JOURNAL OF ECONOMIC THEORY 72, 96–147 (1997)
ARTICLE NO. ET962195

Comparing Four Models of Aggregate Fluctuations due to Self-Fulfilling Expectations*

Stephanie Schmitt-Grohé

Board of Governors of the Federal Reserve System, Washington, DC 20551

Received December 7, 1994; revised January 3, 1996

This paper compares four equilibrium business cycle models with increasing returns to scale production technologies that allow for aggregate fluctuations due to self-fulfilling expectations. Necessary and sufficient conditions for the existence of stationary sunspot equilibria are derived. Numerical examples demonstrate that the degree of increasing returns necessary for the existence of stationary sunspot equilibria lies in the upper range of available empirical estimates. The paper also shows that persistent fluctuations are not a necessary property of these four models when the only source of fluctuations is changes in people's expectations about the future path of the economy. *Journal of Economic Literature* Classification Numbers: E32, D43. © 1997 Academic Press

1. INTRODUCTION

This paper compares four models that allow for endogenous equilibrium fluctuations due to self-fulfilling expectations. First, it compares the parameter values required by each model for such equilibria to exist and discusses their empirical plausibility. Second, it analyzes the models' ability to explain the observed co-movements of macroeconomic time series at business-cycle frequencies in the absence of any shocks to fundamentals.

The four models analyzed in this paper share as a common feature increasing returns to scale production technologies and differ in the behavior of marginal costs and market structure. Two of the models have constant marginal cost schedules whereas the other two have decreasing marginal cost schedules. Both models with constant marginal costs are models of variable markups, and both models with decreasing marginal costs are models of fixed markups. In the models with constant marginal

* I thank Michael Woodford, Martín Uribe, Lars Peter Hansen, three anonymous referees, and seminar participants at NYU, the Federal Reserve Board, UC–Riverside, UCLA, and the 1994 NBER Summer Institute's Workshop on "Macroeconomic Complementarities" for comments. The opinions expressed herein are those of the author and do not necessarily reflect views of the Board of Governors of the Federal Reserve System.

0022-0531/97 $25.00

costs endogenous fluctuations do not exist unless the markup is variable, and in the models with constant markups endogenous fluctuations do not exist unless marginal costs are decreasing. In one of the models with decreasing marginal costs, increasing returns are external to the firm and product markets are perfectly competitive, whereas in the three other models, increasing returns are internal to the firm and product markets are imperfectly competitive.

The first model is a modification of Galí's [11] model of variable markups due to variations in the composition of aggregate demand; the modifications include increasing returns to scale due to fixed costs and a more general preference specification. I dropped the constant returns assumption because it implies pure profits on average, which are not observed in the U.S. economy. I show that the modified model still allows for expectations driven fluctuations and, moreover, that it does so for more realistic parameter configurations. The second model of constant marginal cost, I analyze, is Rotemberg and Woodford's [20] model of countercyclical markups due to implicit collusion. Rotemberg and Woodford conjectured that for certain parameterizations of their model the rational expectations equilibrium might be indeterminate; I establish the indeterminacy result explicitly and show that stationary sunspot equilibria exist for about the same degree of market power as required in the modified Galí model.

The two models with decreasing marginal costs are similar to Hornstein's [13] and Baxter and King's [3] models of the business-cycle. Benhabib and Farmer [4] and Farmer and Guo [10] have pointed out that these models may display indeterminacy of the rational expectations equilibrium and have explored the characteristics of an expectations driven business-cycle in these two models.

The four models are presented in a unified framework that allows a uniform calibration.[1] All are calibrated using the parameter values of King *et al.* [14], which imply a zero profit share. Besides the parameters appearing in the standard real business-cycle model, each model has one to three additional free parameters, including the markup and the elasticity of the markup, in the two models with variable markups, or the degree of homogeneity of the production function in the models with decreasing marginal costs. Then the paper asks for which values of these additional parameters stationary sunspot equilibria exist. A unified framework and uniform calibration are desirable because, in general, the minimum markup required to generate endogenous fluctuations as well as the dynamic properties of expectations driven cycles will depend on the specific parameterization adopted. I find that for the King, Plosser, and Rebelo

[1] For example, Galí assumes that the period utility is linear in consumption whereas Farmer and Guo work with logarithmic preferences over consumption.

calibration the degree of returns to scale necessary for endogenous fluctuations lies in the upper range of available empirical estimates. Also, stationary sunspot equilibria exist for smaller markups in models of variable markups than in models of fixed markups. In all four models the minimum degree of returns to scale necessary for local indeterminacy depends to a great extend on the elasticity of labor supply and the labor share and is decreasing in both parameters.

I also compute the relative standard deviations, contemporaneous correlation, and serial correlation for several macroeconomic aggregates implied by the theoretical models when the only source of uncertainty is revisions in expectations. Previous authors have shown and stressed that models of endogenous business-cycles correctly predict the observed high autocorrelation of output in the absence of highly serially correlated shocks. The paper shows that this prediction is not robust to small perturbations in any one of the two additional free parameters mentioned above. The serial correlation of output varies continuously from -1 to $+1$ for values of those parameters taken from a very narrow range.

The paper is organized in six sections. Following this Introduction, Section 2 presents the four endogenous business-cycle models. For each model, except the implicit collusion model, I analytically derive necessary and sufficient conditions for the existence of stationary sunspot equilibria and numerically determine for which values of the additional parameters stationary sunspot equilibria can occur, in the case of the King, Plosser, and Rebelo parameter values. Section 3 discusses empirical evidence on the size of markups and returns to scale which is then used in Section 4 to evaluate the empirical plausibility of the minimum degree of increasing returns necessary to render the rational expectations equilibrium indeterminate. Section 5 compares the predicted co-movements of output, employment, investment, and the real wage when the business-cycle is driven solely by revisions in expectations. Section 6 concludes the paper.

2. FOUR ENDOGENOUS BUSINESS-CYCLE MODELS

2.1. *A Common Building Block: The Basic Real Business-Cycle Model*

The economies analyzed below are identical to the basic real business-cycle (RBC) model in all respects except for the market structure and the production technology.[2] Product markets may be imperfectly competitive, and firms will have internal or external increasing returns to scale production technologies.

[2] See King *et al.* [14] for a more detailed description of this model.

The economy in the basic RBC model consists of a large number of identical, infinite-lived households. The representative household seeks to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, H_t),$$

where C_t denotes consumption of the household in period t, and H_t denotes hours supplied to the market by the household in period t. The period utility function, $U(\cdot, \cdot)$, is concave, increasing in the first argument and decreasing in the second. The representative household's budget constraint can be written as

$$C_t + E_t \frac{q_{t+1}}{q_t} W_{t+1} \leqslant w_t H_t + W_t,$$

where w_t denotes the wage rate in terms of the consumption good in period t and W_t is a random variable denoting the value of the asset portfolio of the household in units of the consumption good at the beginning of period t. The stochastic process $\{q_t\}$ represents a pricing kernel for contingent claims.[3]

The first order conditions for the household's problem of allocating time and consumption are

$$U_c(C_t, H_t) = \lambda_t \tag{1}$$

and

$$-\frac{U_H(C_t, H_t)}{U_c(C_t, H_t)} = w_t, \tag{2}$$

where λ_t denotes the marginal utility of wealth in period t. One can solve (1) and (2) for C_t and H_t as functions of w_t and λ_t

$$C_t = C(\lambda_t, w_t) \tag{1a}$$

$$H_t = H(\lambda_t, w_t). \tag{2a}$$

The market clearing condition states that output in period t, Y_t, must equal additions to the capital stock in period t and consumption in period t

$$Y_t = C_t + K_{t+1} - (1-\delta) K_t. \tag{3}$$

The capital stock depreciates at the rate δ, $\delta \in (0, 1)$, each period.

[3] A security whose pay-out in period $t+1$ in terms of $t+1$ consumption is a random variable $\{x_{t+1}\}$ has a period t value of $E_t[q_{t+1}x_{t+1}/q_t]$ in units of the period t consumption good.

Households have access to a complete set of frictionless asset markets. From the household's optimal portfolio choice, the asset pricing kernel must satisfy $q_t = \beta^t \lambda_t$. Physical capital must be priced using this pricing kernel, and the price of capital is one. Thus,

$$1 = E_t\{(\beta\lambda_{t+1}/\lambda_t)(u_{t+1} + 1 - \delta)\}, \tag{4}$$

where u_t is the user cost or rental rate of capital in period t. Eqs. (1)–(4) will be part of the set of equilibrium conditions in each of the four models described below.

I restrict the analysis to stationary equilibria near a deterministic steady state and approximate a stationary equilibrium by the solution to the log-linearized equilibrium conditions.[4] For example, I use the notation $\hat{H}_t$ to denote log deviations (of H_t) from steady state. The log-linearized equilibrium conditions can be written as

$$\hat{C}_t = \varepsilon_{CW}\hat{w}_t + \varepsilon_{C\lambda}\hat{\lambda}_t, \tag{5}$$

$$\hat{H}_t = \varepsilon_{HW}\hat{w}_t + \varepsilon_{H\lambda}\hat{\lambda}_t, \tag{6}$$

$$\hat{Y}_t = s_C\hat{C}_t + \frac{s_I}{\delta}[\hat{K}_{t+1} - (1-\delta)\hat{K}_t], \tag{7}$$

and

$$\hat{\lambda}_t = E_t\left\{\hat{\lambda}_{t+1} + \frac{r+\delta}{1+r}\hat{u}_{t+1}\right\}, \tag{8}$$

where $s_C(\equiv C/Y)$ is the steady state consumption share and $s_I(\equiv \delta K/Y)$ is the steady state investment share.[5] In steady state, (3) implies that

$$s_C + s_I = 1. \tag{9}$$

The coefficient r denotes the steady state (quarterly) rate of return on equity. From (4) one sees that in steady state $(r+\delta) = u$ and hence that

$$s_I = \delta s_K/(r+\delta), \tag{10}$$

where $s_K(\equiv uK/Y)$ denotes the share of capital in value added. In the standard RBC model the labor share, $s_H(\equiv wH/Y)$, and the capital share add up to one

$$s_H + s_K = 1. \tag{11}$$

[4] See Woodford [22] for a justification of this method.

[5] The omission of time subscripts indicates steady state values.

Below, I impose a zero profit condition which implies that (11) still holds. Finally, I assume that the preference specification has to be compatible with long-run balanced growth.[6] This compatibility requires, as King *et al.* [14] show, that there exists a $\rho > 0$ such that (1a) is homogeneous of degree one in $(w, \lambda^{-1/\rho})$ and that (2a) is homogeneous of degree zero in $(w, \lambda^{-1/\rho})$. Rotemberg and Woodford [20] show that these homogeneity assumptions imply further that

$$\varepsilon_{HW} - \rho\varepsilon_{H\lambda} = 0, \tag{12}$$

$$\varepsilon_{CW} - \rho\varepsilon_{C\lambda} = 1, \tag{13}$$

and

$$\varepsilon_{CW} = \frac{\rho - 1}{\rho} \frac{s_H}{s_C} \varepsilon_{HW}. \tag{14}$$

There are then only two free preference parameters, the (Frisch) elasticity of hours with respect to wage, ε_{HW}, and the intertemporal elasticity of substitution, $1/\rho$, which with the labor and consumption share determine the other (Frisch) elasticities.

To summarize, the linearization and the parameter restrictions involve 11 steady state parameters: $(\varepsilon_{HW}, \varepsilon_{H\lambda}, \varepsilon_{CW}, \varepsilon_{C\lambda}, \rho, s_H, s_K, s_I, s_C, \delta, r)$ of which 5 are free and the remaining 6 are determined by restrictions (9)–(14). I now discuss the values assigned to those parameters because this particular calibration will be used in the numerical work in the rest of the paper. The 5 calibrated parameters are the labor share in value added, s_H, the steady state quarterly rate of return on equity, r, the quarterly depreciation rate on capital, δ, the (Frisch) elasticity of labor supply with respect to wage, ε_{HW}, and the intertemporal elasticity of substitution, ρ.

I assign the same values as do King *et al.* [14] because other authors have also used this set of parameter values as a benchmark calibration. For example, Rotemberg and Woodford [21] adopt it in an analysis of several dynamic general equilibrium models with imperfectly competitive product markets. The value assigned to the labor share is 0.58, computed as the average ratio of total employee compensation to GNP for 1948–1986. The share of capital by (11) is 0.42. The value for the rate of return on capital is 6.5 percent per year, which is the average real return to equity from 1948 to 1981. The depreciation rate is set at 10 percent per year. The implied investment share by (10) then is 0.25, and by (9) the consumption share is 0.75. For the intertemporal elasticity of substitution ρ, King *et al.* choose a value of one. In their model, the labor supply elasticity is pinned down by assuming a particular functional form of the period utility function

[6] This paper does not discuss growth, however.

TABLE 1

Calibration

	Definition	CAD model	IC model	IR model	EXT model	Parameter description
s_H	$\frac{wH}{Y}$	0.58	0.58	0.58	0.58	Labor share in GDP
r	$u-\delta$	0.016	0.016	0.016	0.016	Quaterly real interest rate
δ		0.024	0.024	0.024	0.024	Quarterly depreciation rate
ε_{HW}		4	4	4	4	Labor supply elasticity
ρ		1	1	1	1	Intertemporal elasticity
s_K	$\frac{uK}{Y}$	0.42	0.42	0.42	0.42	Capital share in GDP
s_I	$\frac{\delta s_K}{r+\delta}$	0.25	0.25	0.25	0.25	Investment share in GDP
s_C	$\frac{C}{Y}$	0.75	0.75	0.75	0.75	Consumption share in GDP
$\varepsilon_{H\lambda}$		4	4	4	4	Income elasticity of labor supply
ε_{CW}		0	0	0	0	Wage elasticity of consumption
$\varepsilon_{C\lambda}$		-1	-1	-1	-1	Income elasticity of consumption
$\tilde{\gamma}$		2.35	2.35	2.35	2.35	Aggregate returns to scale
μ		2.35	2.35	2.35	1	Steady state markup
ε_μ	[a]	-0.25	0.7	—	—	Elasticity of the markup
η		1	1	2.34	1	Degree of homogeneity of $F(K, H)$
α		—	0.9	—	—	Probability of collusion to last

[a] In the CAD model ε_μ denotes the elasticity of the markup with respect to the investment share ($\varepsilon_\mu = \partial \ln \mu(s_t)/\partial \ln s_t$). In the IC model ε_μ denotes the elasticity of the markup with respect to the ratio of the present discounted value of future profits to current aggregate demand ($\varepsilon_\mu = \partial \ln \mu(X_t/Y_t)/\partial \ln(X_t/Y_t)$).

which implies that this elasticity is just the ratio of the steady state fraction of leisure time to the steady state fraction of time worked, which in the United States during 1948–1986 was equal to four (Table 1 lists all parameters).

I now use this building block as well as its linearization in four models with production technologies involving increasing returns to scale and with imperfectly competitive product markets and then analyze under which conditions these models will allow for endogenous fluctuations.

2.2. *The Composition of Aggregate Demand Model (CAD Model)*

This model is based on Galí [11]. In it, N firms produce N different intermediate goods, and each firm is a monopolist. There are two final goods, which are produced with homogeneous of degree one production functions which take a vector of the N intermediate goods as inputs. One final good is bought by households for consumption; the other is bought by the monopolists for investment to increase the capital stock. In particular, I assume that the final goods production functions are of the Dixit–Stiglitz [8] type,

$$f(x) = N^{1/(1-\sigma)} \left[\sum_{i=1}^{N} x_i^{(\sigma-1)/\sigma} \right]^{\sigma/(\sigma-1)}, \qquad \sigma > 0, \tag{15}$$

where σ is the elasticity of substitution between intermediate goods, x_i is the ith element of an $N \times 1$ vector x of intermediate inputs, and $f(x)$ is the quantity of final goods obtained.

The key assumption of this model is that consumption and investment goods are produced with different elasticities of substitution: Consumption goods are produced with an elasticity of substitution of $\sigma > 0$ and investment goods are produced with an elasticity of substitution of $\eta > 0$. Each intermediate good producer sets the price, p_t^i, for his good. Suppose a consumption good producer wants to produce C_t units of the final good. From his cost minimization problem one finds that the demand faced by intermediate good i producer, c_t^i, is

$$c_t^i = \left[\frac{p_t^i}{[N^{-1/\sigma} \sum_{j=1}^{N} p_t^{j(1-\sigma)}]^{1/(1-\sigma)}} \right]^{-\sigma} C_t \equiv D^i(p_t^1, ..., p_t^N)\, C_t.$$

The intermediate good producer faces a similar demand from the sector producing investment goods. Note that $D(\cdot)$ is homogeneous of degree zero in prices and that $D^i(1, 1, ..., 1) = 1/N$. I assume that intermediate good producers cannot price-discriminate between investment and consumption demand. Then the total demand is given by the sum of consumption and investment. I restrict the analysis to a symmetric equilibrium in which all firms charge the same price, $\bar{p}$. From the functional form of the investment and consumption good production functions it follows that the price of one unit of the investment as well as the consumption good will equal $\bar{p}$.

