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Optimal capacity expansion for multi-product, multi-machine manufacturing systems with stochastic demand

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We consider a discrete-time capacity expansion problem involving multiple product families, multiple machine types, and non-stationary stochastic demand. Capacity expansion decisions are made to strike an optimal balance between investment costs and lost sales costs. Motivated by current practices in the semiconductor and other high-tech industries, we assume that only minimal amounts of finished-goods inventories are held, due to the risk of obsolescence. We assume that when capacity is in short supply, management desires to ensure that a minimal service level for all product families is obtained. Our approach uses a novel assumption that demand can be approximated by a distribution whose support is a collection of rays emanating from a point and contained in real multi-dimensional space. These assumptions allow us to solve the problem as a max-flow, min-cut problem. Computational experiments show that large problems can be solved efficiently.

1. Introduction

The semiconductor industry has become one of the leading industries in the US economy. A 1998 study commissioned by the Semiconductor Industry Association shows that chip sales increased 15% annually from 1987 to 1996, moving the industry from twentieth to fourth on the list of top manufacturing industries (Anon, 1998). One of the features of the semiconductor manufacturing process is intensive capital investment. A modern fabrication facility (fab) being built today costs more than a billion dollars. More than 60% of the total cost is solely attributed to the cost of tools. In addition, in most existing fabs millions of dollars are spent on tool procurement each year to accommodate changes in technology. Therefore, making efficient usage of current tools and carefully planning the purchase of new tools are of great importance.

However, capacity planning decisions in the semiconductor industry are very challenging. Some of the main reasons for this are as follows.

- The consumer electronics market: In the consumer electronics business, product design cycles and life cycles are rapidly shrinking. Competition is fierce and the pace of product innovation is high. Consequently, the demand for new semiconductor products is becoming increasingly difficult to predict.
- Rapid changes in technology: In recent years the semiconductor industry has seen line width shrinkages approximately once every 18 months. Fabs dedicated to 300 millimeter wafers have been recently announced by most large semiconductor manufacturers. These and other technological advances require companies to continually replace many of their tools that are used to manufacture semiconductor products.
- Long procurement lead time: The lead time for procuring a new tool usually ranges from 3 months to a year. Many of the most expensive tools require 18 months. As a result, plans for expanding and upgrading capacity must be made based on demand forecasts reaching 2–3 years into the future. These demand forecasts are subject to a very high degree of uncertainty.
- High cost of tools: It is reported that, in 1996, the industry reinvested 23% of its total revenue in capacity expenses, mostly (60–70%) for tool purchases. It is also projected that the cost of semiconductor manufacturing tools will be steadily increasing in the future. Thus, a small improvement in the tool purchase plan could lead to huge impact on the manufacturers.

In this paper, we will propose a discrete-time, finite-horizon analytical model for optimizing tool purchase plans under non-stationary stochastic demand. Capacity expansion decisions strike an optimal balance between lost sales costs and discounted tool purchase costs. Motivated by current practices in the semiconductor and other high-tech industries we assume that backorders are negligible,

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however, when semiconductor companies lack capacity they often postpone delivery dates to their clients. This is formally treated like a backorder. However, deferred delivery dates frequently prompt the client to defer subsequent orders, so the end result can be closer to lost sales than to backorders. We also assume that negligible amounts of finished-goods inventories are held due to the risk of obsolescence. This assumption leads to important simplifications. This assumption closely approximates the practices of many semiconductor companies, but not all of them. There are semiconductor companies that hold a substantial semiconductor finished-goods inventory in order to smooth out production.

We aggregate products into the same product family if they consume identical or nearly proportional amounts of capacity on each of the key tool groups. For example, two logic products might have four and five layers of metal respectively. One wafer of the four-layer product might consume nearly the same amount of capacity on key tool groups as 0.8 wafers of the five-layer product. If so, for long range capacity planning purposes, they could be aggregated into a single product family by appropriately adjusting the units of measure.

We use a novel demand model. If there are K different product families, then the demand $\mathbf{D}(t)$ is a random vector in K-dimensional real space, with a finite mean. In each time period t we assume that a collection of rays emanates from a point. These rays are a subset of the non-negative orthant of K-dimensional real space. Most of our results depend on an assumption that the random demand vector $\mathbf{D}(t)$ has support on these rays. Computational results suggest that our demand model can effectively approximate a multi-dimensional demand vector with support on all of $\Re_+^K$.

Whereas capacity decisions are made in period 1, production decisions are adaptively made at each period. One consequence of the no-backorder, no-inventory assumption is that lost sales can be computed by comparing production in a period to the demand in the same period. We assume that when capacity is in short supply, management desires to ensure a minimal service level for all product families. Specifically, we assume that if $\mathbf{D}(t)$ lies on one of the given rays as described above, and if there is not enough capacity to meet the demand in period t, then the point on the ray that is capacity-feasible and as close to $\mathbf{D}(t)$ as possible will correspond to production in period t. If the rays emanate from the origin, this strategy will equalize the fill rate of the different product families. If they emanate from a different point, this policy will equalize a service measure that is calibrated differently for different product families.[1]

In a classical approach to stochastic programming models, a continuous distribution is approximated by a finite collection of points sampled from the distribution. If we had taken this approach, it would be easy to formulate our problem as a stochastic integer linear programming problem. However, a large number of points would be necessary to adequately model the demand. Our approach is different. By using distributions that are supported on rays instead of points, the number of rays needed to model a continuous distribution may be smaller. We prove that our formulation of this problem may be transformed to a max-flow network problem, which can be efficiently solved by available software.

The rest of this paper is organized as follows. In Section 2, related literature is reviewed. Section 3 presents our demand model, which is used to formulate our problem in Section 4. In Section 5 we provide details of the numerical tests, and we conclude in Section 6.

2. Literature review

In this section we present a brief discussion of stochastic capacity expansion problems, especially single-location models. An extensive survey of early work, capturing application areas and multi-location models as well as single-location models, is provided in Luss (1982). A discussion of the models that appeared after Luss (1982) is given in Çakanyıldırım *et al*. (1999).

A general approach to discrete-time capacity planning models under uncertainty is stochastic programming (Wets, 1989; Birge and Louveaux, 1997). Generally demand uncertainty is represented in terms of demand scenarios. Eppen *et al.* (1989), Escudero *et al.* (1993) and Chen *et al.* (1998) use scenarios in studying capacity planning for manufacturing industries. However, in order to make the scenario-based model solvable, only a limited number of scenarios can be used. Consequently it is usually impossible to accurately model multi-dimensional demand processes.

Karabuk and Wu (1999) treat both demand and capacity as uncertain. A multi-stage stochastic program is formulated. Demand and capacity uncertainty is incorporated through a scenario structure. Then the problem is decomposed to reflect a decentralized decision-making process. For the service industries Berman *et al.* (1994) provide a stochastic programming model to expand capacities at multiple loosely coupled locations.

Another approach for discrete-time models is Markov Decision Processes (MDPs). Using MDPs, Bhatnagar *et al.* (1999) lay out a capacity expansion model capturing machine disposals and inventory holding. Like stochastic programming, MDP is a general purpose tool. It can provide great flexibility in modeling; however, such flexibility makes it extremely difficult (or impossible) to find solutions to problems of realistic size and complexity.

If time is a continuous variable rather than a discrete one, the capacity expansion problem may be formulated as an optimal control problem. For a single product with a stochastic demand process, Davis *et al.* (1987) regulate the

[1] See Fig. 2.

expansion rate with an investment rate function which is the control variable. As soon as the cumulative investment reaches the random price of a discrete capacity unit, that unit becomes available. For multiple products with deterministic demand rates, Sethi *et al.* (1995) study machine capacity expansion when machines can break down randomly. They propose to plan capacities using the mean available capacity numbers. When breakdowns and repairs happen sufficiently fast, they show that the cost of their expansion plan asymptotically converges to the optimal cost.

In the economics community, the capacity expansion problem has been recently addressed by Dixit (1997), and Eberly and Van Mieghem (1997). They established structural properties for expansion and contraction policies for multiple factors contributing to capacity. The latter paper provides a closed-form solution for the optimal policy in the case of iid stochastic processes and stationary costs. For certain capacity expansion cases, Rocklin *et al.* (1984) establish the optimality of (s, S) policies.

There have been efforts to convert stochastic expansion problems to equivalent deterministic problems. One of them is Bean *et al.* (1992), which provides such a conversion by modifying the interest rate when demand is either a Brownian motion or a transformed birth-death process.

