Proof. First, we consider the mechanisms of intermediaries. By symmetry of a **k**-tree, the anticipated reports of the agents in the same tier follow the same distribution, and hence they have the same virtual value function. Therefore, comparing the virtual values is equivalent to directly comparing bids, and the reporting function can be written as:

$$Y_\ell^*(\mathbf{w}_\ell) = \max_{I_c \in \mathcal{C}(I_\ell)} \psi_{W_{\ell,c}}(w_{\ell,c}) = \psi_{W_{\ell,c}}\left(\max_{I_c \in \mathcal{C}(I_\ell)} w_{\ell,c}\right).$$

Next, we consider payments and allocations. By Corollary 1, the equilibrium mechanisms can alternatively be expressed as follows:

$$\begin{aligned} Q_{\ell,c}^*(\boldsymbol{\omega}_\ell) =& \mathbf{1}\left\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\right\}, \\ X_{\ell,c}^*(\boldsymbol{\omega}_\ell) =& P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\mathbf{1}\left\{w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})\right\}. \end{aligned}$$

where $P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) = \psi_{W_{\ell,c}}^{-1}\left(\max\left(\max_{I_{c'} \in \mathcal{C}_\ell \setminus \{I_c\}} \psi_{W_{\ell,c'}}(w_{\ell,c'}), P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right)\right)$ and $P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})$ is the payment of I_ℓ to her upstream intermediary, $I_u \in \mathcal{U}(I_\ell)$.[6] Using that the anticipated reports are identically distributed for all agents in the same tier we obtain

$$P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c}) = \max\left(\max_{I_{c'} \in \mathcal{C}_\ell \setminus \{I_c\}} w_{\ell,c'}, \psi_{W_{\ell,c}}^{-1}\left(P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})\right)\right).$$

Note that this payment is the same as the payment with the reporting function and the reserve price of Theorem 1 in Balseiro et al. (2017).

Also note that the intermediary allocates the impression to the downstream agent $I_c \in \mathcal{C}(I_\ell)$ when $w_{\ell,c} \geq P_{\ell,c}(\boldsymbol{\omega}_{\ell,-c})$ which corresponds to the case that intermediary I_ℓ acquires the impression from her

[6] In Corollary 1, we have $P_{s,\ell}(\boldsymbol{\omega}_{s,-\ell})$ instead of $P_{u,\ell}(\boldsymbol{\omega}_{u,-\ell})$ because the intermediary connects to the seller. Here, the intermediary could connect either to another intermediary or the seller. However, the result would still follow because the upstream mechanism in either case would be a strategy-proof mechanism.

upstream agent and an agent I_c is the winner. Then, it follows that the mechanism of intermediary I_ℓ is the same as the mechanism provided in Theorem 1 in Balseiro et al. (2017).

Second, we consider the seller. Using the fact that the virtual values are same for agents in the same tier, the seller's mechanism is given by

$$
\begin{aligned}
Q^*_{s,c}(\boldsymbol{\omega}_s) =& \mathbf{1}\left\{ w_{s,c} \geq \max\left(\max_{c' : I_{c'} \in \mathcal{C}_s \setminus \{I_c\}} w_{s,c'}, z_{W_{s,c}} \right) \right\}, \\
X^*_{s,c}(\boldsymbol{\omega}_s) =& \begin{cases} \max\left(\max_{c' : I_{c'} \in \mathcal{C}_s \setminus \{I_c\}} w_{s,c'}, z_{W_{s,c}} \right) & \text{if } w_{s,c} \geq \max\left(\max_{c' : I_{c'} \in \mathcal{C}_s \setminus \{I_c\}} w_{s,c'}, z_{W_{s,c}} \right), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

This mechanism is simply a second-price auction with an optimal reserve price of $z_{W_{s,c}}$ for downstream agent $I_c \in \mathcal{C}(I_s)$, which is the same as the seller's mechanism in Theorem 1 in Balseiro et al. (2017). □