In a symmetric equilibrium in which the number of firms is large and all other firms except firm i charge $\bar{p}$, marginal revenue of the firm producing intermediate good i can be approximated by

$$MR_t^i(y^i) = p_t^i(y_t^i) \left(1 - \frac{y_t^i N / Y_t}{\sigma(p_t^i/\bar{p})^{-\sigma}(1-s_t) + \eta(p_t^i/\bar{p})^{-\eta} s_t} \right),^{7} \tag{16}$$

[7] The approximation is that $[N^{-1/\sigma} \sum_{j=1}^{N} p_t^{j(1-\sigma)}]^{1/(1-\sigma)} = \bar{p} N^{-1/\sigma}$ when all firms but firm i charge $\bar{p}$.

where Y_t is the level of aggregate demand in period t, $(1-s_t)$ and s_t are the shares of, respectively, consumption and investment in aggregate demand in period t, that is,

$$s_t = \frac{K_{t+1} - (1-\delta)\,K_t}{Y_t}. \tag{17}$$

In a symmetric equilibrium, firm i will also charge $\bar{p} = 1$, firm level output is proportional to aggregate output, $y^i_t = Y_t/N$, and by (16) marginal revenue for any firm is

$$MR^i_t(y^i) = 1 - \frac{1}{\sigma(1-s_t) + \eta s_t}.$$

I assume that the elasticity of demand, $\sigma(1-s_t) + \eta s_t$, is always greater than one, so that marginal revenue is positive. The elasticity of demand faced by a firm is not time-invariant unless the elasticity of demand for investment goods equals that of consumption goods and in general depends on the composition of aggregate demand.

All intermediate goods producer have access to the same technology, which is described by

$$Y^i_t = F(K^i_t, H^i_t) - \phi,^8$$

where K^i_t are the capital services rented in period t by firm i, and H^i_t is the number of hours hired by firm i in period t. I assume that $F(\cdot, \cdot)$ is concave, strictly increasing in both arguments, and homogeneous of degree one. The last assumption implies that marginal cost are independent of the scale of production. Throughout the paper, I assume that the elasticity of substitution between capital and labor is unity. Firms have internal increasing returns due to the fixed cost $\phi > 0$.

Profit maximization of firms producing intermediate goods requires that marginal revenue is equated to marginal cost,

$$\frac{\sigma(1-s_t) + \eta s_t - 1}{\sigma(1-s_t) + \eta s_t} = MC_t,$$

[8] When modeling a growing economy the relative importance of the fixed cost would vanish under the current specification. To avoid this, one could either assume that the fixed cost also grows over time, or, alternatively, one could follow Rotemberg and Woodford [21] and assume that along a balanced growth path, output per firm remains constant but the number of varieties of intermediate goods grows over time.

which implies that the markup of prices over marginal cost, μ_t, is

$$\mu_t = \frac{\sigma(1-s_t)+\eta s_t}{\sigma(1-s_t)+\eta s_t - 1}. \tag{18}$$

Unless the elasticities of substitution for investment and consumption demand are equal, the markup is variable over time and depends on the share of investment in aggregate demand. Finally, labor demand is given by

$$F_H(K_t^i, H_t^i) = \mu_t w_t$$

and the demand for capital by

$$F_K(K_t^i, H_t^i) = \mu_t u_t.$$

Profits of firm i are given by $Y_t^i - w_t H_t^i - u_t K_t^i = Y_t^i - F(K_t^i, H_t^i)/\mu_t$. If fixed costs, ϕ, were zero and $\mu > 1$, then firms would on average earn pure profits. This is not observed in U.S. postwar data.[9] I require, therefore, pure profits to be zero in the steady state. This is the case if $Y^i = \phi/(\mu - 1)$, and can be brought about through the endogenous determination of N. I assume that in steady state the number of firms N (or the number of differentiated goods) is such that profits are zero for each firm. Such an assumption seems reasonable because persistent pure profits would lead to entry through the introduction of new goods and persistent losses would cause exit of firms and the number of differentiated goods would decrease.

The equilibrium conditions in terms of aggregates in a symmetric equilibrium are described next. Aggregate value added is determined by

$$Y_t = NF(K_t/N, H_t/N) - N\phi. \tag{19}$$

Aggregate factor demands are related to factor prices through the relations

$$F_H(K_t/N, H_t/N) = \mu_t w_t \tag{20}$$

and

$$F_K(K_t/N, H_t/N) = \mu_t u_t, \tag{21}$$

where the markup μ_t is determined by (18).

These equations describe the production side of the CAD model. The other parts of the model are the same as in the common building block described above. A rational expectations equilibrium is the set of stochastic processes for the endogenous variables $\{K_t, H_t, C_t, \lambda_t, w_t, u_t, Y_t, \mu_t, s_t\}$

[9] See Rotemberg and Woodford [21], Section 4, and also Hall [12] for evidence.

satisfying (1)–(4) and (17)–(21). It can then be shown that a steady state exists. When $\mu=1$ and $\phi=0$, these are the equilibrium conditions of the standard RBC model.

Next, I compare the degree of increasing returns in this economy with the steady state markup. One measure of firm level increasing returns to scale is

$$\gamma \equiv \frac{F_K(K^i, H^i)\, K^i + F_H(K^i, H^i)\, H^i}{Y^i} = \frac{F(K^i, H^i)}{F(K^i, H^i) - \phi} = \mu,$$

which measures the percentage increase in firm level output for each percentage increase in the firm's factor inputs. From the zero profit assumption it follows that in this model the degree of internal increasing returns to scale, γ, equals the markup of prices over marginal cost. Thus the calibration of the steady state μ will also imply the calibration of the degree of increasing returns. Looking ahead to the other models presented, the same will be true in the implicit collusion model and the internal increasing returns model with decreasing marginal costs where steady state profits are also assumed to be zero.

Dynamics around the steady state. Given a steady state, I approximate a stationary equilibrium involving small fluctuations around it by the solution to a log-linear approximation of the equilibrium conditions. The log-linearization of (1)–(4) is given in (5)–(8), and the log-linearized equilibrium conditions of (17)–(21) can be written as

$$s_I \hat{s}_t = s_C(\hat{Y}_t - \hat{C}_t) \tag{22}$$

$$\hat{\mu}_t = \varepsilon_\mu \hat{s}_t \tag{23}$$

$$\hat{Y}_t = \mu s_K \hat{K}_t + \mu s_H \hat{H}_t \tag{24}$$

$$s_K(\hat{K}_t - \hat{H}_t) = \hat{\mu}_t + \hat{w}_t \tag{25}$$

$$s_H(\hat{H}_t - \hat{K}_t) = \hat{\mu}_t + \hat{u}_t. \tag{26}$$

The coefficients s_H, s_K, s_I, s_C are defined as in Section 2.1. The assumption of zero profits in steady state implies that $s_H + s_K = 1$; therefore, restrictions (9)–(14) still hold. The steady state version of (18) defines the steady state markup as

$$\mu \equiv \frac{\sigma s_C + \eta s_I}{\sigma s_C + \eta s_I - 1}, \tag{27}$$

and the elasticity of the markup with respect to the investment share, ε_μ, is (from 18)

$$\varepsilon_\mu \equiv \frac{(\eta-\sigma)\, s_I}{\sigma s_C+\eta s_I-1}\,\frac{(-1)}{\sigma s_C+\eta s_I}. \tag{28}$$

Note that if $\eta=\sigma$, then the markup is constant. Perhaps, it is more straightforward to calibrate the model in terms of the markup, μ, and the elasticity of the markup, ε_μ, rather than in terms of the elasticity of substitution between intermediate goods in the production of consumption and investment goods, σ and η, respectively. The markup appears in the other models discussed as well and hence makes the calibration more comparable; moreover, it has been empirically estimated.

I previously assumed that $\sigma s_C+\eta s_I>1$, which implies that $\mu>1$. Next, I explore what the restrictions σ, $\eta>0$ and $\mu>1$ imply for permissible values of ε_μ. Solving (27) and (28) for σ and η as functions of μ and ε_μ, one gets

$$\sigma=\frac{\mu}{\mu-1}\left[1+\frac{\varepsilon_\mu}{\mu-1}\right] \qquad \text{and} \qquad \eta=\frac{\mu}{\mu-1}\left[1-\frac{\varepsilon_\mu}{\mu-1}\frac{s_c}{s_I}\right].$$

Then the restrictions $\sigma>0$ and $\mu>1$ imply that $1-\mu<\varepsilon_\mu$, while $\eta>0$ and $\mu>1$ imply that $s_I/s_C(\mu-1)>\varepsilon_\mu$. Combining these restrictions, one gets that the permissible values for ε_μ have to satisfy

$$1-\mu<\varepsilon_\mu<s_I/s_C(\mu-1). \tag{29}$$

To sum up, the approximation uses 13 steady state parameters, 11 of which are identical to those introduced in the baseline RBC model; the two additional parameters are μ and ε_μ. The parameters have to satisfy restrictions (9)–(14) and (29) and hence there are 7 free parameters.

The log-linearization of the nine equilibrium conditions can be reduced to a system of two difference equations of the form

$$\begin{pmatrix} E_t\hat{\lambda}_{t+1} \\ \hat{K}_{t+1}\end{pmatrix}=M\begin{pmatrix}\hat{\lambda} \\ \hat{K}_t\end{pmatrix}. \tag{30}$$

The coefficients of the matrix M are functions of all 13 steady state parameters. As Woodford [22] shows stationary sunspot equilibria (s.s.e.) exist if and only if both eigenvalues of M are less than one in modulus. In that case, all stationary solutions to (30) can be expressed as a bivariate stochastic process of the form

$$\begin{pmatrix} \hat{\lambda}_{t+1} \\ \hat{K}_{t+1}\end{pmatrix}=M\begin{pmatrix}\hat{\lambda} \\ \hat{K}_t\end{pmatrix}+\begin{pmatrix} v_{t+1} \\ 0\end{pmatrix},$$

where $\{v_t\}$ is a white noise stochastic process and the realization v_t is part of the time t information set.

Necessary and sufficient conditions for stationary sunspot equilibria. In this section, I analytically derive the necessary and sufficient conditions for the existence of stationary sunspot equilibria. I do this not for so general a preference structure as thus far described, but for one in which ρ is set to one. This assumption implies by (12)–(14) that $\varepsilon_{CW}=0$, $\varepsilon_{C\lambda}=-1$, and $\varepsilon_{H\lambda}=\varepsilon_{HW}$. The determinant of M is given as

$$\det(M)=(1+r)\left(1+\frac{(r+\delta)\left(\mu-1+\varepsilon_\mu\frac{s_C}{s_I}\right)}{(r+\delta)\left(1-\varepsilon_\mu\frac{s_C}{s_I}\right)+(1-\delta)\left(1-\frac{s_H\varepsilon_{HW}}{1+\varepsilon_{HW}}\left(1-\mu\varepsilon_\mu\frac{s_C}{s_I}\right)\right)}\right). \tag{31}$$

If both eigenvalues of M are less than one in modulus, the determinant of M is also less than one in modulus. One can rewrite (31) as

$$\det(M)=(1+r)\left(1+x\left(\equiv\frac{\text{Numerator}}{\text{Denominator}}\right)\right).$$

If the elasticity of the markup, ε_μ, is zero, then x is positive and the determinant of M is greater than one, so that stationary sunspot equilibria cannot exist. That is, this model requires a variable markup to make endogenous fluctuations possible. The same will be true in the implicit collusion model. Sections 2.4 and 2.5, however, are examples of economies in which stationary sunspot equilibria exist despite a constant markup.[10] I will show that a necessary condition for existence of stationary sunspot equilibria is $\varepsilon_\mu<0$. The elasticity of the markup is negative only if the elasticity of investment demand, η, is greater than the elasticity of consumption demand, σ. Galí [11] presents some empirical evidence on the relation between markups and the investment share that supports a negative elasticity of the markup with respect to the investment share. A negative markup elasticity further implies a counter-cyclical markup, as is observed in the U.S. economy as well. Rotemberg and Woodford [19] present evidence, mostly in aggregate data, on the countercyclical behavior of markups, whereas Bils [6] estimates countercyclical markups using two-digit industry level data.

[10] For this model a variable markup is essential for the existence of s.s.e., but increasing returns to scale are not. For example, it can be shown that if the production technology exhibits constant returns to scale ($\phi=0$, as in Galí [11]), then the theoretical possibility of s.s.e. remains.

The economic intuition I can offer for this necessary condition for the existence of s.s.e. is the following. Suppose agents in this economy expect future markups to be low, then employment will he higher tomorrow for any given capital stock, and hence the expected rate of return on capital will be higher. Another factor increasing the expected rate of return on capital is the lower markup itself. This implies that current investment is high. For certain parameter values this effect may be so strong that the current investment share is also higher. If the elasticity of the markup with respect to the investment share is sufficiently negative, then the expectation of low future markups reduces current markups by even more, and expectations are self-fulfilling.

I assume that $\varepsilon_{HW} \geqslant \frac{1}{2}$ and that

$$\frac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}} > \frac{(r+\delta)}{(1-\delta)}, \tag{32}$$

which will be true if labor supply is sufficiently elastic.[11] Then the necessary and sufficient condition for the existence of stationary sunspot equilibria is that

$$\max[1-\mu, f_2^{\mathrm{CAD}}(\mu)] < \varepsilon_\mu < \min[f_1^{\mathrm{CAD}}(\mu), f_3^{\mathrm{CAD}}(\mu)]. \tag{33}$$

The functions $f_1^{\mathrm{CAD}}(\mu)$, $f_2^{\mathrm{CAD}}(\mu)$, and $f_3^{\mathrm{CAD}}(\mu)$ are defined in Appendix A, where (33) is derived. Figure 1 gives a graphical idea of this condition. Both $f_1^{\mathrm{CAD}}(\mu)$ and $f_3^{\mathrm{CAD}}(\mu)$ are monotonically increasing for $\mu > 1$, and as $\mu \to \infty$ they converge to some negative constant; therefore, if there exist (μ, ε_μ) pairs which satisfy (33), then it must be the case that $\varepsilon_\mu < 0$.

The RHS of (33) is an increasing function of the markup and the LHS is a decreasing function of the markup. However, for markups close to one, the RHS is smaller than the LHS, so that their intersection defines the smallest markup for which endogenous fluctuations arise. I call this threshold markup $\mu_{\min}^{\mathrm{CAD}}$. Let $\mu_1^* = \max\{\mu\colon f_1^{\mathrm{CAD}}(\mu) = 1-\mu,\ \mu\colon f_1^{\mathrm{CAD}}(\mu) = f_2^{\mathrm{CAD}}(\mu)\}$ and similarly $\mu_3^* = \max\{\mu\colon f_3(\mu) = 1-\mu, \mu\colon f_3^{\mathrm{CAD}}(\mu) = f_2^{\mathrm{CAD}}(\mu)\}$, then

$$\mu_{\min}^{\mathrm{CAD}} = \max(\mu_1^*, \mu_3^*) = \mu_{\min}^{\mathrm{CAD}}(r, \delta, s_H, \varepsilon_{HW}). \tag{34}$$

Figure 1 is drawn assuming $\mu_3^* > \mu_1^*$ and hence $\mu_{\min}^{\mathrm{CAD}} = \mu_3^*$. $\mu_{\min}^{\mathrm{CAD}}$ is a function of the interest rate, the depreciation rate, the labor share, and the labor supply elasticity. It can be shown that $\mu_{\min}^{\mathrm{CAD}}$ is decreasing in ε_{HW} and s_H. To provide a quantitative insight into (33) and (34) I present a numerical example, which uses the King *et al.* [14] calibration described in Section 2.1. Figure 2 shows for which pairs of values of μ and ε_μ stationary sunspot equilibria exist.

[11] This is surely the case of greatest interest, because with a low elasticity of labor supply, the model will not predict fluctuations that involve significant cyclical variation in hours.

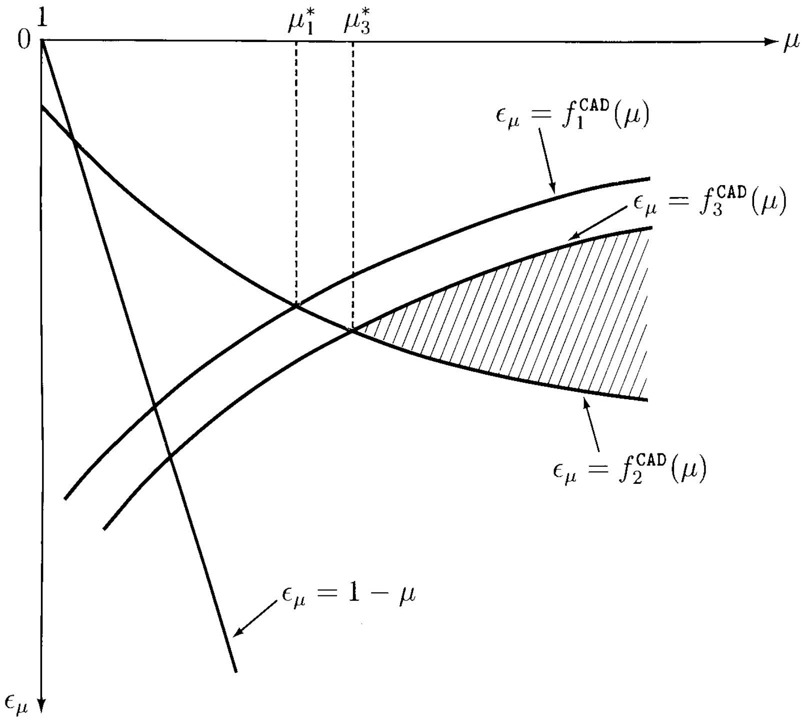

FIG. 1. CAD model with log-utility, necessary and sufficient conditions for the existence of stationary sunspot equilibria.