Over long horizons, with the introduction of new technologies, new types of machines become available for purchase. Rajagopalan *et al.* (1998) study the replacement of old machines with new ones, under both certain and uncertain technology arrival times, and with deterministic, nondecreasing demand. They show some structural properties of the optimal solution and exploit those with a dynamic program.

We now turn to the capacity expansion studies specifically targeting the semiconductor industry. Although it is a deterministic rather than a stochastic capacity planning model, we mention the Capacity optimization Planning System (CAPS) (Bermon and Hood, 1999) because it does an excellent job of capturing the capacity relationships between products and tools. Bard *et al.* (1999) formulate the capacity expansion problem as a nonlinear integer mathematical program that finds a tool-set configuration to minimize average cycle time, within a given budget. Their model is based on the assumption of a small number of products, and it uses queuing network approximations. Chou and Everton (1996) use simulation to study tool capacity and operator availability at development wafer fabs. Angelus *et al.* (1997) consider capacity expansion with fixed costs, and stochastically increasing and correlated demand. They show that the optimal expansion policy is (s, S) type where both parameters depend on the most recently observed demand. Swaminathan (2000) captures demand uncertainty with a two-stage stochastic mixed integer linear program. Since this problem becomes very large even with a few scenarios, he provides Lagrangian-based bounds and heuristics. It is subsequently extended in Swaminathan (2002) to capture production planning decisions.

Treating tool purchase times as continuous decision variables, Çakanyıldırım *et al.* (1999) provide a nonlinear program to minimize the expected lost sales costs and capacity costs, for a fab producing a single product but experiencing uncertain demand. In the single product case, the analysis is simplified after observing that the optimal sequence of machine purchases is independent of demands. However, this is not true for the multi-product case because the allocation of capacity to products is not straightforward.

The current paper extends Çakanyıldırım *et al.* (1999) to the multi-product discrete-time case by proposing a specific capacity allocation scheme. It presents a practical solution approach for a problem similar to the one studied in Swaminathan (2000, 2002). Unlike the other models described above, our model is capable of dealing with problem instances of realistic scale, i.e., large numbers of time periods, products and tools. It captures the uncertainty in non-stationary demand using an innovative model. When combined with our capacity allocation scheme, this demand model is not only tractable but much more compelling than using a small number of demand scenarios in order to represent a multi-dimensional random vector.

3. Notation and demand model

We use the following notation. Note that in the rest of this paper, uppercase letters are used to represent random variables. The corresponding lowercase letter is used to represent a realization of a random variable. Thus $\mathbf{d}(t)$ is a specific realization of $\mathbf{D}(t)$. Vectors and matrices are in boldface.

t = time index; $t \in \{1, \ldots, T\}$ where T is the time horizon we need to consider in the model;
m = tool type index; $m \in \{1, \ldots, M\}$;
k = product family index; $k \in \{1, \ldots, K\}$;
$\lambda(m)$ = purchase lead time for a type m tool;
$\mathbf{D}(t)$ = random demand vector in period t, $\mathbf{D}(t) \in \Re^K$;
$\mathbf{u}$ = utilization matrix; the (m, k)th component represents the units of capacity-time required at a type m tool if one unit of product family k is produced;
$\boldsymbol{\kappa}(t)$ = capacity at time period t; $\boldsymbol{\kappa}(t) \in \Re^M$. Also, $\boldsymbol{\kappa}_m(t)$ is the capacity on tool type m in period t;
$\mathbf{P}(t)$ = production plan in period t; $\mathbf{P}(t) \in \Re^K$. Specifically $\mathbf{P}_k(t)$ is the production of product family k in period t.

To evaluate the average lost sale cost, we need to know the distribution of the random demand vector $\mathbf{D}(t) \in \Re^K$. We use the following novel approach to model demand.

Assumption 1. (Ray-based demand model) The random demand $\mathbf{D}(t)$ can be expressed as

$$\mathbf{D}(t) = \mathbf{b}(t) + \Delta_{I(t)}(t)\boldsymbol{\phi}_{I(t)}(t)$$

where $\mathbf{b}(t)$ is a deterministic vector in $\Re_+^K$, $I(t)$ is a discrete random variable with finite support such that $p_i(t)$ is the probability that $I(t) = i$, $\phi_i(t)$ is a deterministic unit-norm vector in $\Re_+^K$, and $\Delta_i(t)$ is a continuous non-negative random scalar, independent of $I(t)$.

In each time period t, a finite number of vectors $\phi_i(t)$ define a collection of rays that emanate from $\mathbf{b}(t)$ and are contained in $\Re_+^K$. The support of $\mathbf{D}(t)$ is the union of these rays. (See Fig. 1.) The random variable $I(t)$ determines the ray on which $\mathbf{D}(t)$ lies. After the ray has been selected (i.e., if we condition on $I(t) = i$), the continuous random variable $\Delta_i(t)$ determines how far one moves from $\mathbf{b}(t)$ in the direction $\phi_i(t)$ before reaching the demand $\mathbf{D}(t)$. Thus $\Delta_{I(t)}(t) = |D(t) - b(t)|$. If we set $\mathbf{b}(t) = \mathbf{0}$, then the product mix in the demand vector is determined by the random variable $I(t)$, because $\mathbf{D}_k(t)/\mathbf{D}_1(t) = \phi_{ki}(t)/\phi_{1i}(t)$ for all product families k. With $\mathbf{b}(t) = \mathbf{0}$, the magnitude of the demand vector is $|\mathbf{D}(t)| = \Delta_{I(t)}(t)$. The magnitude is correlated with the product mix because the distribution of $\Delta_i(t)$ depends on i.

In the remainder of this section, we discuss one motivation for introducing the ray-based demand model (Assumption 1). We also describe a practical approach for obtaining parameters for the ray-based demand model. Consider a demand vector $\mathbf{D}^*$ that has a continuous density with support in $\Re_+^K$. Suppose we want to compute an approximation to $E[f(\mathbf{D}^*)]$ for some function f. (In the capacity planning problem presented in this paper, the function f corresponds to the lost sales in period t.) One common approach is to approximate $E[f(\mathbf{D}^*)]$ with $(1/|S|)\sum_{s\in S} f(s)$, where S is a set of points sampled from $\mathbf{D}^*$. The variance of the estimate is $(1/|S|)$ var $f(\mathbf{D}^*)$. This classical approximation is often embedded in optimization algorithms, usually formally by assuming that $\mathbf{D}(t)$ has support on S.

Our ray-based demand model can be derived from an improved estimate of $E[f(\mathbf{D}^*)]$, based on a variance-reduction technique called conditioning. Define $\mathbf{\Phi} = \mathbf{D}^*/|\mathbf{D}^*|$, and define $\Delta_{\mathbf{\Phi}}^*$ by $\mathbf{D} = \Delta_{\mathbf{\Phi}}^* \times \mathbf{\Phi}$. The conditioning approach is viable because we can analytically evaluate $E_{\Delta_\phi^*}[f(\Delta_\phi^* \times \phi)]$ for any given unit vector ϕ (see Section 4). We randomly sample points $s \in S$ from $\mathbf{D}^*$, and we compute $\phi_s = s/|S|$. The ray-based estimate of $E[f(\mathbf{D}^*)]$ is

$$\frac{1}{|S|}\sum_{s\in S} E_{\Delta_{\phi_s}^*}[f(\Delta_{\phi_s}^* \times \phi_s)]. \tag{1}$$

This is an unbiased estimator of $E[f(\mathbf{D}^*)]$. The variance of this estimator is

$$\frac{1}{|S|} \text{ var }_{\mathbf{\Phi}}[E_{\Delta_{\mathbf{\Phi}}^*}[f(\Delta_{\mathbf{\Phi}}^* \cdot \mathbf{\Phi})]].$$

The standard conditioning inequality states that (Law and Kelton, 2000)

$$\begin{aligned} &\text{var }_{\mathbf{\Phi}}[E_{\Delta_{\mathbf{\Phi}}^*}[f(\Delta_{\mathbf{\Phi}}^* \times \mathbf{\Phi})]], \\ &= \text{var }[f(\mathbf{D}^*)] - E_{\mathbf{\Phi}}\big[\text{Var}_{\Delta_{\mathbf{\Phi}}^*}[f(\Delta_{\mathbf{\Phi}}^* \times \mathbf{\Phi})]\big], \\ &\le \text{var }[f(\mathbf{D}^*)]. \end{aligned}$$

Thus, the ray-based estimate (1) of $E[f(\mathbf{D}^*)]$ has a lower variance than the more traditional estimate $(1/|S|)\sum_{s\in S} f(s)$.

