Convolution of a truncated normal and a centered normal variable

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Proposition 0.1. *Let* $X \sim N(0, s^2)$ *and* $Y \sim TN(\mu, \sigma, a, b)$*, independent*[1]

Then, $V = X + Y$ *is distributed according to the density*

$$f(v) = \gamma e^{-\frac{(v-\mu)^2}{2(s^2+\sigma^2)}} \left[\Phi(\frac{v-a-\alpha}{\beta}) - \Phi(\frac{v-b-\alpha}{\beta}) \right]$$

where

- $\alpha = \frac{s^2(v-\mu)}{s^2+\sigma^2}$, $\beta^2 = \frac{s^2\sigma^2}{s^2+\sigma^2}$
- $\gamma = \frac{\sqrt{2\pi}\beta}{2\pi s\sigma(\Phi(d)-\Phi(c))}$
- $c = \frac{\mu-b}{\sigma}$, $d = \frac{\mu-a}{\sigma}$

Proof. To see this, take $X \sim N(0, s)$ and $Y \sim TN(\mu, \sigma, a, b)$, independent. Using the convolution formula, the density of $V = X + Y$ is given by

$$f(v) = \int_{u=-infty}^{\infty} f_X(u) f_Y(v-u) du$$

But

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{x^2}{2s^2}} \\ f_Y(y) &= \frac{1}{\sqrt{2\pi\sigma^2}(\Phi(d) - \Phi(c))} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \mathbf{1}_{y\in[a,b]} \end{aligned}$$

Define $\gamma' = \frac{1}{\sqrt{2\pi\sigma^2}(\Phi(d)-\Phi(c))} \cdot \frac{1}{\sqrt{2\pi s^2}}$. Then, for $v - u \in [a, b]$

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[1] $N(0, s)$ is a centered normal with variance s^2, $TN(\mu, \sigma, a, b)$ is the truncation of a normal with parameters (μ, σ^2) between a and b

$$f_X(u)f_Y(v-u) \quad = \quad \gamma' e^{-\frac{x^2}{2s^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

But

$$\frac{u^2}{2s^2} + \frac{(v-u-\mu)^2}{2\sigma^2} \quad = \quad \frac{s^2+\sigma^2}{2s^2\sigma^2}u^2 - \frac{2u(v-\mu)}{2\sigma^2} + \frac{(v-\mu)^2}{2\sigma^2}$$

Define $\beta^2 = \frac{s^2\sigma^2}{s^2+\sigma^2}$

$$\begin{aligned} &= \quad \frac{u^2}{2\beta^2} - \frac{2u\frac{(v-\mu)}{\frac{s^2+\sigma^2}{s^2}}}{2\beta^2} + \frac{\left(\frac{(v-\mu)}{\frac{s^2+\sigma^2}{s^2}}\right)}{2\beta^2} - \frac{\left(\frac{(v-\mu)}{\frac{s^2+\sigma^2}{s^2}}\right)^2}{2\beta^2} + \frac{(v-\mu)^2}{2\sigma^2} \\ &= \quad \frac{(u-\alpha)^2}{2\beta^2} + \frac{(v-\mu)^2}{2(s^2+\sigma^2)} \end{aligned}$$

$\alpha = \frac{(v-\mu)}{\frac{s^2+\sigma^2}{s^2}}$

Hence,

$$f_X(u)f_Y(v-u) = \gamma' e^{-\frac{(u-\alpha)^2}{2\beta^2}} e^{-\frac{(v-\mu)^2}{2(s^2+\sigma^2)}}$$

Now,

$$\begin{aligned} \int_{v-b}^{v-a} e^{-\frac{(u-\alpha)^2}{2\beta^2}} du \quad &= \quad \beta \int_{\frac{v-b-\alpha}{\beta}}^{\frac{v-a-\alpha}{\beta}} e^{-\frac{z^2}{2}} dz \\ &= \quad \beta\sqrt{2\pi}\left[\Phi(\frac{v-a-\alpha}{\beta}) - \Phi(\frac{v-b-\alpha}{\beta})\right] \end{aligned}$$

Therefore, the distribution we want is

$$\begin{aligned} f(v) \quad &= \quad \gamma'\beta\sqrt{2\pi}\left[\Phi(\frac{v-a-\alpha}{\beta}) - \Phi(\frac{v-b-\alpha}{\beta})\right] e^{-\frac{(v-\mu)^2}{2(s^2+\sigma^2)}} \\ &= \quad \gamma\left[\Phi(\frac{v-a-\alpha}{\beta}) - \Phi(\frac{v-b-\alpha}{\beta})\right] e^{-\frac{(v-\mu)^2}{2(s^2+\sigma^2)}} \end{aligned}$$

□