Circles and crosses correspond to (μ, ε_μ) pairs for which stationary sunspot equilibria cannot occur. A circle indicates that, for this (μ, ε_μ) pair, one eigenvalue of M is outside the unit circle and the other is inside, so that the steady state is a saddle point. By contrast, a cross indicates that both eigenvalues of M are greater than one in modulus, so that the steady state is a source. Finally, dots represent (μ, ε_μ) pairs for which both eigenvalues of M are less than one in modulus, so that the steady state is a sink and s.s.e. exist as a result of the indeterminacy of the rational expectations equilibrium. The three solid lines in Fig. 2 represent the analytically derived necessary and sufficient conditions for s.s.e. given in (33).[12]

The minimum markup that makes endogenous fluctuations possible, $\mu_{\min}^{CAD}$, equals 1.75, (and the corresponding elasticity is -0.254) and there are many $(\mu, \varepsilon_\mu : \mu > \mu_{\min}^{CAD})$ such that the CAD model allows for endogenous fluctuations. For the (s_H, ε_{HW}) parameterization in Rotemberg and Woodford [20] (which assumes a higher labor share of 0.75 and a

[12] In this numerical example, $\min(f_1^{CAD}(\mu), f_3^{CAD}(\mu)) = f_3^{CAD}(\mu)$.

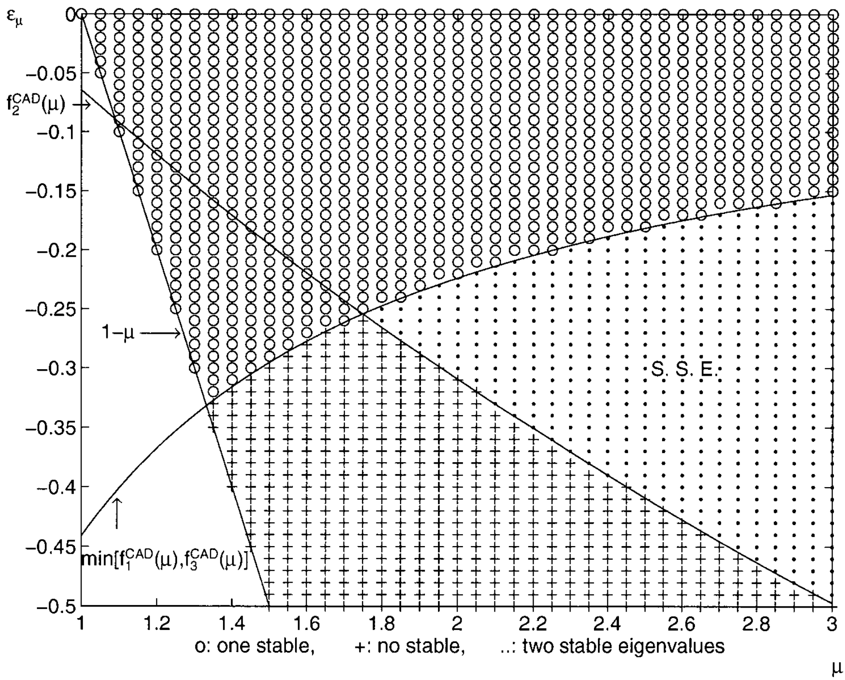

FIG. 2. CAD model, numerical example with King–Plosser–Rebelo parameter values.

lower labor supply elasticity of 1.3 than the King *et al.* calibration) endogenous fluctuations exist only for markups above 1.84 (and ε_μ near -0.15). For the (s_H, ε_{HW}) parameterization used in Farmer and Guo [10] (which assumes a labor share of 0.7 and perfectly elastic labor supply) endogenous fluctuations occur for a markup of just 1.37 and an elasticity of -0.08. In this calibration, a markup of 1.37 and ε_μ of -0.08 correspond to $\eta = 7.3$ and $\sigma = 2.9$, so that the elasticities of investment and consumption demand are still very different, even though ε_μ is small.

The empirical evidence on the size of markups reviewed in Section 3 suggests that a markup of 1.37 may be consistent with empirically plausible values, whereas markups of 1.84 or 1.75 seem inconsistent with empirically realistic values. There is little empirical evidence on the elasticity of the markup with respect to the investment share with which to evaluate whether the ε_μ values required for s.s.e. are at all plausible. Galí [11] estimates ε_μ to be negative, between -0.25 and -0.4.[13]

[13] He reports estimates for the semi-elasticity $\partial \log(\mu_t)/\partial s_t$ to be between -1 and -1.6. With an average investment share of 25 percent, this estimate implies values of ε_μ between -0.25 and -0.4.

TABLE 2

$\mu_{\min}^{CAD}$ as a Function of ε_{HW} and s_H

	ε_{HW}					
s_H	1	2	3	4	6	∞
0.55	2.30	1.98	1.86	1.80	1.73	1.59
0.60	2.20	1.90	1.78	1.72	1.65	1.51
0.65	2.11	1.82	1.70	1.64	1.58	1.44
0.70	2.03	1.74	1.63	1.57	1.51	1.37
0.75	1.96	1.68	1.57	1.51	1.45	1.31
0.80	1.90	1.62	1.51	1.45	1.39	1.25

Table 2 shows $\mu_{\min}^{CAD}$ for various pairs of the labor share and the (Frisch) labor supply elasticity. It illustrates that $\mu_{\min}^{CAD}$ is quite sensitive to the labor share and to the labor supply elasticity. One reason why greater values of the labor share and of the labor supply elasticity make local indeterminacy more likely could be that, for a given expected decline in tomorrow's markup, the higher s_H and ε_{HW} the greater is the associated increase in the expected rate of return.[14] With a greater increase in the expected rate of return an increase in the current investment share is more likely, lowering the current markup and making the expectations of the future markup decline true today. By contrast, $\mu_{\min}^{CAD}$ almost remains constant when the depreciation or the interest rate is varied. For any combination of an annual interest rate between 4 percent and 9 percent and an annual depreciation rate between 5 percent and 12 percent, $\mu_{\min}^{CAD}$ is always between 1.72 and 1.77.

Differences from Galí [11]. The model presented differs from Galí [11] in two respects. The first difference concerns my assumption of increasing returns to scale. In Galí firms produce with a constant returns to scale production function (in our notation, $\phi = 0$), so as to show that the indeterminacy of the rational expectations equilibrium does not rely on increasing returns production technologies. However, constant returns in combination with equilibrium markups greater than 1 imply that there are positive profits on average. In Galí, the markup is calibrated to be 2.8, which implies that profits are 64 percent of value added.[15] Consistently positive profits, particularly of this magnitude, are not observed in the U.S.

[14] From equilibrium in the labor market it follows that the change in equilibrium hours in response to a lower markup is increasing in s_H and ε_{HW}, see (6) and (25); from (26) it follows that the interest rate increases more for higher values of s_H given $\hat{H}_t$.

[15] Profits over value added are given by $(\mu - 1)/\mu$, which is 0.64 for a markup of 2.8 assumed in Galí [11].

economy. The calibration further implies that the labor share in total cost is only 7.6 percent and that the labor share in value added is not even 3 percent.[16] These values are clearly outside the range of parameter values estimated elsewhere for the share of labor in value added; the typical range of values is 0.5 to 0.8. With constant returns to scale, the steady state profit share, s_π, is related to the markup as $s_\pi = (\mu - 1)/\mu$. If one calibrates the steady state profit share, the markup is no longer a free parameter, and an upper limit on the observed profit share implies an upper limit for the markup. For example, $s_\pi < 0.1$ implies that $\mu < 1.11$. By contrast, assuming increasing returns due a fixed cost and imposing the zero profit condition does not imply any restrictions on the size of the steady state markup. These calibration problems motivated the introduction of an increasing returns technology into the model presented above because it allows for zero profits despite an average markup greater than one.

The second difference concerns the preferences of the representative agent. Galí considers the special case in which the period utility function is linear and separable in consumption, that is, $U_c(C_t, H_t) = 1$. This preference specification is not compatible with long-run balanced growth.[17] In this section, I show that endogenous fluctuations still exist under the linear preference specification once one imposes zero profits (and introduces a fixed cost). However, the parameter values required for such equilibria are no longer empirically plausible.

The equilibrium conditions as well as their linear approximation are unchanged except that the linearizations of (1) and (2) are $\hat{\lambda}_t = 0$ and $\hat{H}_t = \varepsilon_{HW} \hat{w}_t$. One can write the system of two first order linear expectational difference equations again as

$$\begin{pmatrix} E_t \hat{H}_{t+1} \\ \hat{K}_{t+1} \end{pmatrix} = M \begin{pmatrix} \hat{H}_t \\ \hat{K}_t \end{pmatrix},$$

where

$$M = \begin{pmatrix} 1 + \varepsilon_{HW}^{-1} & -1 \\ 0 & 1 \end{pmatrix}^{-1} \times \begin{pmatrix} 0 & 0 \\ \delta s_H(\mu + 1/\varepsilon_\mu) - (1 + \varepsilon_{HW}^{-1})/\varepsilon_\mu & \delta s_K(\mu + 1/\varepsilon_\mu) + 1 - \delta \end{pmatrix}.$$

[16] In Galí [11] the calibrated values are $\delta = 0.025$, $r = 0.016$, $s_I = 0.2$, and $\mu = 2.8$. The share of capital in value added then is $s_K = s_I(r + \delta)/\delta = 0.33$. The production function is assumed to be Cobb–Douglas, $K^\alpha H^{(1-\alpha)}$, which implies that $\alpha = \mu s_K = 0.924$ and, hence, $s_H = (1 - \alpha)/\mu = 0.027$.

[17] It is nevertheless attractive because it makes the necessary and sufficient conditions for the existence of stationary sunspot equilibria very easy to derive.

One eigenvalue of the matrix M is equal to zero, and the second eigenvalue then is given by the trace of M:

$$tr(M) = \lambda_2 = 1 + \delta(\mu - 1) - \delta(\mu + \varepsilon_\mu^{-1}) \frac{s_H}{1 + \varepsilon_{HW}}.$$

If the elasticity of the markup with respect to the investment share is negative, $\varepsilon_\mu < 0$, then the modulus of λ_2 is less than one if and only if

$$0 > \frac{1}{\mu(\xi - 1) - \xi} \equiv h(\mu) > \varepsilon_\mu, \qquad \text{where} \quad \xi \equiv \frac{1 + \varepsilon_{HW}}{s_H} > 1.^{18} \tag{35}$$

Recall that $\varepsilon_\mu > 1 - \mu$, combining this inequality with (35) shows that endogenous fluctuations will exist only if the elasticity of the markup and the markup itself satisfy

$$h(\mu) > \varepsilon_\mu > 1 - \mu. \tag{36}$$

This inequality will be satisfied only if $\xi > 1.25$. Figure 3 shows (μ, ε_μ) pairs satisfying (36). The markup has to be above two (as in Galí's original model), and the elasticity of the markup has to be less than minus one. The left-hand side of (36) is decreasing in ξ, and for any ξ between 1 and 1.25, there exist ε_μ satisfying (36). One parameter configuration that would fall in this range is a labor share of more than 80 percent and an inelastic labor supply. Hence, s.s.e. may exist even when labor supply is inelastic (as in Galí's original model). However, a labor share of more than 80 percent in value added is on the high end of estimates using postwar U.S. data. When labor supply is indeed elastic, the labor share has to be even higher for s.s.e. to be possible. In particular, for the calibration in Galí which assumes a labor cost share of 7.6 percent and a unitary labor supply elasticity ($\varepsilon_{HW} = 1$), the model would not allow for endogenous fluctuations. Similarly, for the (s_H, ε_{HW}) pairs used in King *et al.* (0.58,4), Woodford and Rotemberg (0.75, 1.3), and Farmer and Guo (0.7, ∞), s.s.e. would not be possible.

If the elasticity of the markup with respect to the investment share is positive, then s.s.e. exist for (μ, ε_μ) pairs satisfying

$$\frac{s_I}{s_C}(\mu - 1) > \varepsilon_\mu > g(\mu) \equiv \frac{1}{\mu(\xi - 1) - \xi + 2\xi/\delta} \qquad \text{if} \quad \mu(\xi - 1) - \xi < 0$$

$$\min\left(\frac{s_I}{s_C}(\mu - 1), \frac{1}{\mu(\xi - 1) - \xi}\right) > \varepsilon_\mu > \frac{1}{\mu(\xi - 1) - \xi + 2\xi/\delta} \qquad \text{else.}$$

[18] The eigenvalue will always be real and positive.

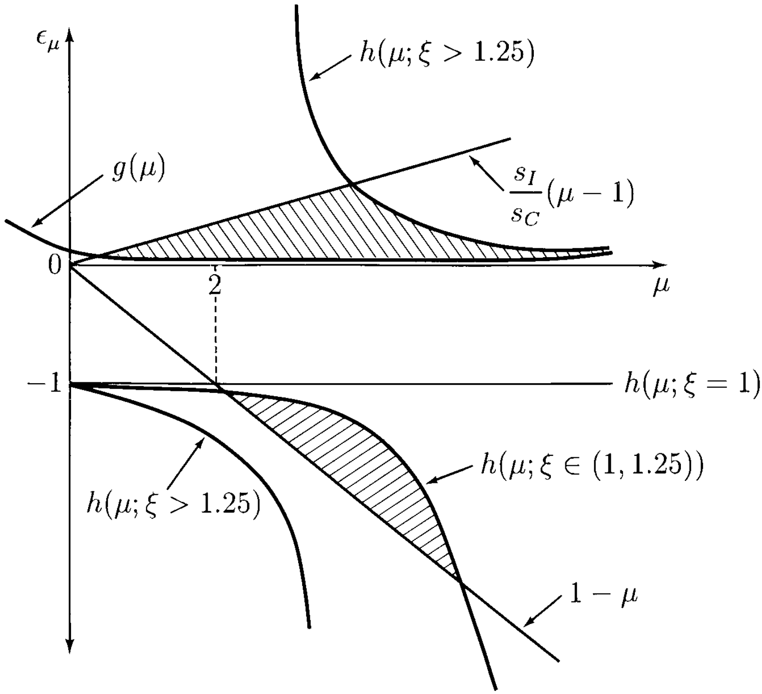

FIG. 3. CAD model with linear utility, necessary and sufficient conditions for the existence of stationary sunspot equilibria.

As can be seen from Fig. 3, in this case s.s.e. exist for any feasible calibration of the labor share, the depreciation rate, and the labor supply elasticity. The required markup is just 1.05 for the King *et al.* baseline calibration. However, if the elasticity of the markup with respect to the investment share is positive, then the model predicts a countercyclical investment share while one observes a pro-cyclical investment share.[19]

Therefore, I consider the existence of s.s.e. under this preference specification only a theoretical possibility which is not very interesting when the objective is to calibrate the model to the U.S. economy. By contrast, log-utility which is more desirable for independent reasons, makes endogenous fluctuations more likely. This raises the question whether preference parameterizations assuming even higher degrees of risk aversion will further lower $\mu_{\min}^{\mathrm{CAD}}$. One can search for $\mu_{\min}^{\mathrm{CAD}}$ numerically given different

[19] For the log-utility specification, $\varepsilon_\mu < 0$ is a necessary condition for s.s.e., in contrast with the current (and Galí's) finding that s.s.e. exist for both positive and negative elasticities of the markup with respect to the investment share. This difference is a consequence of the different preference specification.

values of $1/\rho$ and the other parameters of the model. So far I have discussed the case that $\rho = 1$ and found that for the King *et al.* baseline calibration the markup had to be greater than 1.75 to allow for the existence of s.s.e. I have shown that when $\rho = 0$ stationary sunspot equilibria do not exist for the baseline King *et al.* calibration (for $\varepsilon_\mu < 0$). Searching over (μ, ε_μ) pairs given ρ and other parameter values to find the smallest markup such that both roots of M are stable I find that lowering the intertemporal elasticity indeed lowers $\mu_{\min}^{\text{CAD}}$ at least for values greater than 1/3. Then s.s.e. exist for markups near 1.5.

2.3. *The Implicit Collusion Model (IC Model)*

In the Rotemberg and Woodford [20] model, the economy is organized into I industries and within each industry are M firms. There is only one final good which is produced in three stages. First, firms combine labor and capital to produce an intermediate firm-specific good. Then the output of firms within one industry is combined to produce an industry-specific intermediate good. Finally, those industry outputs are the factor inputs in the final good production. The second and third stage use constant returns to scale technologies, while the production technology of an individual firm is subject to increasing returns to scale.

The difference of this model to the previous one lies in the determination of the equilibrium markup. In the Rotemberg and Woodford model, firms within an industry collude. However, the collusive agreement is not enforceable; so the arrangement is implicit in the sense that a firm deviating from the collusive agreement will be punished.