To see the relationship between the ray-based estimate of $E[f(\mathbf{D}^*)]$ and our ray-based demand model, assume that we obtain the parameters for $\mathbf{D}(t)$ from $\mathbf{D}^*$ in the following manner.

Step 1. $\mathbf{b}(t) = \mathbf{0}$;
Step 2. $P(I(t) = i) = p_i = \frac{1}{|S|}$;
Step 3. $\{\frac{s}{|s|} : s \in S\} = \{\phi_i(t) : 1 \le i \le |S|\}$; and
Step 4. the distribution of $\Delta_i(t)$ is equal to that of $\Delta_{\phi_i}^*$ (see Appendix A).

Then

$$\begin{aligned} E[f(\mathbf{D}(t))] &= E_{I(t)}\{E[f(\mathbf{D}(t)) \mid I(t) = i]\}, \\ &= \frac{1}{|S|}\sum_i E_{\Delta_i(t)}[f(\Delta_i(t)\phi_i(t))], \end{aligned}$$

which is the ray-based estimate of $E[f(\mathbf{D}^*(t))]$. Thus the ray-based demand model can arise from the variance reduction technique called conditioning. In this paper, we formally assume that the demand is ray-based; i.e., that Assumption 1 holds.

In Section 5, we will generate parameters for the ray-based demand model. We will assume that $\mathbf{D}^*(t)$ is a multivariate lognormal random variable with known parameters. We will set $\mathbf{b}(t) = \mathbf{0}$. We will obtain the $\phi_i(t)$'s by drawing an appropriate number of random samples from $\mathbf{D}^*(t)$ and normalizing them. We will obtain $\Delta_i(t)$ from the conditional distribution of $\mathbf{D}^*(t)$ given that the demand lies on the ray corresponding to $\phi_i(t)$ (see Appendix A). We could set $p_i(t) = 1/r$ where $r = |\text{supp}(I(t))|$ is the number of rays. However, the recommended procedure is to adjust the $p_i(t)$'s so that the mean vector of the ray-based demand $\mathbf{D}(t)$ coincides with the mean vector $\boldsymbol{\mu}(t) = E[\mathbf{D}^*(t)]$. This

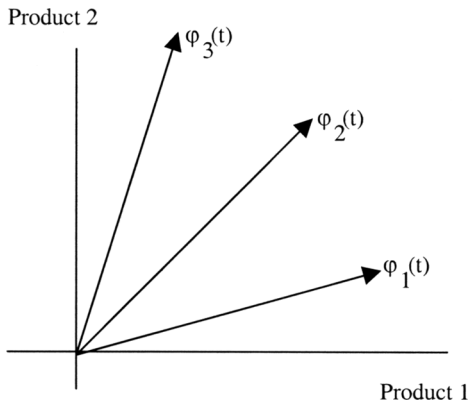


Fig. 1. The support of $\mathbf{D}(t)$ with $\mathbf{b}(t) = \mathbf{0}$.

can be done, for example, by solving the following quadratic program:

$$\text{Minimize} \quad \sum_i \left(\frac{1}{r} - p_i(t)\right)^2,$$

such that

$$\boldsymbol{\mu}(t) = \mathbf{b}(t) + \sum_i p_i(t)\mu_i(t)\boldsymbol{\phi}_{I(t)}(t),$$

where $\mu_i(t)$ is the expected value of $\Delta_i(t)$.

4. Optimization model

In this section we provide an integer programming formulation of the capacity expansion problem. We consider the problem in multiple discrete time periods. Our objective is to find the optimal tool purchase plan, so as to minimize the sum of lost sale costs and tool purchase costs.

4.1. *Assumptions*

We assume that all tool purchase decisions are made at the beginning of time period 1. Thus, the capacity $\boldsymbol{\kappa}(t)$ is based on the distribution of the demand $\mathbf{D}(t)$, which is available at the time of planning, but not on the realization $\mathbf{d}(t)$ of $\mathbf{D}(t)$. In practice a dynamic rolling horizon approach would be utilized; thus new plans would be generated periodically, and only the earliest portion of the computed plans would be implemented. We note that both the capacity $\boldsymbol{\kappa}(t)$ and the production plan $\mathbf{P}(t)$ are intermediate variables that depend on the tool purchase decisions. The production plan $\mathbf{P}(t)$ in period t also depends on the realization $\mathbf{d}(t)$ of demand $\mathbf{D}(t)$ in period t, as shown in the following assumptions.

Assumption 2. In each time period t, the following events are assumed to happen in the given order

1. observe the state;
2. newly-installed tools become available for production;
3. demand is observed;
4. production decisions are made, and production occurs;
5. costs are incurred.

We have assumed that both inventories and backorders are negligible. This assumption has an important consequence, namely, that after tool purchase decisions have been made the problem decomposes by time periods. Thus, after $\boldsymbol{\kappa}(t)$ has been selected for all t, the production $\mathbf{P}(t)$ in period t can be selected independently of $\mathbf{P}(t-1)$, $\mathbf{P}(t+1)$, etc.

According to Assumption 2, the production plan $\mathbf{P}(t)$ is made after the demand $\mathbf{D}(t)$ has been observed. There are many ways to choose $\mathbf{P}(t)$. In our model, we use a service-based policy called proportional production which, as we described in the Introduction, achieves an equal fill rate for all product families if $\mathbf{b}(t) = \mathbf{0}$. Proportional production is illustrated in Fig. 2. If demand $\mathbf{d}(t)$ falls in the shaded region, then production is equal to demand. If the realized demand is outside of the shaded region, $\mathbf{p}(t)$ will be the production plan and there is a shortfall. Mathematically we can define our proportional production plan as follows.

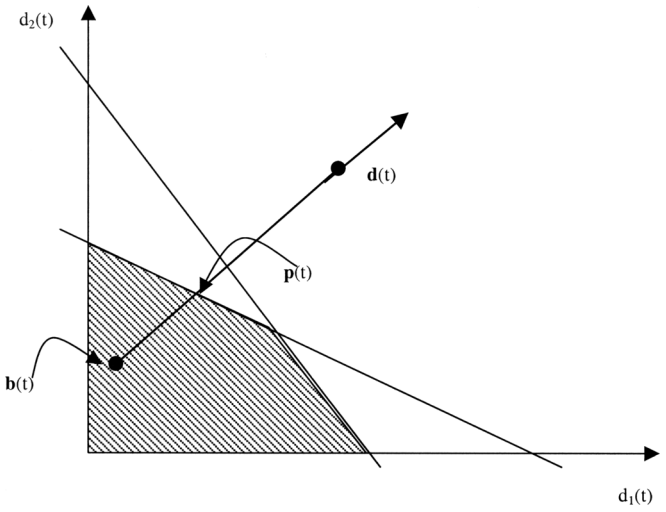


Fig. 2. Proportional production.

Assumption 3. (Proportional Production) We apply proportional production in each time period, i.e.,

$$\mathbf{P}(t) = \mathbf{b}(t) \min\left[\sigma^*_{I(t)}(t), \Delta_{I(t)}(t)\right]\boldsymbol{\phi}_{I(t)}(t),$$

where

$$\sigma^*_i(t) = \max\{\sigma : \mathbf{u}[\mathbf{b}(t) + \sigma\boldsymbol{\phi}_i(t)] \leq \boldsymbol{\kappa}(t)\}.$$

Consequently, we observe that

1. If $\mathbf{b}(t) = \mathbf{0}$, the ratio of unsatisfied demand to total demand is the same across all products. Thus, the fill rates among the different products are equalized. If $\mathbf{b}(t) \neq \mathbf{0}$, then the fill rates are different from product to product.
2. The production plan $\mathbf{P}(t)$ always lies on the same ray as the demand $\mathbf{D}(t)$.
3. The maximum production plan coefficient $\sigma^*_{I(t)}(t)$ is determined by the demand direction $\boldsymbol{\phi}_{I(t)}(t)$ and the current capacity $\boldsymbol{\kappa}(t)$.

We point out that there are other plausible alternatives to proportional production. For example, we could select the production plan that maximizes the revenue obtained.

