$N$ cups, $M$ balls. Denote $p_{k}$ as the probability that exactly $k$ cups are non-empty. Let $X$ be the number of non-empty cups. Then we have

$$
E[X]=\sum_{k=1}^{M} k p_{k}
$$

## Finding $p_{k}$

Choose $k$ cups, $\binom{N}{k}$ possibilities. Then choose a partition of the $M$ balls into $k$ non-empty subsets. This is a Stirling number of the second kind ${ }^{1}\left\{\begin{array}{c}M \\ k\end{array}\right\}$ possibilities. Now arrange the partition in some order, $k$ ! possibilities. The total number of possibilities for $M$ balls into $N$ cups is $N^{M}$. So this probability is

$$
p_{k}=\frac{\binom{N}{k}\left\{\begin{array}{c}
M \\
k
\end{array}\right\} k!}{N^{M}}
$$

## Answer

$$
E[X]=\sum_{k=1}^{M}\binom{N}{k}\left\{\begin{array}{c}
M \\
k
\end{array}\right\} \frac{k \cdot k!}{N^{M}}
$$

For instance, for $N=5$ cups and $M=4$ balls, we have $E[X]=\frac{369}{125} \approx 2.95$. Check it out on Wolfram Alpha

[^0]
[^0]:    ${ }^{1}$ Check wikipedia. The formula is $\left\{\begin{array}{l}n \\ k\end{array}\right\}=\frac{1}{k!} \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^{n}$