The production function for firm i in industry j is given by

$$y_t^{ij} = F(K_t^{ij}, H_t^{ij}) - \phi,$$

where $F(\cdot, \cdot)$ is homogeneous of degree one in K_t^{ij}, the capital stock rented by firm i, and in H_t^{ij}, the number of hours of labor hired by firm i in period t, which implies that marginal cost are independent of scale and the same for each firm in the economy. I assume that firms take as given the level of marginal cost, as well as the prices charged by other industries and the level of aggregate demand in period t, Y_t.

From the assumption of linear homogeneity for the two aggregation stages in the production of the final good, it follows that the demand faced by firm i in industry j is homogeneous of degree one in the level of aggregate demand and homogeneous of degree zero in prices. If I restrict the analysis to a symmetric equilibrium in which firms in all industries except industry j charge the common price, p_t, the demand for firm i in industry j can be expressed as

$$y_t^{ij} = D\left(\frac{p_t^{ij}}{p_t}, \frac{p_t^j}{p_t}\right) Y_t, \tag{37}$$

where p_t^{ij} is the price charged by firm i in industry j in period t, p_t^j is the price charged by all other firms in industry j in period t, and Y_t is the level of aggregate demand. As marginal costs are common across industries, one can write

$$y_t^{ij} = D\left(\frac{\mu_t^{ij}}{\mu_t}, \frac{\mu_t^j}{\mu_t}\right) Y_t,$$

where μ_t^{ij} is the markup charged by firm i in industry j, μ_t^j is the markup charged by all other firms in industry j, and μ_t is the markup charged by all other industries. If one further chooses p_t as the numeraire and sets $p_t = 1$, then $1/\mu_t$ equals marginal costs. Ignoring the fixed cost, (gross profits in period t for firm i in industry j equal

$$\pi_t^{ij} = (p_t^{ij} - MC_t)\, y_t^{ij} = \frac{\mu_t^{ij} - 1}{\mu_t} D\left(\frac{\mu_t^{ij}}{\mu_t}, \frac{\mu_t^j}{\mu_t}\right) Y_t.$$

Rotemberg and Woodford assume that firms succeed in implementing the symmetric equilibrium that maximizes the present discounted value of expected future profits for each firm in industry j, taking as given the stochastic process for $\{\mu_t\}$, $\{Y_t\}$ and the pricing kernel $\{q_t\}$. This equilibrium can be sustained if the punishment for a deviator is as severe as possible. Given the possibility of exit, the severest punishment is earning zero profits forever after deviating. Thus, the present discounted value of profits of a deviating firm is given by

$$\pi_t^D \equiv \max_{\mu_t^{ij}} \pi_t^{ij}.$$

The present discounted value of future profits for firm i in industry j, if there are no deviations expected in the future, X_t^i, is

$$X_t^i \equiv E_t\left\{\sum_{j=1}^{\infty} \alpha^j \frac{\beta^j \lambda_{t+j}}{\lambda_t} \frac{\mu_{t+j}^i - 1}{\mu_{t+j}} D\left(\frac{\mu_{t+j}^i}{\mu_{t+j}}, \frac{\mu_{t+j}^i}{\mu_{t+j}}\right) Y_{t+j}\right\},$$

where the asset pricing kernel, q_{t+j}/q_t, is replaced with $\beta^j \lambda_{t+j}/\lambda_t$ and α is the probability that the collusive agreement will not be renegotiated. Firms will not deviate from the collusive agreement so long as the equilibrium markup under collusion satisfies the incentive compatibility constraint

$$\pi_t^D \leqslant \pi_t^i + X_t^i, \tag{38}$$

where π_t^i is the single period profit of firm i in industry j if all firms, including firm i, charge p_t^i while all other industries charge p_t. Rotemberg and

Woodford consider only cases in which (38) is always binding. Thus firms are indifferent between the additional profits from deviating in the present and the future loss of X_t^i. In a symmetric equilibrium, all industries charge the same price, p_t, and hence the markup is the same for all industries. By (37), each firm sells $D(1, 1)\, Y_t$. I assume further that $D(1, 1) = 1/I \cdot M$, so that aggregate output in a symmetric equilibrium is related to firm level output as $Y_t = I \cdot M \cdot y_t$ and

$$X_t = E_t \left\{ \sum_{j=1}^{\infty} \alpha^j \frac{\beta^j \lambda_{t+j}}{\lambda_t} \frac{\mu_{t+j} - 1}{\mu_{t+j}} Y_{t+j} \right\}. \tag{39}$$

Holding with equality (38) can be solved for the equilibrium markup as a function of X_t/Y_t only, and one has

$$\mu_t = \mu(X_t/Y_t), \tag{40}$$

where $\mu(\cdot)$ is increasing in X_t/Y_t.

A rational expectations equilibrium is the set of stochastic processes for the endogenous variables $\{K_t, H_t, C_t, \lambda_t, w_t, u_t, Y_t, \mu_t, X_t\}$ satisfying equations (1)–(4), (19)–(21), and (39)–(40). Rotemberg and Woodford show conditions under which a steady state exists. I assume as before that in steady state each firm makes zero net profits—that is, $Y - wH - uK = 0$ or $Y = 1/\mu(Y + \phi)$.

The degree of firm level increasing returns to scale is given by

$$\gamma \equiv \frac{F_K(K^{ij}, H^{ij})\, K^{ij} + F_H(K^{ij}, H^{ij})\, H^{ij}}{Y_{ij}} = \frac{F(K^{ij}, H^{ij})}{F(K^{ij}, H^{ij}) - \phi} = \mu,$$

where the last equality again follows from the assumption that in steady state net profits are zero. Thus, in the IC model as well as in the CAD model the degree of firm level increasing returns to scale equals the markup.

Dynamics around the steady state. Linearizing (39)–(40) around the steady state gives

$$\hat{\lambda}_t + \hat{X}_t = E_t \left\{ \hat{\lambda}_{t+1} + \left(1 - \frac{\alpha}{1+r}\right)\left(\frac{1}{\mu - 1}\hat{\mu}_t + \hat{Y}_t\right) + \frac{\alpha}{1+r}\hat{X}_t \right\} \tag{41}$$

and

$$\hat{\mu}_t = \varepsilon_\mu(\hat{X}_t - \hat{Y}_t). \tag{42}$$

The assumption of zero profits in steady state implies that $s_H + s_K = 1$; then restrictions (9)–(14) still hold, and the coefficients s_H, s_K, s_I, s_C are defined as in Section 2.1. Beyond the baseline King *et al.* parameters there are three additional parameters: the steady state markup μ; the elasticity of the markup with respect to X/Y, ε_μ, which Rotemberg and Woodford show has to satisfy $0<\varepsilon_\mu<\mu-1$; and the probability that collusion will continue, α. The system of linearized equilibrium conditions consists of (5)–(8), (24)–(26), and (41)–(42). It can be reduced to a system of three linear difference equations similar in form to (30). In this model, one has one predetermined state variable, K_t, and two non-predetermined state variables, μ_t and H_t. The steady state has the saddle path property only if the number of eigenvalues of the Jacobian matrix M with modulus less than one is exactly equal to one. This is not necessarily the case in this model. The calibration considered by Rotemberg and Woodford [20] implies that the steady state has the saddle path property. Stationary sunspot equilibria exist if the number of eigenvalues of M with modulus less than one is greater than one. In that case a stationary solution to the difference equation is given by

$$\begin{pmatrix} \hat{\mu}_{t+1} \\ \hat{H}_{t+1} \\ \hat{K}_{t+1} \end{pmatrix} = M \begin{pmatrix} \hat{\mu}_t \\ \hat{H}_t \\ \hat{K}_t \end{pmatrix} + \begin{pmatrix} v^1_{t+1} \\ v^2_{t+1} \\ 0 \end{pmatrix},$$

where v^1_t and v^2_t are i.i.d. mean zero random variables.[20]

The economic interpretation of these stationary sunspot equilibria is similar to the one given for the CAD model. Suppose agents expect the markup to be lower tomorrow, this raises expected labor demand for any given capital stock, higher employment will increase the return on capital and further the lower markup by itself raises the expected return on capital. This will lead to increased investment demand today and in turn raise current output and lower the current markup, as long as Y_t increases by more than X_t.

A numerical example. For this model I do not analytically derive necessary conditions for the existence of stationary sunspot equilibria because the analytical eigenvalues of the 3×3 matrix M become too messy. Instead, I present a numerical example. As in the previous model, I ask for what value for the markup and its elasticity—given King *et al.* values for all other parameters—does this economy allow for stationary fluctuations due to revisions in agents expectations of the future path of the economy. Figure 4 presents regions for the values of the markup, μ, and the elasticity of the markup, ε_μ, such that s.s.e. exist and such that such equilibria do not exist. In this numerical example, 11 parameters (ε_{HW}, $\varepsilon_{H\lambda}$, ε_{CW}, $\varepsilon_{C\lambda}$, ρ, s_H,

[20] If M has only two stable roots, then $v^2_t=0$.

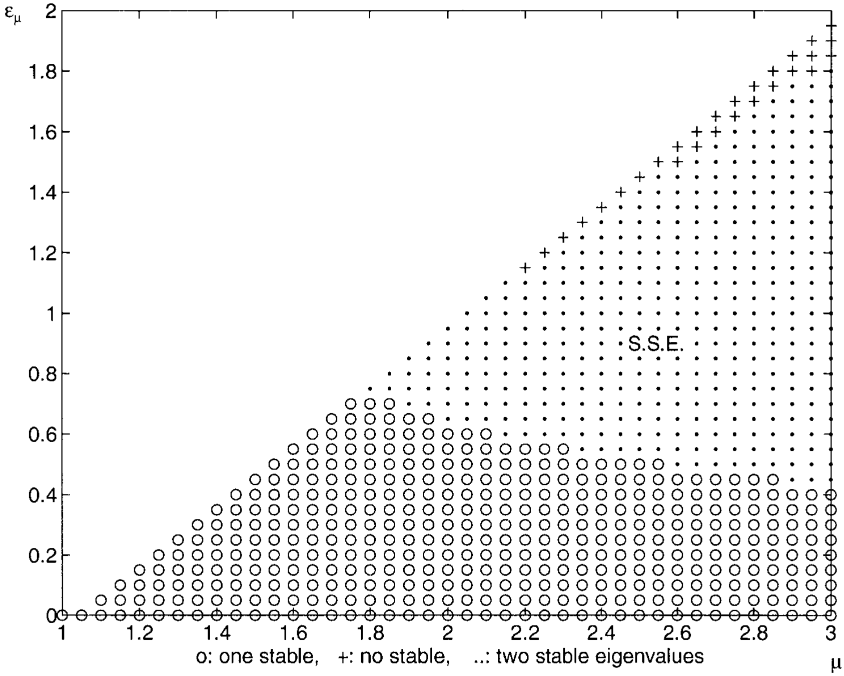

FIG. 4. IC model, numerical example with King–Plosser–Rebelo parameter values.

s_K, s_I, s_C, δ, r) are again assigned the baseline King *et al.* values (see also Table 1). The remaining parameter, α, is set to 0.9, a value taken from Rotemberg and Woodford. The motivation given in Rotemberg and Woodford for this particular parameter value is that it guarantees that the incentive compatibility constraint holds with equality for a relatively small number of firms in each industry (10 firms in their calibration).

As in Fig. 2, circles and crosses in Fig. 4 correspond to (μ, ε_μ) pairs for which s.s.e. do not exist, and dots correspond to (μ, ε_μ) pairs for which stationary sunspot equilibria do exist. The difference between a circle and a cross is that circles indicate that the steady state is a saddle, (as for example in the Rotemberg and Woodford calibration which assumes $\mu = 1.2$ and $\varepsilon_\mu = 0.19$) and crosses indicate that the steady state is a source. Figure 4 shows that there exist indeed many (μ, ε_μ) pairs such that the steady state of this economy is a sink. For those (μ, ε_μ) pairs exactly two eigenvalues of the matrix M were less than one in absolute value, so that in a stationary solution the vector v_{t+1} can have at most one non-zero element.

In this particular example the smallest markup for which s.s.e. exist, given the other parameter values, is 1.754. This value is very close to the minimum markup necessary to allow for endogenous fluctuations in the CAD model. In the CAD model, I have an analytical expression for $\mu_{\min}^{\mathrm{CAD}}$, whereas here I found $\mu_{\min}^{\mathrm{IC}}$ with a numerical search. For this particular markup s.s.e. exist only if the elasticity of the markup is very close to its upper bound of $\mu - 1$. For markups above 1.754 s.s.e. exist even for smaller elasticities away from this upper bound. In fact, for markups above 2.3 the steady state becomes unstable when ε_{μ} approaches that upper bound, unstable in the sense that, if a predetermined state variable is not at its steady state value, then the model spirals off forever away from this steady state.

For the (s_H, ε_{HW}) pair used in Farmer and Guo $(0.7, \infty)$, endogenous fluctuations are possible for markups greater than 1.37. Again, this value is similar to the minimum markup in the CAD model for this parameterization of labor share and elasticity of labor supply. Thus, the CAD model and the IC model both allow for s.s.e. for plausible values for the degree of internal increasing returns to scale for some common labor share and labor elasticity calibrations.

2.4. *A Model with Increasing Returns and Decreasing Marginal Costs (IR Model)*

The structure of this model is similar to that of the CAD model. Two final goods are produced with homogeneous of degree one production functions of the Dixit–Stiglitz type from N intermediate inputs (see (15)). In contrast to the CAD model, investment goods and consumption goods are produced with a common elasticity of substitution, $\sigma > 1$. Firms producing intermediate goods have market power and behave as monopolistic competitors. They produce with increasing returns to scale technologies. I call this the IR model, even though all models presented have increasing returns technologies. A second difference to the CAD model is that marginal costs are no longer independent of scale. The production function of firm i is given by

$$Y_t^i = F(K_t^i, H_t^i) - \phi,$$

where Y_t^i is output of firm i in period t and $F(\cdot, \cdot)$ is increasing in capital rented by firm i in period t, K_t^i, and in hours of labor hired by firm i in period t, H_t^i. Further, $F(\cdot, \cdot)$ is homogeneous of degree $\eta > 0$, where η need not equal 1, and $\phi > 0$ denotes a fixed cost term. Hornstein [13] uses a similar production technology in a model with monopolistic competition to analyze the importance of productivity shocks in accounting for output and employment volatility. Benhabib and Farmer [4] also work with this production technology (assuming fixed costs are zero) and show that, for some $\eta > 1$, this model may allow for endogenous fluctuations. Farmer and

Guo [10] exploit this indeterminacy to analyze the predicted character of business cycles due to self-fulfilling expectations.

The demand faced by an intermediate good producing firm i is the sum of the demand from consumption and investment goods producing firms. As the elasticity of substitution for different intermediate inputs in production of investment and consumption goods is the same the total demand for intermediate inputs is given by

$$y_t^i = \left[\frac{p_t^i}{[N^{-1/\sigma} \sum_{j=1}^{N} p_t^{j(1-\sigma)}]^{1/(1-\sigma)}} \right]^{-\sigma} Y_t, \qquad \sigma > 1,$$

where y_t^i is the demand for firm i's good in period t, Y_t is the level of total aggregate demand in period t, N is the number of intermediate goods, and p_t^i is the price set by firm i in period t. I assume that the elasticity of substitution, σ, is greater than one to ensure that marginal revenue is positive. The profit maximizing firm sets marginal revenue equal to marginal cost. The markup of price over marginal cost will, therefore, be constant and related to the elasticity of substitution, σ, as

$$\mu_t = \mu = \frac{\sigma}{\sigma - 1} > 1. \tag{43}$$

The firm's profit maximization problem is well defined, if at the intersection of marginal revenue and marginal cost the slope of the marginal revenue schedule is less than that of the marginal cost schedule. The slope of the marginal revenue schedule is given by $p_t^{i'}(y_t^i)/\mu$, which is negative. The slope of marginal cost schedule is given by $(1/\eta - 1)\, MC(y_t^i)/(y_t^i + \phi)$, where $MC(y_t^i)$ denotes marginal costs as a function of y_t^i. Marginal costs are decreasing in firm level output, if $\eta > 1$. The slope of the marginal revenue function will be less than the slope of the marginal cost curve at their intersection as long as

$$\frac{1-\mu}{\mu} < \frac{1-\eta}{\eta} \frac{y_t^i}{y_t^i + \phi}. \tag{44}$$

In a symmetric equilibrium, each firm will employ the same amount of factor inputs, that is $K_t = NK_t^i$ and $H_t = NH_t^i$, where K_t denotes the aggregate capital stock in period t and H_t denotes total hours in period t. The demand for labor and capital by an individual firm is again given by (20) and (21), where now μ takes the constant value given in (43). Finally, aggregate output (from the functional form of the aggregator) is given by (19). Consumer behavior and market clearing conditions are as in the baseline RBC model. An equilibrium is then the set of stochastic processes for the endogenous variables $\{K_t, H_t, w_t, u_t, Y_t, C_t, \lambda_t, \mu_t\}$ that satisfy (1)–(4), (19)–(21), and (43), given N.