4.2. *Lost sales costs*

The demand distribution (Assumption 1) and proportional production (Assumption 3) allow us to derive a simple expression for the lost sales. Given a demand $\mathbf{d}(t)$ which is a realization of the random vector $\mathbf{D}(t)$, the production plan $\mathbf{p}(t)$ is fixed. Assume that the production plan $\mathbf{p}(t)$ is applied and that the lost sales cost is linear in the quantity of unsatisfied demand. Note that we do not produce more than the demand, so there are no inventory costs. As we mentioned

earlier, substantial inventories of finished goods constitute a risk that many companies prefer to avoid. Then the lost sales cost is $\mathbf{c} \times [\mathbf{d}(t) - \mathbf{p}(t)]^+$. The kth component of vector $\mathbf{c}$ is the unit lost sales cost for the kth product. Consequently the expected lost sales cost in period t will be

$$E_{D(t)}\{\mathbf{c} \times [\mathbf{D}(t) - \mathbf{P}(t)]^+\}.$$

Let $n_{\Delta_i(t)}(\cdot)$ be the partial expectation, defined by

$$n_{\Delta_i(t)}(\sigma) = E_{\Delta_i(t)}(\Delta_i(t) - \sigma)^+.$$

Then the expected lost sales cost in t can be rewritten as

$$\sum_i p_i(t) \times n_{\Delta_i(t)}(\sigma_i^*(t))\mathbf{c} \times \phi_i(t). \tag{2}$$

A proof of Equation (2) is provided in Appendix B.

4.3. *Variables and constraints*

Having defined a convenient expression for the lost sales, we now formulate the variables and constraints of our integer programming model. We also relate the variables to our ability to meet the demand. We use (m, n) to index tool purchases, where (m, n) represents the nth purchase of a type m tool. We also write $j \sim (m, n)$ where j $(1 \le j \le J)$ indexes purchases of all tools of any type. Define a binary variable $x(j, t)$ as follows:

$$x(j, t) = \begin{cases} 1 & \text{if tool } j \text{ is available for production by time } t, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Obviously $x(j, t)$ is non-decreasing with respect to time, i.e.,

$$x(j, t) \le x(j, t + 1)\ \forall j, t. \tag{4}$$

We refer to Equation (4) as *time monotonicity* in the rest of this paper. Since $\lambda(m)$ is the purchase lead time for a type m tool, if $t \le \lambda(m)$ then the value of $x(j, t)$ is fixed because the purchase decision has already been made.

The most intricate part of the formulation is relating the lost sales cost, Equation (2), to the variables $x(j, t)$. Intuitively, the variable $x(j, t)$ determines whether or not the purchase of tool j has been made by time period t. However, the purchase of tool j may not increase the actual capacity at all. This can happen if the capacity of a type m tool is currently the tight constraint, and the purchase of tool j increases the capacity of a type $m'(\neq m)$ tool. In order to understand the actual capacity, we denote the demand ray that has direction $\phi_i(t)$ as demand ray i. For each time period t and each demand ray i, we need to understand which tool is currently limiting the capacity. We let $\sigma(i, j, t)$ be the distance along ray i from $\mathbf{b}(t)$ to the point where tool j limits capacity. For $j \sim (m, n)$, we have

$$\sigma(i, j, t) = \max\{\sigma : \mathbf{u}_{m,:}[\mathbf{b}(t) + \sigma\phi_i(t)] \le \kappa(j)\},$$

where $\mathbf{u}_{m,:}$ is the mth row of utilization matrix $\mathbf{u}$ and $k(j)$ is the capacity of the type m tool group just before $j \sim (m, n)$ is available. Thus, when tool j becomes available for production, the capacity of the type m tool group jumps from $k(j)$ to $k(j')$, where $j' \sim (m, n + 1)$.

Suppose, at time period t, that the realized demand falls on ray i and is

$$\mathbf{d}(t) = \mathbf{b}(t) + \delta_i(t)\phi_i(t). \tag{5}$$

Lemma 1. *Let Equation* (5) *hold and let* $\sigma(i, j, t) < \delta_i(t) \le \sigma(i, j', t)$ *where* $j \sim (m, n)$ *and* $j' \sim (m, n + 1)$. *Then the capacity of the type m tool group is adequate to meet the demand* $\mathbf{d}(t)$ *if and only if* $x(j, t) = 1$.

Proof. A direct consequence of the definitions of $\sigma(i, j, t)$ and $x(j, t)$. ■

Along demand ray i, the actual capacity is determined by the sequence in which the values $\{\sigma(i, j, t) : 1 \le j \le J\}$ fall. We sort $\{\sigma(i, j, t) : 1 \le j \le J\}$ in increasing order. We denote by $\gamma_t^i(j)$ the *immediate successor* of tool j along ray i; i.e., we write

$$\gamma_t^i(j) = j',$$

if (i, j', t) comes immediately after (i, j, t) in the sorted list. We also define the *precedence set* $\pi_t^i(j)$ of tool j along demand ray i as follows:

$$\pi_t^i(j) = \{j' : j' = j \text{ or } (i, j', t) \text{ comes before } (i, j', t) \text{ in the sorted list}\}.$$

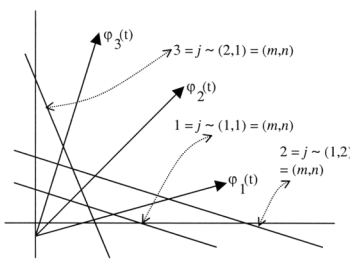
Figure 3 illustrates the definition of γ and π using an example with two product families, two types of tools and three demand rays. The definition of π implies that $\sigma(i, j', t) \le \sigma(i, j, t)$ if and only if $j' \in \pi_t^i(j)$.

If Equation (5) holds, then in order to meet the demand, every type of tool must have adequate capacity. Assume that

$$\sigma(i, j, t) < \delta_i(t) < \sigma\big(i, \gamma_t^i(j), t\big). \tag{6}$$

By Lemma 1, to meet the demand we must have

$$x(j', t) = 1 \quad \forall j' \text{ subject to } \sigma(i, j', t) < \delta_i(t), \text{ i.e., } \forall j' \in \pi_t^i(j).$$



$\gamma_t^i(j)$

i \ j	1	2	3
1	2	--	1
2	3	--	2
3	2	3	--

$\pi_t^i(j)$

i \ j	1	2	3
1	{1,3}	{1,2,3}	{3}
2	{1}	{1,2,3}	{1,3}
3	{1}	{1,2}	{1,2,3}

$z(i,j,t)$

i \ j	1	2	3
1	1	0	1
2	1	0	1
3	1	0	0

Fig. 3. $\gamma_t^i(j), \pi_t^i(j)$ and $z(i, j, t)$ when $x(1, t) = x(3, t) = 1$ and $x(2, t) = 0$.

To capture this, we define another group of binary indicator variables $z(i, j, t)$ by

$$z(i,j,t) = \begin{cases} 1 & \text{if } x(j',t) = 1\ \forall j' \in \pi_j^i(j), \\ 0 & \text{otherwise.} \end{cases}$$

Thus if Equations (5) and (6) hold, we can meet demand if and only if $z(i, j, t) = 1$. Figure 3 illustrates $z(i, j, t)$ for the case when the first and third purchases have been made, i.e., $x(1, t) = x(3, t) = 1$ and $x(2, t) = 0$. The decision variables $z(i, j, t)$ satisfy the following two properties:

1. Along demand ray i, the precedence set of j is contained in the precedence set of $\gamma_t^i(j)$, so
$$z(i,j,t) \geq z\big(i, \gamma_t^i(j), t\big). \tag{7}$$
2. By definition of the precedence set, $j \in \pi_t^i(j)$. Therefore,
$$z(i,j,t) \leq x(j,t). \tag{8}$$

Subsequently we refer to Equation (7) as *ray monotonicity* and to Equation (8) as *consistency*. Note that $\sigma(i, j, t) \leq \sigma(i, j', t)$ implies $z(i, j, t) \geq z(i, j', t)$.

Lemma 2. *Let* $\mathbf{x} = (x(j,t), \forall j, t)$ *and* $\mathbf{z} = (z(i,j,t), \forall i, j, t)$. *Assume that the demand in period t satisfies Equation* (5) *where* $\sigma(i, j', t) < \delta_i(t) \leq \sigma(i, \gamma_t^i(j'), t)$.

(i). *Suppose that the vectors* $\mathbf{x}$ *and* $\mathbf{z}$ *satisfy Equations* (4), (7) *and* (8). *If* $z(i, j', t) = 1$, *then there is adequate capacity to meet the demand in period t.*

(ii). *Conversely, if there is adequate capacity to meet the demand in period t, then there exist* $\mathbf{x}$ *and* $\mathbf{z}$ *satisfying Equations* (4), (7) *and* (8), *with* $z(i, j', t) = 1$.