Again, I assume that in steady state the number of firms N is such that profits are zero for each firm,

$$y^i = wH^i + uK^i,$$

or that $y^i/(y^i+\phi) = \eta/\mu$. This assumption also implies that in steady state condition (44) is satisfied as long as $\eta < \mu$. Then it can be shown that a unique steady state exists.

The degree of firm level internal increasing returns to scale in steady state is equal to

$$\gamma \equiv \frac{F_K(K^i, H^i)\, K^i + F_H(K^i, H^i)\, H^i}{Y^i} = \frac{\eta F(K^i, H^i)}{F(K^i, H^i) - \phi} = \mu.$$

The last equality follows from our zero profits assumption. In Benhabib and Farmer [4] the degree of firm level increasing returns to scale, γ, is equal to the degree of homogeneity of the production function, η, and is less than the steady state markup because firms make pure profits on average. By contrast, in the model presented here the markup and the degree of increasing returns to scale are equal to each other. This is supported by Morrison [15], who finds γ equals μ even when both are estimated independently. The motivation to introduce fixed costs, $\phi > 0$, in this model is to allow for zero profits in steady state despite $\eta < \mu$. I find this desirable for two reasons: First, as discussed above, pure profits are not observed on average in the U.S. economy, and second, I want to parameterize and calibrate this economy as closely as possible to the other three economies, which all assume a zero profit share.[21]

Dynamics around the steady state. To check whether s.s.e. can exist for this model I again log-linearize the equilibrium conditions around the steady state. The linearization of (1)–(4) is unchanged, and the linearization of (20) and (21) is no longer given by (25) and (26) because $F(\cdot, \cdot)$ is not homogeneous of degree one. It can be written as

$$(\eta - 1)(s_K \hat{K}_t + s_H \hat{H}_t) + s_K(\hat{K}_t - \hat{H}_t) = \hat{w}_t \tag{45}$$

$$(\eta - 1)(s_K \hat{K}_t + s_H \hat{H}_t) + s_H(\hat{H}_t - \hat{K}_t) = \hat{u}_t. \tag{46}$$

The capital and labor share, s_K and s_H, are defined as in Section 2.1. The zero profit assumption implies that $s_K + s_H = 1$, hence, restrictions (9)–(14) on the steady state parameters still hold. The steady state markup is given by (43) as a function of σ. The steady state parameters appearing in the

[21] The calibration in Farmer and Guo [10] assigns a value of 7 percent for the steady state profit share.

approximation are those 11 first introduced in the baseline RBC model, the steady state markup, μ, and the degree of homogeneity of the production function $F(\cdot,\cdot)$, η, where $\eta<\mu$. If $\eta=1$, then (45) and (46) are identical to equations (25) and (26) of the CAD model assuming a constant markup in the latter model (that is, $\varepsilon_\mu=0$). If $\mu=\eta=1$, then this model is equivalent to the baseline RBC model. The system of seven linear expectational difference equations can be reduced to a system of two equations similar to (30), in which the matrix M is now a function of the 13 steady state parameters just described.

Necessary and sufficient conditions for stationary sunspot equilibria. The determinant of the Jacobian matrix M for this model is[22]

$$\det(M)=(1+r)\left(1+\frac{\mu-1}{1+(1-\delta)/(r+\delta)(1-(\eta s_H\varepsilon_{HW}/(1+\varepsilon_{HW}))}\right)$$

$$\equiv(1+r)\left[1+x\left(\equiv\frac{\text{Num.}}{\text{Denom.}}\right)\right]. \tag{47}$$

From (47) one can see immediately that endogenous fluctuations will not exist if the labor supply is perfectly inelastic, that is, $\varepsilon_{HW}=0$.[23] A necessary condition for the existence of s.s.e. is not only that labor is elastically supplied but also that

$$\varepsilon_{HW}^{-1}<\eta s_H-1. \tag{48}$$

Since $\varepsilon_{HW}\geqslant 0$ and $s_H<1$, (48) implies that $F(\cdot,\cdot)$ has to be homogeneous of some degree greater than one, $\eta>1$. Hence, decreasing marginal cost is a necessary condition for the existence of s.s.e.. This confirms the finding from the two previous models: if $\eta=1$, s.s.e. do not exist unless the markup is variable. Equation (48) is equivalent to the necessary and sufficient condition for s.s.e. reported in Benhabib and Farmer [4] for a continuous time version of this model without fixed costs.[24]

This necessary condition has the following economic interpretation. The slope of the aggregate labor demand curve (in contrast to the labor demand curve of an individual firm) in a symmetric equilibrium is ηs_H-1,

[22] Again, I assume that the period utility function is $U(C,H)=\log(C)+v(H)$ (this is equivalent to $\rho=1$).

[23] The denominator of x is then positive, and the determinant is greater than one, so that at least one eigenvalue of M has to be greater than one in modulus.

[24] The condition reported by Benhabib and Farmer is that $\varepsilon_{HW}^{-1}<\beta-1$. They assume $F(K,H)=K^\alpha H^\beta$, so that $\eta=(\alpha+\beta)$. The definition of the labor share is $s_H\equiv wH/Y=F_HH/(\mu Y)$ by $(20)=\beta F/(\mu Y)=\beta/\eta$ (using the zero profit condition). Then $\eta s_H-1=\beta-1$.

TABLE 3

$\mu_{\min}^{\mathrm{IR}}$ as a Function of s_H and ε_{HW}

	ε_{HW}					
s_H	1	2	3	4	6	∞
0.55	6.56	3.68	2.94	2.60	2.29	1.93
0.60	5.01	2.85	2.38	2.22	2.07	1.76
0.65	3.83	2.47	2.19	2.04	1.90	1.62
0.70	3.10	2.29	2.02	1.89	1.76	1.50
0.75	2.88	2.13	1.88	1.76	1.64	1.40
0.80	2.69	1.99	1.76	1.65	1.54	1.31

whereas the slope of the labor supply curve is ε_{HW}^{-1}. The necessary condition says that this aggregate "labor demand" has to be upward sloping and steeper than the labor supply curve. For higher values of ε_{HW} and s_H, (48) is satisfied for lower values of η—that is, higher values of ε_{HW} or s_H reduce the degree of returns to scale necessary for the existence of s.s.e.. This point is illustrated in Table 3.

The intuition for the presence of stationary sunspot equilibria is as follows. Suppose that agents observe a stationary and positively serially correlated sunspot variable and use it as a coordination device associating a high realization of the sunspot variable with a high level of output. If agents observe a positive realization of the sunspot variable today, then they expect a high realization for tomorrow and therefore expect the level of output tomorrow to be high. This expectation lowers their current marginal utility of income, λ_t and leads to a leftward shift of the labor supply schedule. If the elasticity of output with respect to labor is such that the labor demand curve is upward sloping and steeper than the labor supply curve, the result of this shift in labor supply is that current employment and hence current output rises. For sufficiently large returns to scale the current increase in output exceeds that expected for tomorrow, and s.s.e. are possible.

As in Section 2.2, I assume that (32) holds. The necessary and sufficient condition for stationary sunspot equilibria is then given by

$$\min[\mu, f_2^{\mathrm{IR}}(\mu)] > \eta > \max[f_1^{\mathrm{IR}}(\mu), f_3^{\mathrm{IR}}(\mu)] \tag{49}$$

The definition of $f_1^{\mathrm{IR}}(\mu), f_2^{\mathrm{IR}}(\mu), f_3^{\mathrm{IR}}(\mu)$ as well as the derivation of this condition is given in Appendix B. Figure 5 gives a graphic idea of (49): $f_1^{\mathrm{IR}}(\mu)$, $f_2^{\mathrm{IR}}(\mu)$ and $f_3^{\mathrm{IR}}(\mu)$ are linear functions of μ, monotonically increasing with a slope of less than one; and $f_1^{\mathrm{IR}}(1), f_2^{\mathrm{IR}}(1), f_3^{\mathrm{IR}}(1) > 1$, so that they eventually cross the 45-degree line from above. Further, they all intersect at the same point, called μ^{**}.

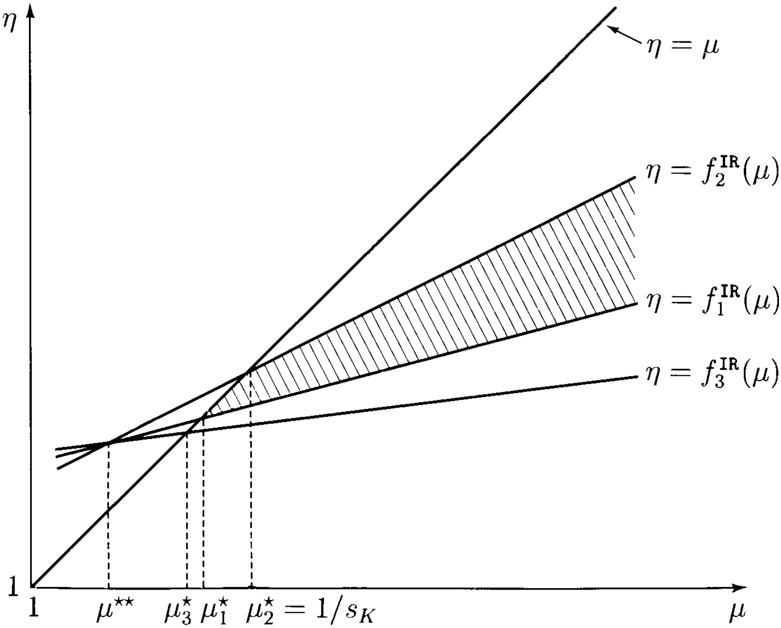

FIG. 5. IR model with log-utility, necessary and sufficient conditions for the existence of stationary sunspot equilibria.

I am interested in how large the markup has to be to allow for stationary equilibrium fluctuations in response to realizations of a sunspot variable. From (49) it follows that the markup has to be greater than

$$\mu_{\min}^{\mathrm{IR}} = \mu_{\min}^{\mathrm{IR}}(r, \delta, s_H, \varepsilon_{HW}) = \begin{cases} \max(\mu_1^*, \mu_3^*) & \text{if } \max(\mu_1^*, \mu_3^*) < \mu_2^* \\ \mu^{**} & \text{if } \mu_2^* < \max(\mu_1^*, \mu_3^*) < \mu^{**} \end{cases} \quad (50)$$

where μ_i^* is the intersection of $f_i^{\mathrm{IR}}(\cdot)$ with the 45-degree line. At $\mu = \mu_{\min}^{\mathrm{IR}}$ the RHS of (49) equals the LHS of (49); $\mu_{\min}^{\mathrm{IR}}$ is decreasing in s_H and ε_{HW} and increasing in r and δ. Equation (50) also gives a lower bound for the degree of homogeneity of the production function, η,

$$\eta > \eta_{\min} = \mu_{\min}^{\mathrm{IR}}.$$

As in the previous sections, I present a numerical example. Figure 6 shows pairs of μ and η, with $1 < \eta < \mu$, for which stationary sunspot equilibria exist, given numerical values for the other 11 parameters. Five of those will be assigned the common baseline RBC model values ($s_H = 0.58$, $\delta = 0.024$, $r = 0.016$, $\varepsilon_{HW} = 4$, and $\rho = 1$ from King *et al.*). The other 6 are pinned

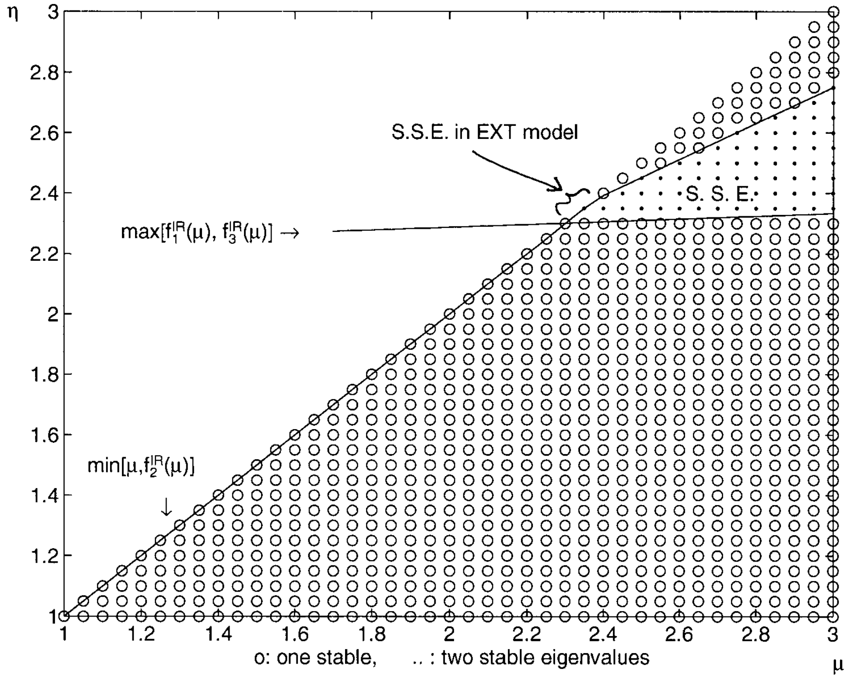

FIG. 6. IR and EXT model, numerical example with King–Plosser–Rebelo parameter values.

down by the steady state restrictions (9)–(14). In Fig. 6, circles again correspond to (μ, η) pairs, such that no stationary equilibrium fluctuation due to sunspot events exist, and dots correspond to (μ, η) pairs, such that s.s.e. exist. The figure also plots the RHS and the LHS of (49).

For this particular calibration $\mu^{IR}_{min}(0.016, 0.024, 0.58, 4) = 2.3$. Indeed, for markups above 2.3, several values exist for the degree of homogeneity, η, so that the calibrated model will display endogenous fluctuations. When η gets large or close to μ, then endogenous fluctuations disappear, and there is exactly one stable eigenvalue. The necessary condition given in (48), $\eta > (1 + \varepsilon_{HW})/(\varepsilon_{HW} s_H) = 2.16$ is not a sufficient condition for local indeterminacy here, whereas in the continuous time version (48) is necessary and sufficient (see Benhabib and Farmer [4]). Farmer and Guo [10] can show that stationary sunspot equilibria exist for markups smaller than 2.3 because they assume a larger labor share, $s_H = 0.7$, and an infinitely elastic labor supply, $\varepsilon_{HW} = \infty$.[25] These parameters imply that $\mu^{IR}_{min}(0.016, 0.024, 0.7, \infty) = 1.50$. Their numerical work, however, assumes

[25] They also assume a smaller interest rate, $r = 0.01$ (= 4 percent annual rate).

a markup of 1.72. Table 3 shows that $\mu^{\mathrm{IR}}_{\min}$ is rather sensitive to the labor share and labor supply elasticity. As in the CAD model, $\mu^{\mathrm{IR}}_{\min}$ almost does not change in response to changes in the depreciation or the interest rate. For any combination of an annual interest rate between 4 percent and 9 percent and an annual depreciation rate between 5 percent and 12 percent, $\mu^{\mathrm{IR}}_{\min}$ is always between 2.23 and 2.34. The evidence on the size of the markup reviewed in Section 3 is not reconcilable with a markup of 2.3, however, a markup of 1.5 seems close to the upper range of plausible markups. Further, endogenous fluctuations exist only for a combination of relatively high markups and large deviations from constant marginal costs.

2.5. *A Model with External Increasing Returns (EXT Model)*

Benhabib and Farmer [4] point out that an alternative model with a productive externality and perfectly competitive product markets (as, for example, Baxter and King [3]) has similar equilibrium conditions as the previous model, and hence s.s.e. may also exist. I call this model the externality, or EXT, model.

The economy consists of N identical firms. The production technology of each firm i is

$$Y^i_t = F(K^i_t, H^i_t, Y_t),$$

where Y^i_t is output of firm i in period t, K^i_t and H^i_t are factor inputs of capital and labor, respectively, by firm i in period t. $F(\cdot, \cdot)$ is assumed to be homogeneous of degree one in capital and labor and homogeneous of some degree, $\nu \in (0, 1)$ in aggregate output, Y_t, which each firm takes as given. Labor demand by firm i is

$$F_2(K^i_t, H^i_t, Y_t) = w_t,$$

where w_t is the wage rate in period t in units of the final good. The rental rate of capital is equated to the marginal product of capital

$$F_1(K^i_t, H^i_t, Y_t) = u_t.$$

Aggregate output is the sum of output in the N firms

$$Y_t = \sum_{i=1}^{N} Y^i_t.$$

The aggregate capital stock, K_t, is the sum of the N firm level capital stocks, and total hours are the sum of the hours hired by each firm. In a symmetric equilibrium each firm rents the same amount of capital and hires the same number of hours—that is, $K_t^i = K_t/N$ and $H_t^i = H_t/N$. The equilibrium conditions in terms of aggregates then are

$$Y_t = F(K_t, H_t, Y_t) \tag{51}$$

$$F_2(K_t, H_t, Y_t) = w_t \tag{52}$$

$$F_1(K_t, H_t, Y_t) = u_t. \tag{53}$$

The behavior of consumers is identical to the baseline RBC model. An equilibrium is the set of stochastic processes for the endogenous variables $\{K_t, H_t, w_t, u_t, Y_t, C_t, \lambda_t\}$ that satisfy (1)–(4) and (51)–(53). A steady state can be shown to exist.

Firm-level short-run increasing returns, which were greater than one in the models considered so far, are now equal to one:

$$\gamma = \frac{F_1 K^i + F_2 H^i}{F} = 1.$$

While γ measures the percentage increase in firm level output per one percent increase in factor inputs by that firm, aggregate increasing returns, $\tilde{\gamma}$, measures the percentage increase in aggregate output per one percent increase in aggregate factor inputs. Because of the externality, aggregate increasing returns are no longer equal to firm level increasing returns, but greater:

$$\tilde{\gamma} = \frac{F_1 K^i + F_2 H^i}{F(1 - NF_3)} = \frac{1}{1 - v} > \gamma. \tag{54}$$

Perfectly competitive product markets in this economy imply that profits are zero for each firm and that prices equal marginal cost. However, in equilibrium, marginal cost are decreasing and not constant as they were in the CAD and IC model. This model is the only one in which prices equal marginal cost so that the markup is one and returns to scale exceed the steady state markup. In the previous three models, aggregate returns to scale were equal to the steady state markup as a result of the zero profit assumption.

Dynamics around the steady state and necessary and sufficient conditions for stationary sunspot equilibria. After log-linearizing (51)–(53) around the steady state, one has

$$\hat{Y}_t = \tilde{\gamma} s_K \hat{K}_t + \tilde{\gamma} s_H \hat{H}_t \tag{55}$$

$$s_K(\hat{K}_t - \hat{H}_t) + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \hat{Y}_t = \hat{w}_t \tag{56}$$

$$s_H(\hat{H}_t - \hat{K}_t) + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \hat{Y}_t = \hat{u}_t. \tag{57}$$

The steady state labor and capital shares are defined as in Section 2.1. The definition of the degree of aggregate increasing returns, $\tilde{\gamma}$, is given by (54); and from the homogeneity properties of $F(\cdot, \cdot, \cdot)$ $\tilde{\gamma}$ is equal to $1/(1-v)$. The complete set of linearized equilibrium conditions consists of (5)–(8) and (55)–(57). It involves 12 parameters, 11 of which (ε_{HW}, $\varepsilon_{H\lambda}$, ε_{CW}, $\varepsilon_{C\lambda}$, ρ, s_H, s_K, s_I, s_C, δ, r) were introduced in Section 2.1; the only new free parameter is $\tilde{\gamma} > 1$. If $\tilde{\gamma} = 1$, the model is identical to the baseline RBC model.

In the IR model $\tilde{\gamma} = \gamma = \mu$. If one sets $\mu = \eta$ in the IR model, equilibrium is not well defined; however, the linearized equilibrium conditions are then identical to (55)–(57). In that sense, the EXT model is the limit of the IR model as $\eta \to \mu$. I exploit this to find the minimum necessary degree of aggregate increasing returns for local indeterminacy, $\tilde{\gamma}^{\text{EXT}}_{\min}$, from (49) to be

$$\tilde{\gamma}^{\text{EXT}}_{\min} = \max(\mu_1^*, \mu_3^*). \tag{58}$$

In the case that $\mu_2^* < \max(\mu_1^*, \mu_3^*)$, s.s.e. do not exist in the EXT model. From the numerical example in Section 2.4, $\tilde{\gamma}^{\text{EXT}}_{\min} = 2.3$ in the case of the King–Plosser–Rebelo calibration. One can represent (58) in Fig. 6, (μ, η) pairs, such that $\mu = \eta$, refer to the economic environment of the EXT model, while (μ, η) pairs such that $\mu > \eta$ refer to the IR model. In Fig. 6 sunspot driven fluctuations are possible only for a small range of $\tilde{\gamma}$ values and outside that narrow range endogenous fluctuations do not exist. Baxter and King [3] estimate $\tilde{\gamma}$ in aggregate data between 1.1 and 1.81 assuming a labor share of 0.54. This evidence combined with that discussed in the next section on the degree of increasing returns in manufacturing data suggests that aggregate increasing returns of 2.3 or greater are empirically not realistic.

3. EMPIRICAL EVIDENCE ON THE SIZE OF MARKUPS AND RETURNS TO SCALE

In this section I review some of the empirical evidence on the size of markups and returns to scale to determine whether the magnitudes necessary for indeterminacy are empirically plausible. Among others, Rotemberg and Woodford [21] point out that if product markets are imperfectly competitive and materials are used in production, then gross output markups are smaller than value added markups. Gross output markups, μ^g, are related to value added markups as $\mu^v = (1 - s_m)\,\mu^g/(1 - s_m \mu^g)$, where s_m is the share of material cost in gross output revenue. This paper abstracts from the use of materials as factor inputs and works

TABLE 4

Empirical estimates of markups and returns to scale

	Estimated parameter	Estimated value	Material share	μ^v or γ^v
Domowitz *et al.* (1988)	μ^g	1.4–1.7	0.49	>2.3
Morrison (1990)	μ^g, γ^g	1.2–1.4	0.50	1.5–2.3
Hall (1990)	$1/\mu^v$	<0.67		>1.5
	$1/\gamma^v$	<0.67		>1.5
Norrbin (1993)	μ^v	1.46		1.46
	$1/\mu^v$	0.29		3.42
	μ^g	1.05	0.62	1.14
	$1/\mu^g$	0.87	0.62	1.54
Basu and Fernald (1994)	γ^g	1.03–1.09	0.67	1.10–1.33
	γ^v	1.05–1.26		1.05–1.26
Roeger (1995)	$1-1/\mu^v$	0.13–0.38		1.15–1.61

Note. μ^g = gross output markup, μ^v = value added markup, $(\mu^v = (1-s_m)\,\mu^g/(1-s_m\mu^g))$, γ^g = gross output returns to scale, and γ^v = value added returns to scale, $(\gamma^v = (1-c_m)\,\gamma^g/(1-c_m\gamma^g))$.

directly with a value added production function, therefore, the observable counterpart to μ is value added markup. Table 4 presents the empirical evidence. I converted estimates of gross output returns to scale and gross output markups to approximate value added markups assuming that the cost and revenue shares of materials are the same.

Domowitz *et al.* [9] estimate gross output markups between 1.4 and 1.7. This estimate corresponds to value added markups of greater than 2.3.[26] Morrison [15] estimates gross output returns to scale and gross output markups. She finds values of 1.2 to 1.4 for 16 of her 18 industries. Assuming a material share of 0.5, this translates into value added markups between 1.5 and 2.3. Morrison also finds that gross output markups are close to gross output returns to scale, which supports my assumption that returns to scale equal markups. Hall [12] estimates the inverse of value added markups and value added returns to scale. Hall finds value added markups greater than 1.5 for all of the 7 1-digit industries and for 17 of the 21 2-digit industry groups he considers. His markup estimates assume constant returns to scale production functions which could lead to biased estimates. When estimating value added returns to scale, Hall finds values above 1.5 for 6 out of the 7 1-digit and 15 of the 21 2-digit industry groups. Norrbin

[26] Table 1 in Norrbin [16] lists the material share in each of the 2-digit industries included in the Domowitz *et al.* data set. From there, one computes an average material share of 0.49, which I use to compute the value added markup.

[16] uses the Hall data set to estimate value added markups directly rather than the inverse and finds that this lowers the estimates. A precision weighted mean of the estimated inverses of value added markups is 0.29, which implies an average value added markup of 3.42, and a precision weighted mean of the value added markups estimating the inverse is 1.68, whereas the precision weighted mean of value added markups when estimated directly is only 1.46. Norrbin also estimates gross output markups taking into account intermediate inputs and finds values between 1.05 and 1.15 depending on whether gross output markups or their inverse are estimated. For an average material share of 0.62, found in his data set, the implied value added markup lies between 1.14 and 1.54. Norrbin's estimates of gross output and value added markups assume a constant returns to scale production function and might therefore be biased. Roeger [18] uses the difference between the primal and dual productivity measure to estimate value added markups in the Hall data set. He estimates markups significantly greater than one for all 24 manufacturing industries he considers; and for 18 out of the 24 industry groups markups are less than 1.6. Like Hall and Norrbin, Roeger assumes a constant returns to scale production function; however, he finds substantially smaller markups.

Basu and Fernald [2] have shown empirically that misspecification errors in existing returns to scale estimates result in an upward bias. Correcting for the various sources of bias (ignoring materials as intermediate inputs, assuming constant returns to scale in the production of value added, estimating the inverse rather than the markup directly, etc.), Basu and Fernald find value added returns to scale of, at most, 1.26. In contrast to the papers cited so far, Basu and Fernald pool the data to estimate a common returns to scale parameter for all 21 manufacturing industries. When estimating gross output returns to scale, they find values between 1.03 and 1.09. Converting those values into value added returns to scale using a material share of 2/3 as observed in their data set translates into value added returns to scale between 1.10 and 1.33. Burnside [7] uses the same data set and a method similar to that of Basu and Fernald and shows that for different sets of instruments value added returns estimates lie between 1.12 and 1.30 in pooled regressions and that weighted averages of industry-specific returns estimates are even lower. Bartelsman, Caballero, and Lyons [1] also find small values for returns to scale. They estimate the parameter v in (54), describing short-run external effects, to be 0.12, which implies that external returns to scale are just 1.14. As the more recent papers (which find lower degrees of returns to scale) correct for various sources of bias that might have contaminated the earlier (substantially higher) estimates of returns to scale, I use the former as the benchmark against which I evaluate the empirical plausibility of indeterminacy of equilibrium in the four models analyzed.

4. COMPARING THE DEGREE OF RETURNS TO SCALE NECESSARY FOR THE EXISTENCE OF STATIONARY SUNSPOT EQUILIBRIA

A critical issue for the claim that theories based on endogenous rather than exogenous fluctuations can account for the observed co-movements of U.S. business cycles is whether the degree of aggregate increasing returns or the level of average markups necessary for such equilibria to exist is quantitatively plausible and, in particular, not too high. In this section, I compare the minimum level of (aggregate) returns to scale, $\tilde{\gamma}_{\min}$, necessary to make endogenous fluctuations possible. I phrase the discussion in terms of this parameter rather than the minimum necessary markup because $\tilde{\gamma}$ is a common parameter across all four models whereas the markup is not. In the models with internal increasing returns, that is the CAD, IC, and IR model, aggregate returns to scale equal firm level returns to scale and the latter equal markups. Hence, $\tilde{\gamma}_{\min}$ is given by (34) in the CAD model, by (50) in the IR model, and by (58) in the EXT model.

Table 5 shows $\tilde{\gamma}_{\min}$ for three alternative calibrations of the labor share and labor supply elasticity. For the King *et al.* calibration $\tilde{\gamma}_{\min}$ is equal to 1.75 and 2.3, for the constant and decreasing marginal cost models respectively. These numbers are well within the range of values Domowitz, Morrison, or Hall find for a typical industry. However, they are substantially larger than the more recent empirical evidence that, as I have argued above, should be used as the benchmark against which to judge the empirical plausibility of the degree of returns required for indeterminacy of equilibrium. Therefore, I conclude that, at least for the King *et al.* calibration, endogenous business cycles are—for all four models discussed here—only a theoretical possibility and do not occur for realistic parameterizations of the degree of returns to scale.

The four models will allow for stationary sunspot equilibria for substantially lower levels of aggregate increasing returns for a higher labor share and a higher labor supply elasticity. As shown above, in general, $\tilde{\gamma}_{\min}$ is a function of the depreciation rate, the interest rate, the labor share in value added, and the (Frisch) elasticity of labor supply and is decreasing in the two latter arguments. When labor supply is perfectly elastic ($\varepsilon_{HW} = \infty$) and the labor share is 0.7, $\tilde{\gamma}_{\min}$ falls to 1.37 and 1.5 for the constant and decreasing marginal cost models respectively. These magnitudes are empirically more realistic or at least close to the upper range of the relevant empirical estimates. Both the higher labor share and the more elastic labor supply contribute to the decline in $\tilde{\gamma}_{\min}$. For example, keeping the labor share at 0.58 as in the baseline calibration but allowing for infinitely elastic labor supply reduces $\tilde{\gamma}_{\min}$ to 1.55 and 1.78 for the CAD and IR model respectively. Alternatively, increasing the labor share to 0.7 and keeping the labor supply elasticity at 4 reduces $\tilde{\gamma}_{\min}$ to 1.57 and 1.89.

TABLE 5

$\tilde{\gamma}_{\min}$ for Different (s_H, ε_{HW}) Pairs

	$s_H = 0.58$ $\varepsilon_{HW} = 4$	$s_H = 0.7$ $\varepsilon_{HW} = \infty$	$s_H = 0.75$ $\varepsilon_{HW} = 1.3$
CAD model	1.75	1.37	1.84
IC model	1.75	1.37	1.84
IR model	2.30	1.50	2.53
EXT model	2.30	1.50	2.53

All four models require a high labor supply elasticity and a high labor share to bring $\tilde{\gamma}_{\min}$ down to empirically plausible magnitudes. The third column of Table 5 shows that for a labor supply elasticity of just 1.3 (as calibrated by Rotemberg and Woodford [20]) and a relatively high labor share of 75 percent, $\tilde{\gamma}_{\min}$ is 1.84 for the variable markup models and 2.53 for the constant markup models. Both values are hardly realistic empirically.

In the three calibrations presented in Table 5, both models of variable markups seem to be equal in the degree of increasing returns they require to make s.s.e. possible. As shown above, both models with decreasing marginal cost require exactly the same degree of returns to scale for s.s.e.: $\tilde{\gamma}^{\mathrm{IR}}_{\min} = \tilde{\gamma}^{\mathrm{EXT}}_{\min}$. In the case of the King *et al.* calibration, the two models of decreasing marginal costs and constant markups (IR and EXT models) require a roughly 30 percent higher degree of aggregate increasing returns for stationary sunspot equilibria to exist than do the two models of constant marginal costs and variable markups (CAD and IC models). The CAD and IC models will allow for stationary sunspot equilibria for lower levels of aggregate increasing returns than the IR and EXT model also for other calibrations. When labor supply is perfectly elastic and the labor share is 0.7, the fixed markup models require 10 percent higher returns to scale than the variable markup models to make s.s.e. possible. Although, I have not been able to derive generally that $\tilde{\gamma}^{\mathrm{IR}}_{\min} > \tilde{\gamma}^{\mathrm{CAD}}_{\min}$, a comparison between the entries in Table 2 and those in Table 3 suggests it. For all 36 (s_H, ε_{HW}) pairs considered $\mu^{\mathrm{IR}}_{\min} > \mu^{\mathrm{CAD}}_{\min}$. Not only does the variable markup model allow for s.s.e. for smaller markups, but Table 2 also contains many more markup values that are quantitatively reasonable than does Table 3.

Two recent papers, by Benhabib and Farmer [5] and Perli [17], put these results into perspective. Both papers show that two-sector extensions of the EXT model can generate local indeterminacy for substantially lower values of returns to scale than does any of the (one-sector) models discussed here. Benhabib and Farmer and Perli further show that local indeterminacy occurs for low (and empirically plausible) values of the degree of increasing returns even when the labor supply elasticity is small.

5. COMPARING THE PREDICTED CHARACTER OF AGGREGATE FLUCTUATIONS DUE TO SELF-FULFILLING EXPECTATIONS

The strategy in the real business-cycle literature has often been to ask whether a single (exogenous) shock can account for the observed co-movements in aggregate data; the same question has been asked of endogenous theories of the business-cycle. Farmer and Guo [10] argue that their model of aggregate fluctuations due to revisions of expectations can explain the contemporaneous correlations of output, investment, consumption, and employment in U.S. time series with about the same degree of precision as the standard RBC model. This section considers whether, if aggregate fluctuations are due solely to revisions of expectations, the predicted relative volatilities, persistence and cyclicality of output, employment, investment, and the real wage resemble those observed in (detrended) U.S. time series.

To generate those numerical predictions all four models are calibrated using the baseline King, Plosser, and Rebelo values listed in Table 1. The level of aggregate increasing returns is set as small as possible while still allowing for s.s.e. in all four models. The value assigned is $\tilde{\gamma}=2.35$. The set of statistics is shown in Table 6. The table reports these statistics also for linear detrended U.S. data and for the standard RBC model assuming a persistent technology shock (Columns 1 and 2 are taken from Tables 5 and 6 in King *et al.* [14]). All four models can replicate the fact that consumption is less volatile than output and that investment is more volatile than output. As Hornstein [13], for example, has noted, introducing monopolistic competition and increasing returns into a technology shock driven RBC model reduces the relative volatility of hours substantially, particularly in an increasing returns economy with constant markups and constant marginal costs. Here both models with constant marginal costs and variable markups overpredict the relative volatility of hours whereas both models of decreasing marginal cost and constant markups underpredict it.