Proof. (i) follows directly from Equations (3), (7), (8), and Lemma 1. To prove (ii), assume that there is adequate capacity and that x is given by Equation (3). Note that Equation (3) implies Equation (4). From Equation (5), Lemma 1 implies that $x(j, t) = 1$ if $\sigma(i, j, t) < \delta_i(t)$. Set $z(i, j, t) = 1$ if $\sigma(i, j, t) < \delta_i(t)$ and $z(i, j, t) = 0$ otherwise. It is easy to verify that $\mathbf{z}$ satisfies Equations (7) and (8). ■

4.4. *The objective function*

We now relate the expected lost sales cost to the variables $\mathbf{z}$. Suppose that the demand $\mathbf{D}(t)$ lies on demand ray i (i.e., we condition on $I(t) = i$). Furthermore, suppose that in period t, all the tools in the precedence set of tool j' along demand ray i are available but the immediate successor j_o of j' is not. In other words,

$$z(i, j', t) = 1 \quad \text{and} \quad z(i, j_o, t) = 0 \quad \text{where} \quad j_o = \gamma_t^i(j').$$

If we make the purchase of tool j_o, we increase $x(j_o, t)$ and $z(i, j_o, t)$ from zero to one, and then by Equation (2) and Lemma 2, the expected lost sales cost will decrease by

$$\nu(i,j,t) = \big[n_{\Delta_i(t)}(\sigma(i,j',t)) - n_{\Delta_i(t)}(i, \sigma\big(\gamma_t^i(j')\big), t)\big]\mathbf{c} \times \phi_i(t).$$

Therefore, the lost sales cost incurred in period t, given that the demand ray is $i = I(t)$, is

$$\sum_{\{j: z(i,j,t)=0\}} \nu(i,j,t) = \sum_j \nu(i,j,t) - \sum_j \nu(i,j,t) z(i,j,t). \tag{9}$$

We now consider tool purchases costs. Define $\mu(j, t)$ to be the decrease in tool purchase cost by deferring the purchase of tool j from time period t to $t + 1$, including the effects of discounting. Let $\mu(j, T)$ include the fixed cost of purchasing tool j. Under Equation (3), the total cost of the purchase of tool j is $\sum_t \mu(j, t)x(j, t)$.

4.5. *Formulation*

Our objective function trades off expected lost sales costs against tool purchase costs. The constraints are time monotonicity, Equation (4), ray monotonicity, Equation (7) and consistency, Equation (8). Mathematically, using Equation (9) to formulate our objective, we have the following ILP model:

$$\text{(ILP1)} \quad \min \sum_t \Bigg\{ \sum_j [\mu(j,t)x(j,t) - \sum_{i,j} [p_i(t)\nu(i,j,t)z(i,j,t)] \Bigg\}$$

such that

$$x(j,t) \leq x(j,t+1)\ \forall j, t, \tag{4}$$
$$z(i,j,t) \leq x(j,t)\ \forall i, j, t, \tag{8}$$
$$z\big(i, \gamma_t^i(j), t\big) \leq z(i,j,t)\ \forall i, j, t, \tag{7}$$
$$x(j,t), z(i,j,t) \quad \text{binary}\ \forall i, j, t. \tag{10}$$

Clearly (ILP1) is equivalent to a min-cut problem, which is a dual to a max-flow problem in the capacity expansion network (see Fig. 4). For simplicity, $T = 3$, the rays are i and i', and the tool purchases are j and j'. Also assume no purchase lead time; i.e., $\lambda(j) = \lambda(j') = 0$. Along those rays and in all time periods, j and j' fall in the same sequence. Thus $\gamma_t^i(j') = (i, j, t)$ and $\gamma_t^{i'}(j') = (i', j, t)\ \forall t$. $\{(i, j', 1), (i', j, 1), (i', j', 1)\}$. This network has four levels of nodes in the network. The first level contains only one node: *SOURCE*. The fourth level also contains only one node: *SINK*. Every node in the second level corresponds to a variable $z(i, j, t)$, and is called a *z-node*. Every node in the third level corresponds to a variable $x(j, t)$, and is called an *x-node*. We use the variable names as node labels.

The network is constructed to capture the objective function and the constraints in (ILP1). More precisely, the flow on the arc leading from SOURCE to a node $z(i, j, t)$ in the second level is constrained to lie between zero and $p_i(t)\nu(i, j, t)$, the corresponding lost sales cost. These arcs are referred to as *Lost Sales Arcs*. Similarly, the arc from a node $x(j, t)$ in the third level to SINK is a *Purchase Arc*. The flow on the arc must be between zero and $\mu(j, t)$, the purchase cost. In addition, every constraint of the form $u \leq v$ in (ILP1) corresponds to an arc leading from node u

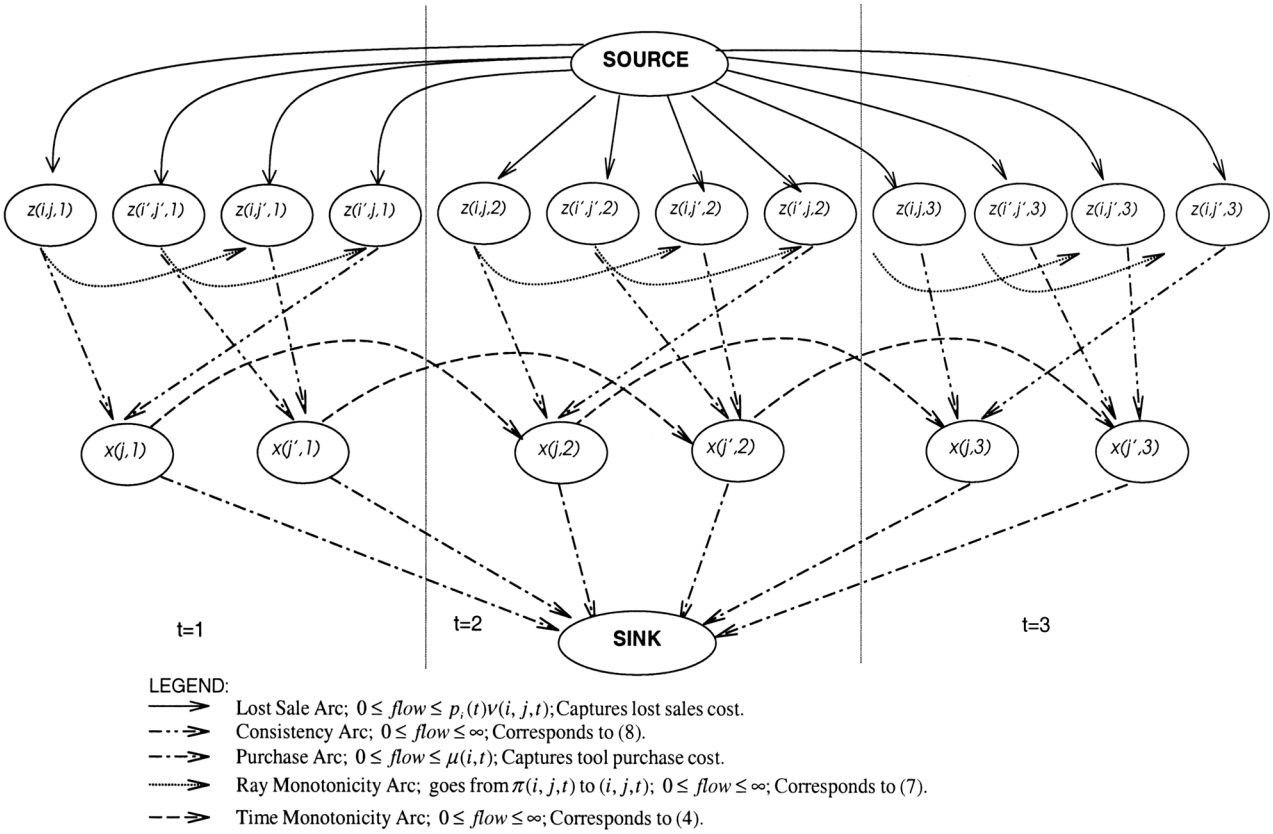


Fig. 4. Capacity expansion network.

to node v. Thus, for example, a *Consistency Arc* leads from $z(i, j, t)$ to $x(j, t)$, a *Time Monotonicity Arc* leads from $x(j, t)$ to $x(j, t + 1)$, and a *Ray Monotonicity Arc* leads from $z(\gamma_t^i(j))$ to $z(i, j, t)$. On all of these arcs, the flow is constrained to lie between zero and ∞. The infinite-capacity arcs ensure that every finite capacity cut in the network corresponds to a feasible tool purchase plan. We have the following result.

Theorem 1. (ILP1) *is equivalent to a minimum-cut problem.*

We note that some variables in the formulation of (ILP1) are fixed because of tool purchase lead times. This results in the deletion of some nodes and arcs in the Capacity Expansion Network. A full discussion and a proof of Theorem 1 are given in Appendix C.