Panel b of Table 6 reports first order serial correlations. The disturbance is serially uncorrelated but still produces highly persistent output fluctuations in all four models. The CAD and IC models correctly predict positive serial correlation for the other variables, as well, whereas the IR and EXT models predict a negative serial correlation for investment and hours. This prediction is highly dependent on the size of $\tilde{\gamma}$ and η; for higher values of $\tilde{\gamma}$,

TABLE 6

a. Relative standard deviation: std(x)/std(output)

	U.S. data	KPR model	CAD model	IC model	IR model	EXT model
Output	1.00	1.00	1.00	1.00	1.00	1.00
Consumption	0.69	0.64	0.35	0.39	0.82	0.91
Investment	1.35	2.31	3.36	3.41	2.32	1.82
Hours	0.52	0.48	0.71	0.70	0.43	0.32
Real wage	1.14	0.69	0.42	0.44	0.83	0.91

b. Autocorrelation coefficient AR(1)

	U.S. data	KPR model	CAD model	IC model	IR model	EXT model
Output	0.96	0.93	0.89	0.71	0.60	0.81
Consumption	0.98	0.99	0.98	0.98	1.00	1.00
Investment	0.93	0.88	0.88	0.66	−0.08	0.16
Hours	0.52	0.86	0.88	0.66	−0.24	−0.12
Real wage	0.97	0.98	0.94	0.88	0.97	0.99

c. Contemporaneous correlation with output

	U.S. data	KPR model	CAD model	IC model	IR model	EXT model
Consumption	0.85	0.82	0.65	0.58	0.84	0.92
Investment	0.60	0.92	0.97	0.96	0.82	0.82
Hours	0.07	0.79	0.85	0.86	0.56	0.42
Real wage	0.76	0.90	0.91	0.86	0.90	0.95

Note. Columns 1 and 2 of Table 6 are taken from Tables 5 and 6 in King *et al.* [14]. Column 2 (KPR model) assumes that the stochastic process for the technology shock follows an AR(1) process with serial correlation of 0.9. The computed model moments are independent, in the case of the exogenous cycle model, of the variance of the technology innovation and in the case of the endogenous cycle models of the variance of the sunspot innovation, $\text{var}(\nu_t)$.

the negative serial correlations disappear. Figure 7 illustrates that persistence in output is also highly dependent on the particular parameterization. It shows how the first order serial correlation of output varies with the elasticity of the markup in the CAD and IC models, and with the degree of homogeneity, η, in the IR model. For the range of values that make s.s.e. possible, this correlation (as a function of a single parameter) varies continuously from -1 to $+1$. In the CAD and IC models output is highly serially correlated (more than 0.5) for most values of ε_μ that make s.s.e. possible and starts to fall almost vertically only when ε_μ becomes small (in absolute value) and approaches the critical value beyond which endogenous fluctuations can no longer exist. This is not the case in the IR model. Here, over a rather small range of η (η is between 2.3 and 2.35, which should not be distinguishable empirically) which make s.s.e. possible first order serial correlation increases almost linearly from -1 to $+1$, and for less than 1/3 of the values of η does the serial correlation exceed 0.5. In the IR model, returns to scale of 2.35 are very close to the minimum degree of increasing returns to scale necessary for indeterminacy, and a parameterization so close to the bifurcation point might not be representative of the relation between η and the serial correlation of output. Therefore, in the right column of Fig. 7, I consider an alternative calibration with a higher labor share ($s_H = 0.7$) and perfectly elastic labor supply. In this calibration, endogenous fluctuations become possible in all models for returns to scale above 1.5. I use a value of 1.6 for returns to scale, which is less close to the bifurcation point of this alternative calibration. Now there are more values of η such that s.s.e exist (if $\mu = 1.6$ then η: $1.5 < \eta < 1.6$ satisfy (49)). The right column shows that, in this case, the IR model predicts serial correlation to exceed 0.5 for about 1/2 of the values for η that make s.s.e. possible.

The behavior of the serial correlation of output is just a mirror image of the behavior of the eigenvalues of the Jacobian matrix of the system of equilibrium conditions. For a given degree of returns to scale, the eigenvalues are a monotonic and continuous function of ε_μ or η respectively, which vary from $+1$ to -1 over the range of values that generate local indeterminacy. For most values inside this range, the (real parts of the) eigenvalues are close to one, and only as ε_μ or η approaches the boundary of that range, do they drop almost vertically to -1. This seems to suggest that the critical parameter determining the propagation mechanism in these models is not so much the degree of market power or the elasticity of the markup, but rather how close the (μ, ε_μ) or the (μ, η) pair is from a bifurcation point. If it is very close, then small perturbations in the markup, as well as the elasticity of the markup or the degree of homogeneity of the production function, can potentially change the dynamic properties of the economy importantly. Although for most parameterizations of the elasticity of the markup and the degree of homogeneity of the production function that make s.s.e. possible output is highly serially correlated when the cycle is driven solely by revisions to expectation, I nevertheless want to stress that persistent output fluctuations are not a necessary property of stationary sunspot equilibria in the models discussed in this paper.

Finally, I want to point out how different the predictions of the IR and EXT models for the relative standard deviation and the autocorrelation are despite the small differences in the assumed parameters. In the IR model $\eta = 2.34$ whereas in the EXT model $\eta = 2.35$. The EXT model predicts a

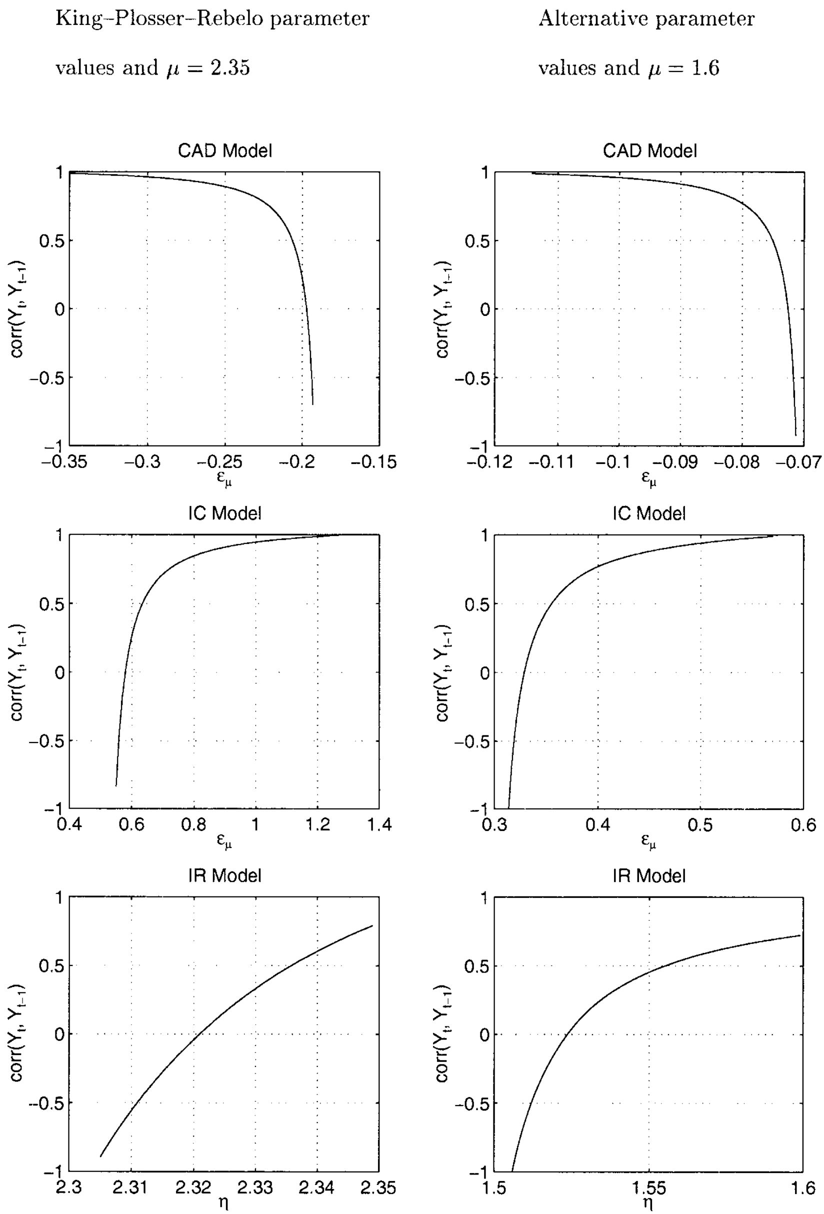

FIG. 7. Serial correlation of output.

lower relative volatility of investment (1.82 vs 2.32) and total hours (0.32 vs 0.43) and predicts a higher volatility of consumption and the real wage. More striking is that the IR model predicts a negative investment serial correlation whereas the EXT model predicts a positive one. Again, this difference is a consequence of this calibration's being so close to the bifurcation point. In Table 7, I repeat Table 6 for ($s_H = 0.7$, $\varepsilon_{HW} = \infty$), the calibration used by Farmer and Guo, and a markup of 1.6. In this case, all four models behave very much alike: the only difference is that the IR model

TABLE 7

Alternative Calibration: $s_H = 0.7$, $\varepsilon_{HW} = \infty$, and $\tilde{\gamma} = 1.6$[a]

a. Relative standard deviation: std(x)/std(output)

	CAD model	IC model	IR model	EXT model
Output	1.00	1.00	1.00	1.00
Consumption	0.24	0.34	0.21	0.32
Investment	4.85	4.48	5.16	4.85
Hours	0.88	0.85	0.89	0.86
Real wage	0.24	0.34	0.21	0.32

b. Autocorrelation coefficient AR(1)

	CAD model	IC model	IR model	EXT model
Output	0.96	0.98	0.45	0.73
Consumption	0.98	0.99	0.94	0.97
Investment	0.96	0.98	0.42	0.69
Hours	0.96	0.97	0.42	0.69
Real wage	0.98	0.99	0.94	0.97

c. Contemporaneous correlation with output

	CAD model	IC model	IR model	EXT model
Consumption	0.67	0.76	0.46	0.57
Investment	0.99	0.97	0.99	0.97
Hours	0.95	0.93	0.98	0.95
Real wage	0.67	0.76	0.46	0.57

[a] CAD model, $\varepsilon_\mu = -0.1$, IC model, $\varepsilon_\mu = 0.55$, IR model, $\eta = 1.55$.

predicts less persistence in aggregate hours, investment, and output than do the endogenous markup models.[27]

6. CONCLUSIONS

This paper compared four equilibrium business cycle models that allow for aggregate fluctuations in the absence of shocks to economic fundamentals. The models depart from standard business cycle models in that firms have market power or increasing returns to scale production functions, or both. On the one hand, the paper shows that the relative volatility, autocorrelation, and contemporaneous correlation properties of macroeconomic aggregates predicted by each of the four endogenous business cycle models are broadly consistent with those actually observed in postwar U.S. data. On the other hand, the paper raises two questions about the empirical plausibility of self-fulfilling expectations as an explanation of actual business cycles; for the four models studied, the degree of market power or returns to scale required for the existence of expectations driven business cycles lies in the upper range of available empirical estimates and persistent output fluctuations are not a necessary property of the predicted business cycle.

APPENDIX A: DERIVATION OF (33)

This appendix derives the necessary and sufficient condition for the existence of stationary sunspot equilibria in the CAD model, which is (33) in the text.

I want to show for which (μ, ε_μ) pairs both eigenvalues of the matrix M defined in (30) are less than one in modulus. This is the case if and only if

$$-1 < \det(M) < 1 \qquad \text{and} \qquad -(1+\det(M)) < tr(M) < 1+\det(M).$$

The determinant of M is given by (31) which one can rewrite as

$$\det(M) = (1+r)\left(1 + x\left(\equiv \frac{\text{Numerator}}{\text{Denominator}}\right)\right).$$

[27] In Farmer and Guo [10] the predicted output correlation is 0.8 while it is 0.42 in my computations. The difference is due to the higher markup and degree of homogeneity of the production function assumed by Farmer and Guo, ($\mu = 1.72$, $\eta = 1.6$) vs ($\mu = 1.6$, $\eta = 1.55$), which was used in Table 7. Further, Farmer and Guo detrend the simulated model data using the Hodrick–Prescott filter before computing the various second moments whereas I do not.

The claim is that, provided $s_H/(1+\varepsilon_{HW}^{-1}) > (r+\delta)/(1-\delta)$, the determinant of M will be less than one in modulus, if and only if,

$$f_1^{\mathrm{CAD}}(\mu) > \varepsilon_\mu > f_2^{\mathrm{CAD}}(\mu) \qquad \text{and} \qquad \varepsilon_\mu < f_5^{\mathrm{CAD}}(\mu) \tag{A1}$$

or if both inequalities hold with the inequality signs reversed, where

$$f_1^{\mathrm{CAD}}(\mu) \equiv \frac{\dfrac{(r+\delta)(1+(1+r)\mu)}{(1-\delta)(2+r)} + \left(1 - \dfrac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}\right)}{\dfrac{r+\delta}{(1-\delta)(2+r)} - \dfrac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}\mu} \frac{s_I}{s_C},$$

$$f_2^{\mathrm{CAD}}(\mu) \equiv \frac{\dfrac{-(r+\delta)(1-(1+r)\mu)}{(1-\delta)\,r} + \left(1 - \dfrac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}\right)}{\dfrac{-(r+\delta)}{(1-\delta)\,r} - \dfrac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}\mu} \frac{s_I}{s_C} \qquad \text{and}$$

$$f_5^{\mathrm{CAD}}(\mu) \equiv \frac{\dfrac{r+\delta}{1-\delta} + 1 - \dfrac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}}{\dfrac{r+\delta}{1-\delta} - \dfrac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}\mu} \frac{s_I}{s_C}.$$

$\varepsilon_\mu - f_5^{\mathrm{CAD}}(\mu) = 0$ describes (μ, ε_μ) pairs such that the denominator of x equals zero. For (μ, ε_μ) pairs such that $\varepsilon_\mu > f_5^{\mathrm{CAD}}(\mu)$ the denominator of x is positive and for (μ, ε_μ) pairs such that $\varepsilon_\mu < f_5^{\mathrm{CAD}}(\mu)$ the denominator of x is negative. At $\mu = 1$, $f_5^{\mathrm{CAD}}(\mu)$ is negative and $f_5^{\mathrm{CAD}}(\mu)$ is monotonically increasing. The limit of $f_5^{\mathrm{CAD}}(\mu)$ as μ approaches infinity is zero. $\varepsilon_\mu - f_1^{\mathrm{CAD}}(\mu) = 0$ describes (μ, ε_μ) pairs such that $\det(M) = -1$. $\det(M) > -1$ for (μ, ε_μ) pairs such that $\varepsilon_\mu > f_1^{\mathrm{CAD}}(\mu)$ and such that the denominator of x is positive and for (μ, ε_μ) pairs such that $\varepsilon_\mu < f_1^{\mathrm{CAD}}(\mu)$ and such that the denominator of x is negative. $f_1^{\mathrm{CAD}}(\mu)$ is negative and greater than $f_5^{\mathrm{CAD}}(1)$, at $\mu = 1$, and monotonically increasing in μ, and as $\mu \to \infty$ it approaches a negative constant from below. $\varepsilon_\mu - f_2^{\mathrm{CAD}}(\mu) = 0$ describes the (μ, ε_μ) pairs such that $\det(M) = 1$. $\det(M) < 1$ for (μ, ε_μ) pairs such that $\varepsilon_\mu > f_2^{\mathrm{CAD}}(\mu)$ and such that the denominator of x is negative and for (μ, ε_μ) pairs such that $\varepsilon_\mu < f_2^{\mathrm{CAD}}(\mu)$ and such that the denominator of x is positive. $f_2^{\mathrm{CAD}}(\mu)$ is strictly decreasing. At $\mu = 1$, $f_2^{\mathrm{CAD}}(\mu)$ is also negative, though above $f_1^{\mathrm{CAD}}(\mu = 1)$. As $\mu \to \infty$, $f_2^{\mathrm{CAD}}(\mu)$ converges to $((2+r)/r)\lim_{\mu\to\infty} f_1^{\mathrm{CAD}}(\mu) < \lim_{\mu\to\infty} f_1^{\mathrm{CAD}}(\mu)$. Therefore, $f_2^{\mathrm{CAD}}(\mu)$ intersects $f_1^{\mathrm{CAD}}(\mu)$ at some $\mu > 1$. $f_5^{\mathrm{CAD}}(\mu)$ will cross $f_1^{\mathrm{CAD}}(\mu)$ from below for some $\mu > 1$ and further it will also intersect $f_2^{\mathrm{CAD}}(\mu)$. It can be shown that these three functions all intersect only once for $\mu > 1$ and also at the same $\mu > 1$. Let's call this point μ^*, then we have that for $\mu < \mu^*$, $f_2^{CAD}(\mu) > f_1^{CAD}(\mu) > f_5^{CAD}(\mu)$ and for $\mu > \mu^*$, $f_5^{CAD}(\mu) > f_1^{CAD}(\mu) > f_2^{CAD}(\mu)$. From here it follows that $\varepsilon_\mu < 0$ is a necessary condition for the determinant to be less than one in modulus and hence for s.s.e. to exist. Combining (A1) with the additional restriction I derived above that ε_μ has to be greater than $1-\mu$, one has that s.s.e. will exist only if

$$f_1^{CAD}(\mu) > \varepsilon_\mu > \max[f_2^{CAD}(\mu), 1-\mu]$$

or

$$f_2^{CAD}(\mu) > \varepsilon_\mu > \max[f_1^{CAD}(\mu), 1-\mu].$$

If one further imposes the restrictions that

$$tr(\mathrm{M}) < 1 + \det(M) \tag{A2}$$

one can rule out that stationary sunspot fluctuations exist for positive values of the denominator of x. That is, none of the (μ, ε_μ) pairs with μ values less than μ^* satisfy (A2). If the denominator of x is negative I show next that then (A2) does not impose any restrictions on (μ, ε_μ) beyond those already implied by the determinant being less than one in absolute value. That is all of the (μ, ε_μ) pairs satisfying $f_1^{CAD}(\mu) > f_2^{CAD}(\mu)$ satisfy (A2). This implies that μ^* is the smallest markup for which endogenous fluctuations may exist. The trace of M can be written as

$$tr(M) = 1 + \det(M) + \frac{(r+\delta)}{\text{denom of } x}\frac{\delta}{s_I}\left((\mu-1)\frac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}\right.$$
$$\left. + s_H s_C + (1 - s_K \mu)\,\varepsilon_\mu s_C\right).$$

Condition (A2) holds if and only if $\varepsilon_\mu < f_5^{CAD}$ and

$$\varepsilon_\mu \begin{cases} < f_4^{CAD}(\mu) & \text{if} \quad \mu > 1/s_K \\ > f_4^{CAD}(\mu) & \text{if} \quad \mu < 1/s_K \end{cases} \tag{A3}$$

$$\text{where} \quad f_4^{CAD}(\mu) \equiv \frac{\dfrac{\mu-1}{s_C}\dfrac{s_H}{1+\varepsilon_{HW}^{-1}} + s_H}{s_K \mu - 1}$$

or if $\varepsilon_\mu > f_5^{\text{CAD}}$ and (A3) holds with the inequality signs reversed. $f_4^{\text{CAD}}(\mu)$ is strictly decreasing, $f_4^{\text{CAD}}(\mu=1) = -1$, positive for $\mu > 1/s_K$ and less than -1 for $1 < \mu < 1/s_K$; further,

$$\lim_{\mu \nearrow 1/s_K} f_4^{\text{CAD}}(\mu) = -\infty$$

$$\lim_{\mu \searrow 1/s_K} f_4^{\text{CAD}}(\mu) = \infty.$$

To see what this implies for allowable (μ, ε_μ) pairs I organize the discussion into four cases, that is $tr(M) < 1 + \det(M)$ only if either

(a) $\varepsilon_\mu < \min(f_5^{\text{CAD}}(\mu), f_4^{\text{CAD}}(\mu))$ for $\mu > 1/s_K$,
(b) $f_4^{\text{CAD}}(\mu) < \varepsilon_\mu < f_5^{\text{CAD}}(\mu)$ for $1 < \mu < 1/s_K$,
(c) $\varepsilon_\mu > \max(f_5^{\text{CAD}}(\mu), f_4^{\text{CAD}}(\mu))$ for $\mu > 1/s_K$, or
(d) $f_5^{\text{CAD}}(\mu) < \varepsilon_\mu < f_4^{\text{CAD}}(\mu)$ for $1 < \mu < 1/s_K$.

to (a): as $f_4^{\text{CAD}}(\mu) > 0 > f_5^{\text{CAD}}(\mu)$ when $\mu > 1/s_K$ this is satisfied by all $\varepsilon_\mu < f_5^{\text{CAD}}(\mu)$.

to (b): from above we that if $\varepsilon_\mu < f_5^{\text{CAD}}(\mu)$, then $\text{abs}(\det(M)) < 1$ only if $\varepsilon_\mu > f_2^{\text{CAD}}(\mu)$. As

$$f_2^{\text{CAD}}(\mu = 1/s_K) = \frac{\delta(1+r)/r/(1-\delta) - s_I/s_C s_H/(1+\varepsilon_{HW}^{-1})}{-(r+\delta)/(1-\delta)/r - 1/s_K s_H/(1+\varepsilon_{HW}^{-1})} > -1$$
$$= f_4^{\text{CAD}}(\mu = 1)$$

$f_2^{\text{CAD}}(\mu) > f_4^{\text{CAD}}(\mu)$ for $1 < \mu < 1/s_K$. So (b) does not impose any restrictions on feasible (μ, ε_μ) pairs beyond $f_5^{\text{CAD}}(\mu) > \varepsilon_\mu > f_2^{\text{CAD}}(\mu)$. $\Rightarrow$ whenever $f_5^{\text{CAD}}(\mu) > \varepsilon_\mu > f_2^{\text{CAD}}(\mu)$ then (a) and (b) are satisfied.

to (c): as $f_4^{\text{CAD}}(\mu) > 0$ for $\mu > 1/s_K$ this can never be satisfied at the same time as $\text{abs}(\det(M)) < 1$. $\Leftrightarrow$ no (μ, ε_μ) pairs satisfy (c) and $-1 < \det(M) < 1$.

to (d): For $1 < \mu < 1/s_K$ I claim that there are no (μ, ε_μ) pairs such that $f_4^{\text{CAD}}(\mu) > \varepsilon_\mu > \max[f_5^{\text{CAD}}(\mu), 1-\mu]$. To prove this I need the additional assumption that $\varepsilon_{HW} \geqslant 0.5$. Consider $s_K \geqslant 1/5$, then one can show that $f_4(\mu) < 1-\mu$ whenever $1 < \mu < 1/s_K$. Consider $s_K \leqslant 1/5$. For $1 < \mu \leqslant 2$, $1 - \mu > f_4(\mu)$, at $\mu = 2$ (using the assumption that $\varepsilon_{HW} \geqslant 0.5$) $f_5^{\text{CAD}}(\mu=2) > -1 > f_4^{\text{CAD}}(\mu=2)$. Since $f_5^{\text{CAD}}(\mu)$ is strictly increasing, we have that $f_4^{\text{CAD}}(\mu) < \max[1-\mu, f_5^{\text{CAD}}(\mu)]$ for $1 < \mu < 1/s_K$. $\Leftrightarrow$ no (μ, ε_μ) pairs satisfy (d), $\varepsilon_\mu > 1 - \mu$, and $-1 < \det(M) < 1$.

It follows from here that $-1 < \det(M) < 1$ and $tr(M) < 1 + \det(M)$ if and only if $f_1^{\text{CAD}}(\mu) > \varepsilon_\mu > f_2^{\text{CAD}}(\mu)$.

To be sure that endogenous fluctuations will in fact occur one also has to impose the restriction that $tr(M) > -(1+\det(M))$, then one finally has sufficiency. Here one can restrict the analysis to the case that the denominator of x is negative, because only then $f_2^{\mathrm{CAD}}(\mu) < f_1^{\mathrm{CAD}}(\mu)$. In that case $tr(M) > -(1+\det(M))$ if and only if

$$\varepsilon_\mu < f_3^{\mathrm{CAD}}(\mu),$$

where

$$f_3^{\mathrm{CAD}}(\mu) \equiv \left\{ \frac{(r+\delta)(1+(1+r)\,\mu)}{(1-\delta)(2+r)} + \left(1 - \frac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}\right) + \frac{(r+\delta)\,\delta/(2s_I)}{(1-\delta)(2+r)} \left[(\mu-1)\frac{s_H}{1+\varepsilon_{HW}^{-1}} + s_H \right] \frac{s_I}{s_C} \right\} \Big/ \left\{ \frac{r+\delta}{(1-\delta)(2+r)} - \frac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}\mu + \frac{(r+\delta)\,\delta/2}{(1-\delta)(2+r)}(s_K \mu - 1) \right\}.$$

$f_3^{\mathrm{CAD}}(\mu)$ is monotonically increasing, at $\mu = 1$ $f_2^{\mathrm{CAD}}(\mu) > f_3^{\mathrm{CAD}}(\mu)$, and $\lim_{\mu\to\infty} f_3^{\mathrm{CAD}}(\mu) < \lim_{\mu\to\infty} f_1^{\mathrm{CAD}}(\mu)$. $\lim_{\mu\to\infty} f_3^{\mathrm{CAD}}(\mu)$ is not necessarily greater than $\lim_{\mu\to\infty} f_2^{\mathrm{CAD}}(\mu)$. However, if one assumes that the steady state capital labor ratio is greater than one,[28] then $\lim_{\mu\to\infty} f_3^{\mathrm{CAD}}(\mu) > \lim_{\mu\to\infty} f_2^{\mathrm{CAD}}(\mu)$ so that there are (μ, ε_μ) pairs such that $f_3^{\mathrm{CAD}}(\mu) > \varepsilon_\mu > f_2^{\mathrm{CAD}}(\mu)$. Collecting all the restrictions I finally have that both eigenvalues of the matrix M are less than one in modulus and $\varepsilon_\mu > 1-\mu$ if and only if

$$\max[1-\mu, f_2^{\mathrm{CAD}}(\mu)] < \varepsilon_\mu < \min[f_3^{\mathrm{CAD}}(\mu), f_1^{\mathrm{CAD}}(\mu)]$$

which is (33).

APPENDIX B: DERIVATION OF (49)

This appendix derives the necessary and sufficient condition for the existence of stationary sunspot equilibria in the IR model, which is (49) in the text.

[28] In quarterly U.S. data this ratio is 10.4.

I am looking for (μ, η) pairs with $\eta < \mu$ such that

$$-1 < \det(M) < 1 \qquad \text{and} \qquad -(1+\det(M)) < tr(M) < 1+\det(M)$$

where the determinant of M is given by (47) and the trace of M is given by

$$tr(M) = 1 + \det(M) + \delta/s_I \frac{(\mu-\eta)\dfrac{s_H\varepsilon_{HW}}{1+\varepsilon_{HW}} + s_C(1-s_K\eta)}{1+\dfrac{1-\delta}{r+\delta}\left(1-\dfrac{s_H\varepsilon_{HW}}{1+\varepsilon_{HW}}\eta\right)}.$$

A necessary condition for $\det(M) < 1$ is

$$1+\frac{1-\delta}{r+\delta}\left(1-\frac{s_H\varepsilon_{HW}}{1+\varepsilon_{HW}}\eta\right) < 0. \tag{B1}$$

Provided (B1) holds $\det(M) > -1$ if and only if

$$\eta > \frac{\dfrac{1+r}{r+\delta} - \dfrac{1+r}{2+r} + \dfrac{1+r}{2+r}\mu}{\dfrac{1-\delta}{r+\delta}\dfrac{s_H\varepsilon_{HW}}{1+\varepsilon_{HW}}} \equiv f_1^{\mathrm{IR}}(\mu), \tag{B2}$$

where f_1^{IR} has the properties

$$f_1^{\mathrm{IR}}(\mu=1) > 1,\ 0 < f_1'^{\mathrm{IR}}(\mu) < 1 \ \left(\text{assuming } \frac{s_H}{1+\varepsilon_{HW}^{-1}} > \frac{r+\delta}{1-\delta}\right).$$

Note that (B2) implies (B1) and that $\eta > 1$. Given (B1) $\det(M) < 1$ if and only if

$$\eta < \frac{\dfrac{1+r}{r}\mu - \dfrac{1+r}{r}\dfrac{\delta}{r+\delta}}{\dfrac{1-\delta}{r+\delta}\dfrac{s_H\varepsilon_{HW}}{1+\varepsilon_{HW}}}. \tag{B3}$$

At $\mu = 1$ the RHS of (B3) is greater than one and the slope is also greater than one, so that (B3) is satisfied for any $1 < \eta < \mu$.

Hence, for $\eta < \mu$, $-1 < \det(M) < 1$ if and only if

$$\eta > f_1^{\mathrm{IR}}(\mu).$$

If (B1) holds then $tr(M) < 1 + \det(M)$ if and only if

$$\eta < \frac{s_C + \frac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}} \mu}{s_C s_K + \frac{s_H \varepsilon_{HW}}{1+\varepsilon_{HW}}} \equiv f_2^{\mathrm{IR}}(\mu), \tag{B4}$$

where $f_2^{\mathrm{IR}}(\mu)$ has the properties

$$f_2^{\mathrm{IR}}(\mu = 1) > 1, \; 0 < f'^{\mathrm{IR}}(\mu) < 1.$$

Finally, given (B1), $tr(M) > -(1 + \det(M))$ if and only if

$$\eta > \frac{\frac{1+r}{r+\delta} - \frac{1+r}{2+r} + \frac{1+r}{2+r}\mu + \frac{\delta}{2s_I(2+r)}\left(s_C + \frac{s_H}{1+\varepsilon_{HW}^{-1}}\mu\right)}{\frac{1-\delta}{r+\delta}\frac{s_H}{1+\varepsilon_{HW}^{-1}} + \frac{\delta}{2s_I(2+r)}\left(s_C s_K + \frac{s_H}{1+\varepsilon_{HW}^{-1}}\right)} \equiv f_3^{\mathrm{IR}}(\mu), \tag{B5}$$

where $f_3^{\mathrm{IR}}(\mu)$ has the properties

$$f_3^{\mathrm{IR}}(\mu = 1) > 1, \; 0 < f_3'^{\mathrm{IR}}(\mu) < 1.$$

Combining (B2)–(B5) I have that both eigenvalues of M are less than one in modulus and $\eta < \mu$ if and only if

$$\min[\mu, f_2^{\mathrm{IR}}(\mu)] > \eta > \max[f_1^{\mathrm{IR}}(\mu), f_3^{\mathrm{IR}}(\mu)],$$

which is (49).

REFERENCES

1. E. Bartelsman, R. Caballero, and R. Lyons, Customer- and supplier-driven externalities, *Amer. Econ. Rev.* **84** (1994), 1075–1084.
2. S. Basu and J. Fernald, "Constant Returns and Small Markups in U.S. Manufacturing," International Finance Discussion Paper 483, Board of Governors of the Federal Reserve System, 1994.
3. M. Baxter and R. G. King, "Productive Externalities and Business Cycles," Discussion Paper 53, Institute for Empirical Macroeconomics, Federal Reserve Bank of Minneapolis, 1991.
4. J. Benhabib and R. E. A. Farmer, Indeterminacy and increasing returns, *J. Econ. Theory* **63** (1994), 19–41.
5. J. Benhabib and R. E. A. Farmer, "Indeterminacy and Sector-Specific Externalities," C. V. Starr Center Research Report 95-02, New York Univ., 1995.

6. M. Bils, The cyclical behavior of marginal cost and price, *Amer. Econ. Rev.* **77** (1987), 647–666.
7. C. Burnside, What do production function regressions tell us about increasing returns to scale and externalities?, unpublished manuscript, Univ. of Pittsburgh, 1994.
8. A. Dixit and J. Stiglitz, Monopolistic competition and optimum product diversity, *Amer. Econ. Rev.* **67** (1977), 297–308.
9. I. Domowitz, G. Hubbard, and B. Petersen, Market structure and cyclical fluctuations in U.S. manufacturing, *Rev. Econ. Statist.* **70** (1988), 55–66.
10. R. E. A. Farmer and J. T. Guo, Real business cycles and the animal spirits hypothesis, *J. Econ. Theory* **63** (1994), 42–72.
11. J. Galí, Monopolistic competition, business cycles, and the composition of aggregate demand, *J. Econ. Theory* **63** (1994), 73–96.
12. R. E. Hall, Invariance properties of Solow's productivity residual, *in* "Growth–Productivity–Unemployment" (P. Diamond, Ed.), pp. 71–112, MIT Press, Cambridge, MA, 1990.
13. A. Hornstein, Monopolistic competition, increasing returns to scale and the importance of productivity changes, *J. Monet. Econ.* **31** (1993), 299–316.
14. R. G. King, C. Plosser, and S. Rebelo, Production, growth and business cycles: I. The basic neoclassical model, *J. Monet. Econ.* **21** (1988), 195–232.
15. C. Morrison, "Market Power, Economic Profitability and Productivity Growth Measurement: An Integrated Structural Approach," NBER Working Paper No. 3355, NBER, 1990.
16. S. Norrbin, The relation between price and marginal cost in U.S. industry: A contradiction, *J. Polit. Econ.* **101** (1993), 1149–1166.
17. R. Perli, Indeterminacy, home production, and the business-cycle: A calibrated analysis, unpublished manuscript, New York Univ., 1995.
18. W. Roeger, Can imperfect competition explain the difference between primal and dual productivity measures? Estimates for U.S. manufacturing, *J. Polit. Econ.* **103** (1995), 316–330.
19. J. J. Rotemberg and M. Woodford, Markups and the business-cycle, *in* "NBER Macroeconomics Annual 1991" (O. J. Blanchard and S. Fisher, Eds.), pp. 63–129, MIT Press, Cambridge, MA, 1991.
20. J. J. Rotemberg and M. Woodford, Oligopolistic pricing and the effects of aggregate demand on economic activity, *J. Polit. Econ.* **100** (1992), 1153–1207.
21. J. J. Rotemberg and M. Woodford, Dynamic General Equilibrium Models with Imperfectly Competitive Product Markets, *in* "Frontiers of Business Cycle Research" (T. F. Cooley, Ed.), pp. 243–293, Princeton Univ. Press, Princeton, NJ, 1994.
22. M. Woodford, Stationary sunspot equilibria: The case of small fluctuations around a deterministic steady state, unpublished manuscript, The Univ. of Chicago, 1986.