5. Numerical testing

The primary goal of our numerical experiments is to test the performance of our algorithm for capacity planning problems of practical size and complexity. The approach we have developed to solve the optimal capacity planning problem is based on Assumption 1. If the demand $\mathbf{D}(t)$ is a continuous

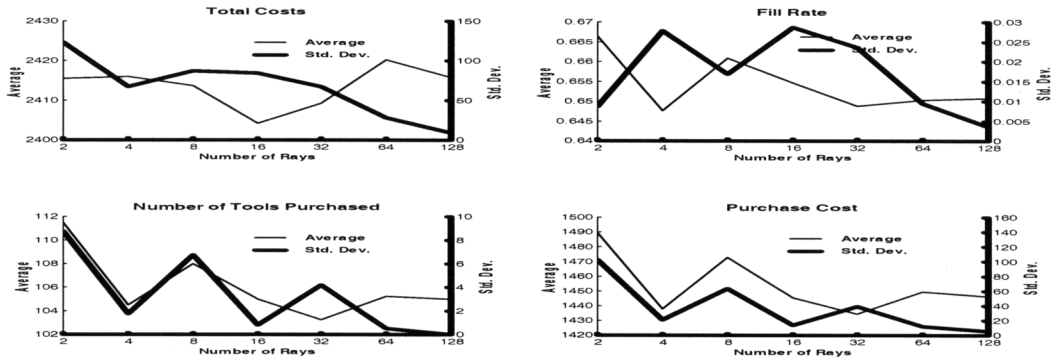


Fig. 5. Base case.

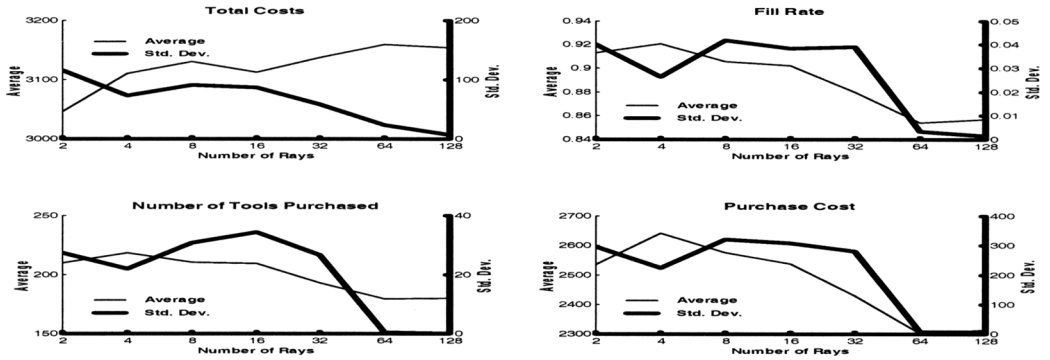


Fig. 6. Double product family case.

random variable, then in accordance with Section 3, we can approximate $\mathbf{D}(t)$ with a distribution which has support on a finite number of demand rays as discussed in the previous section. As the number of demand rays increases, we obtain a more accurate approximation to the true demand. However, the size of the Capacity Expansion Network also increases, requiring more computation. We want to determine how many demand rays are required to approximate a continuous demand distribution effectively.

We combined data from two industrial data sources. We obtained a tool data file from the SEMATECH (SEmiconductor MAnufacturing TECHnology) database. Tool data includes purchase prices and capacities for each tool type. The forecast and demand data files come directly from a semiconductor manufacturer. We used SeDFAM (Çakanyıldırım and Roundy, 1999) to analyze this forecast data and to generate the variance and covariance of future demands.

Based on the properties of the industrial data, we assume that the demand $\mathbf{D}(t)$ has a multi-variate log-normal distribution. For each t, we use the distribution of $\mathbf{D}(t)$ to randomly generate a pre-determined number of future demands. After being normalized, these become our demand rays. Given any demand ray, the distribution and partial expectation of $\Delta_i(t)$ are easy to derive (see Section 3).

We use two steps to compute an optimal expansion plan. A Capacity Expansion Network is constructed in the first step. More specifically, tool data and demand data are read in, and an intermediate network data file is generated using MATLAB. The network file is of DIMACS format (Nguyen and Venkateswaran, 1993) and describes the Capacity Expansion Network. We refer to the first step as NG (Network Generation) hereafter. In the second step we use HIPR, software developed by B. Cherkassky and A. Goldberg, to solve the maximum flow/minimum cut problem (Nguyen and Venkateswaran, 1993). HIPR is an efficient implementation of the push-relabel method, and is designed to run on UNIX/LINUX. It reads the network information from the DIMACS format data file and writes the results to standard output. The second step is called HIPR in the rest of paper.

In our simulation we assume that generation of the purchase plan is done quarterly, and we use a 4-year planning horizon. Demand approximately doubles over the 4-year horizon. The lost sale costs vary across the product families. The average is about \$10 000 per wafer and the deviation is $\pm 10\%$. Purchase prices for tools range from \$100 K to \$15 M. For each tool group the initial capacity is set to be greater than or equal to the expected demand in the first time period.

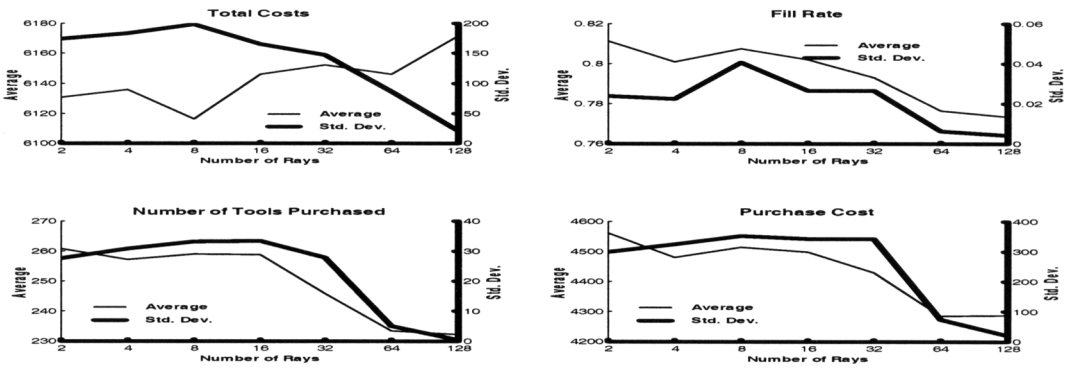


Fig. 7. Double machine type case.

Table 1. Base case (four product families and 43 tool types)

Demand rays	*2*	*4*	*8*	*16*	*32*	*64*	*128*
Nodes (in thousands)	33	55	99	187	363	716	1420
Arcs (in thousands)	73	124	227	431	842	1663	3304
Total cost	2416	2416	2414	2404	2409	2420	2416
Lost sale cost	926	978	941	958	975	970	969
Purchase cost	1490	1438	1473	1446	1434	1450	1447
Number of purchases	112	105	108	105	103	105	105
Fill rate	0.666	0.648	0.661	0.655	0.649	0.650	0.651
NG CPU seconds	63.6	121	234	459	907	1816	3629
HIPR CPU seconds	0.32	0.66	1.48	3.03	6.16	13.17	33.07
File size	1.9	3.3	6.2	12.3	24.3	48.3	99

There are three different cases in our simulation. In the first case, which we call the base case, there are four product families and 43 tool types. In the second one we double the number of product families and keep all other parameters unchanged. Therefore, this case is referred to as the double product family case. In the last case, the number of product families is again 43 but the number of tool types is doubled. This case is called the double tool type case. For each case, we test our algorithm with 2, 4, 8, 16, 32, 64 and 128 demand rays. For each case and for each number of rays, we randomly generate demand rays and run our algorithm four times.

Every time when we run through NG and HIPR, we record the following information: number of rays, number of nodes and arcs in the Capacity Expansion Network, total cost, lost sale cost, purchase cost, total number of tools purchased, fill rate, CPU time spent in NG and HIPR and size of the network files. Some of the above items are easy to understand and some deserve more explanation. The purchase cost and the lost sale cost are the first and second terms of the objective function of (ILP2), respectively. The total cost is the sum of these two costs. The unit of measure for all costs is millions of dollars. The fill rate is the expected percentage of satisfied demand, which is one minus the ratio of the total lost sales cost to the total demand, measured in dollars. All times are measured in seconds. The unit of file size is Megabytes. Our simulations are run on a Sun Microsystems ULTRA 10 workstation with 256 MB of memory.

We are interested in the convergence rate of our algorithm as the number of rays increases. For each case and each number of demand rays we generate and solve four problem instances. Then we compute the mean and standard deviation of the total cost and plot them against the number of rays for each case (see Figs. 5, 6 and 7). The plots of fill rate, total number of tools purchased and purchase cost are obtained similarly. It is remarkable that the total cost and other measures plotted are not more sensitive to the number of demand rays. When 64 rays are used, the maximum ratio of standard deviation to mean in all plots is less than 1/60.

We pay special attention to how fast total cost, fill rate, total number of tools purchased and the purchase cost converge as we increase the number of demand rays. The standard deviation indicates that the number of rays required to achieve convergence is uniformly between 64 and 128. Surprisingly, the solution of the double product family case converges as fast as it does in the other two cases. Note that the demand vector is in $\Re_+^8$ in this case, but only $\Re_+^4$ elsewhere. We expected that more demand rays would be needed to get a satisfactory approximation in the double product family case. However, that did not happen. The fact that capacity is added in discrete rather than continuous increments probably helps.

In Tables 1, 2 and 3, we summarize the results of each case. Since we repeat each case four times, every number in the tables is the average of four data points. Now consider

Table 2. Double product family case (eight product families and 43 tool types)

Demand rays	*2*	*4*	*8*	*16*	*32*	*64*	*128*
Nodes (in thousands)	33	55	99	187	363	716	1420
Arcs (in thousands)	73	125	230	438	854	1687	3352
Total cost	3046	3110	3130	3113	3137	3160	3155
Lost sale cost	511	467	553	574	707	857	841
Purchase cost	2535	2643	2577	2539	2430	2303	2314
Number of purchases	210	219	211	210	193	180	180
Fill rate	0.913	0.920	0.906	0.902	0.880	0.854	0.857
NG CPU seconds	72.6	136	262	515	1023	2034	3927
HIPR CPU seconds	0.63	1.1	2.04	4.89	10.08	24.34	49.09
File size (in MB)	2.1	3.5	6.5	13.0	25.7	51.1	100.0

Table 3. Double machine type case (four product families and 86 tool types)

Demand rays	*2*	*4*	*8*	*16*	*32*	*64*	*128*
Nodes (in thousands)	66	110	198	374	727	1431	2840
Arcs (in thousands)	158	273	502	962	1881	3720	7398
Total cost	6131	6136	6117	6146	6152	6146	6172
Lost sale cost	1569	1657	1602	1648	1721	1861	1885
Purchase cost	4562	4479	4515	4498	4431	4285	4287
Number of purchases	261	257	259	259	246	234	232
Fill rate	0.812	0.801	0.808	0.802	0.793	0.777	0.774
NG CPU seconds	205	395	773	1537	3047	6162	12109
HIPR CPU seconds	1.96	3.20	7.42	15.70	27.19	82.04	169.70
File size (in MB)	4.2	7.3	13.9	26.9	53.4	107.8	218

the running time of both NG and HIPR. The size of the network, both in the number of nodes and the number of arcs, is $O(\eta)$, where $\eta = TRMN$, where T is the number of time periods, R is the number of demand rays, M is the number of tool types and N is the average number of potential tool purchases per tool type. Thus, the number J of tools is $O(MN)$. The running time of NG is $O(\eta)$. Recalling that the time-complexity of max-cut problem with V nodes and $O(V)$ arcs is $O(V^2 \log V)$, (Gallo *et al.*, 1989) the running time of HIPR is at most $O(\eta^2 \log \eta)$. Tables 1, 2 and 3 fully support this time-complexity with respect to R, M, and the lack of dependence on the number of product families.

Also note that the CPU time for running HIPR is much shorter than the CPU time for running NG, especially when the number of rays is large. The MATLAB code used to generate the Capacity Expansion Network is not designed for efficiency. Re-coding NG in C or another suitable programming languages would make it much more efficient than it currently is, and asymptotically more efficient than HIPR. In this paper, we did not implement NG in C for two reasons: (i) this code is intended to be a research prototype; and (ii) after re-coding, the asymptotic running time of the algorithm would be bounded by HIPR, not by NG. The reported CPU times for our capacity planning algorithm are viable.

6. Conclusions

In this paper, we develop a capacity expansion model for multi-product, multi-machine manufacturing systems with stochastic demand. We assume that the demand can be approximated by a distribution whose support is a collection of a finite number of rays. We also make an assumption that capacity is allocated proportionally if it is not sufficient. Those two assumptions are essential in our model formulation. We prove that the optimal capacity planning problem is equivalent to a maximum flow/minimum cut problem. We construct the Capacity Expansion Network and do numerical simulation on industrial data. Our experiment shows that a reasonable number of rays (<128) are sufficient to get a stable plan, and that the CPU times are very predictable and very reasonable for a planning problem of this type.

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Appendices

Appendix A: Derivation of the distribution of $\Delta_i(t)$

Given any demand ray $\phi_i(t)$, the density function of $\Delta_i(t)$ can be expressed as follows:

$$f_{\Delta_i}(r) = \frac{f_{\mathbf{D}(t)}(r\,\phi_i(t))r^{K-1}}{\int_0^\infty f_{\mathbf{D}(t)}(s\phi_i(t))s^{K-1}\mathrm{d}r},$$

where $f_{\mathbf{D}(t)}(\mathbf{x})$ is the density function of $\mathbf{D}(t) \in \Re^K$. The factor r^{K-1} is required because the demand rays spread apart as one moves away from the origin, as is the case in polar coordinates. For example, assume that $\mathbf{D}(t) \in \Re^4$, which has a log-normal multi-variate distribution with parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then

$$f_{\mathbf{D}(t)}(\mathbf{x}) = \prod_{j=1}^{4} x_j^{-1}(2\pi)^{-2}[\mathrm{Det}(\Sigma)]^{-\frac{1}{2}} \times \exp\left[-\frac{1}{2}(\log\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\log\mathbf{x}-\boldsymbol{\mu})\right].$$

Therefore,

$$f_{\Delta_i}(r) = \frac{\sqrt{b}}{\sqrt{2\pi r}}\exp\left[-\frac{1}{2}((\log r)\mathbf{e}+\mathbf{a})'\Sigma^{-1}((\log r)\mathbf{e}+\mathbf{a}) + \frac{d-(c^2/4b)}{2}\right],$$

where

$$\mathbf{a} = \log\phi_i(t) - \boldsymbol{\mu},$$
$$b = \sum_{i,j}(\Sigma^{-1})_{i,j},$$
$$c = \sum_{i,j}(a_i + a_j)(\Sigma^{-1})_{i,j},$$
$$d = \sum_{i,j}(a_i a_j)(\Sigma^{-1})_{i,j},$$

and $\mathbf{e}$ is a vector consisting solely of ones. In other words, the random variable $\Delta_i(t)$ has log-normal distribution with parameters $(-c/2b, 1/b)$. Now we are able to compute the partial expectation.

$$n_{\Delta_i(t)}(r) = \int_r^\infty (x-r)f(x)\mathrm{d}x = \exp\left(-\frac{c}{2b}+\frac{1}{2b}\right)[1-P_1(\log r)] - r[1-P_2(r)],$$

where P_1 is the cumulative density function for the normal distribution with parameters $((-c/2b)+1/b, 1/b)$ and P_2 is the cumulative density function of the log-normal distribution with parameters $(-c/2b, 1/b)$.

Appendix B: Proof of Equation (2)

$$\begin{aligned}
&E_{\mathbf{D}(t)}\{\mathbf{c}\times[\mathbf{D}(t)-\mathbf{P}(t)]^+\}\\
&\quad= E_{I(t)}\left\{E_{\mathbf{D}(t)}\{\mathbf{c}\times[\mathbf{D}(t)-\mathbf{P}(t)]^+ \mid I(t)\}\right\},\\
&\quad= E_{I(t)}\left\{E_{\Delta_i(t)}\{[\Delta_i-\sigma_i^*(t)]^+\mathbf{c}\times\phi_i(t) \mid I(t)=i\}\right\},\\
&\quad= E_{I(t)}\left\{E_{\Delta_i}\left[n_{\Delta_i(t)}(\sigma_i^*(t))\mathbf{c}\times\phi_i(t) \mid I(t)=i\right]\right\},\\
&\quad= \sum_i p_i\left\{n_{\Delta_i(t)}(\sigma_i^*(t))\mathbf{c}\times\phi_i(t)\right\}.
\end{aligned}$$

Appendix C: Detailed proof of Theorem 1

The formulation (ILP1) is simple in appearance. We can eliminate some variables and constraints by considering

lead times. Recall that $\lambda(m)$ is the purchase lead time for a type m tool, and that $x(j, t)$ has already been fixed if $t \le \lambda(m)$. These dependencies can be made explicit. We denote by λ_j the lead time of the purchase of tool j, and assume that the lead time depends on the type of tool; i.e.,

$$\lambda_j = \lambda(m) \quad \text{where } j \sim (m, n).$$

Then, $x(j, t)$ is input data if $t \le \lambda_j$, and it is a variable for $t > \lambda_j$.

As a basis for re-formulating (ILP1), we define subsets of indices which will correspond to variables in the reformulation. Let

$$\mathcal{J} = \{(j, t) : t > \lambda_i \quad \text{and} \quad x(j, \lambda_j) = 0\},$$

index tools j that do not become available by lead time λ_j in the periods after λ_j. Let

$$\mathcal{I} = \left\{(i, j, t) : \text{either } t > \lambda_{j'} \text{ or } x(j', t) = 1 \,\forall j' \in \pi_t^i(j)\right\},$$

index (i, j, t) such that every tool in the precedence set $\pi_t^i(j)$ of tool j along demand ray i is or may be available for purchase in period t. For $(i, j, t) \in \mathcal{I}$, depending on when other tools are purchased, the purchase j of in period t can increase capacity along ray i. We re-formulate (ILP1) as follows.

$$\text{(ILP2)} \quad \min \sum_t \left\{ \sum_j [\mu(j, t)x(j, t)] - \sum_{i,j} [p_i(t)\nu(i, j, t)z(i, j, t)] \right\},$$

such that

$$x(j, t) \le x(j, t + 1) \quad \forall (j, t) \in \mathcal{J}, \tag{A1}$$
$$z(i, j, t) \le x(j, t) \quad \forall (i, j, t) \in \mathcal{I}, \forall (j, t) \in \mathcal{J}, \tag{A2}$$
$$z\left(i, \gamma_t^i(j), t\right) \le z(i, j, t) \quad \forall (i, j, t) \in \mathcal{I}, \tag{A3}$$
$$x(j, t) \quad \text{binary } \forall (j, t) \in \mathcal{J}, \tag{A4}$$
$$z(i, j, t) \quad \text{binary } \forall (i, j, t) \in \mathcal{I}, \tag{A5}$$

Lemma 1. (ILP1) *and* (ILP2) *are equivalent.*

Proof. Note that if $(j, t) \notin \mathcal{J}$ or $(i, j, t) \notin \mathcal{I}$, then the value of $x(j, t)$ or $z(i, j, t)$ is determined by the input data and by Equations (7) and (8). Furthermore, feasible solutions to (ILP1) are also feasible to (ILP2). We claim that every feasible solution to (ILP2) can be augmented by the input data to yield a solution feasible to (ILP1). From the input data, we impute values of $x(j, t)$ for $(j, t) \notin \mathcal{J}$, and also of $z(i, j, t)$ for $(i, j, t) \notin \mathcal{I}$.

We note that the complement of $\mathcal{J}$ is the disjoint union of

$$\mathcal{J}_0^{\mathrm{c}} = \{(j, t)t \le \lambda_j \quad \text{and} \quad x(j, t) = 0\}, \quad \text{and}$$
$$\mathcal{J}_1^{\mathrm{c}} = \{(j, t)x(j, \min(t, \lambda_j)) = 1\}.$$

If $(j, t) \in \mathcal{J}_0^{\mathrm{c}}$, then $x(j, t) = 0$ is input data and Equation (4) holds trivially. If $(j, t) \in \mathcal{J}_1^{\mathrm{c}}$, then by Equation (A1), we get $x(j, t) = 1$. Since $(j, t) \in \mathcal{J}_1^{\mathrm{c}}$ implies $(j, t + 1) \in \mathcal{J}_1^{\mathrm{c}}$, we obtain Equation (4) for $(j, t) \in \mathcal{J}_1^{\mathrm{c}}$. Let

$$\mathcal{I}^{\mathrm{c}} = \left\{(i, j, t) : \exists j' \in \pi_j^i(t) \text{ s.t. } x(j', t') = 0 \text{ and } t \le \lambda_{j'}\right\},$$

be the complement of $\mathcal{I}$. We set $z(i, j, t) = 0$ for all $(i, j, t) \in \mathcal{I}^{\mathrm{c}}$, and show that Equation (8) holds if either $(i, j, t) \in \mathcal{I}^{\mathrm{c}}$ or $(j, t) \notin \mathcal{J}$.

If $(i, j, t) \in \mathcal{I}^{\mathrm{c}}$, then we have $z(i, j, t) = 0$; if $(j, t) \in \mathcal{J}_1^{\mathrm{c}}$, then we have $x(j, t) = 1$. In either case, Equation (8) holds. The only remaining case is $(j, t) \in \mathcal{J}_0^{\mathrm{c}}$, which actually implies $(i, j, t) \in \mathcal{I}^{\mathrm{c}}$ (set $j' = j$ in the definition of $\mathcal{I}^{\mathrm{c}}$). Thus, Equation (8) holds. Furthermore if $(i, j, t) \in \mathcal{I}^{\mathrm{c}}$, then the immediate successor $\gamma_t^i(j)$ of j is also in $\mathcal{I}^{\mathrm{c}}$, satisfying Equation (7). Thus every feasible solution for (ILP2) with the input data and the imputed values of $x(j, t)$ and $z(i, j, t)$ corresponds to a feasible solution to (ILP1).

Note that $x(j, t) = 1$ if $(j, t) \in \mathcal{J}_1^{\mathrm{c}}$. Since $z(i, j, t) = 0$ for $(i, j, t) \in \mathcal{I}^{\mathrm{c}}$ in a feasible solution to (ILP1), the objective of (ILP1) is equal to

$$\sum_{\mathcal{J}} [\mu(j, t)x(j, t)] + \sum_{\mathcal{J}_1^{\mathrm{c}}} \mu(j, t) - \sum_{\mathcal{I}} [p_i(t)\nu(i, j, t)z(i, j, t)],$$

which differs from the objective of (ILP2) by a constant. ■

Theorem 1. (ILP1) *is equivalent to a minimum-cut problem in the Capacity Expansion Network.*

Proof. By the previous lemma, it suffices to show that (ILP2) is equivalent to a minimum-cut problem. Suppose there is a cut of the Capacity Expansion Network which separates SOURCE and SINK. Use R to represent the set of z-nodes which are on the same side of the cut as SINK. Obviously, R can be decomposed by time period into T disjoint subsets R_t, one for each time period. Similarly, S represents the set of x-nodes which are on the same side of the cut as SINK, and $S = \cup_{t=1}^{T} S_t$.

The correspondence between the cut and the original variables is that all the x-nodes and z-nodes on the same side as SOURCE assume one; i.e.,

- $j \notin S_t$ iff $x(j, t) = 1$, and
- $(i, j) \notin R_t$ iff $z(i, j, t) = 1$.

Since the capacity of some arcs is infinite, the capacity of the cut could be infinite. The cut capacity is finite if and only if the following three conditions are satisfied:

1. If node $z(i, j, t) \notin R_t$, then node $x(j, t) \notin S_t$. Equivalently, $x(j, t) \ge z(i, j, t)$.
2. If node $z(\lambda_t^i(j)) \notin R_t$ then $z(i, j, t) \notin R_t$. Equivalently, $z(\gamma_t^i(j)) \le z(i, j, t)$.
3. If node $x(j, t) \notin S_t$ then node $x(j, t + 1) \notin S_{t+1}$. Equivalently, $x(j, t) \le x(j, t + 1)$.

Thus the capacity of the cut is finite if and only if **x** and **z** are feasible for (ILP2). If the cut capacity is finite, then it is equal to

$$\sum_t \left[\sum_{j \notin S_t} \mu(j,t) + \sum_{(i,j) \in R_t} p_i(t) v(i,j,t) \right]$$
$$= \sum_t \sum_{j \notin S_t} \mu(j,t) + \sum_{\mathcal{I}} p_i(t) v(i,j,t)$$
$$- \sum_t \sum_{(i,j) \notin R_t} p_i(t) v(i,j,t).$$

Note that the middle term is constant, and is independent of $x(j,t)$ and $z(i,j,t)$. Therefore, minimizing the cut capacity is equivalent to minimizing

$$\sum_t \sum_{j \notin S_t} \mu(j,t) - \sum_t \sum_{(i,j) \notin R_t} p_i(t) \times v(i,j,t)$$
$$= \sum_{\mathcal{J}} \mu(j,t) x(j,t) - \sum_{\mathcal{I}} p_i(t) v(i,j,t) z(i,j,t).$$

■

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