Proofs of Statements

The electronic companion contains detailed proofs of the theorems in the paper.

EC.1. Proof of Theorem 1

**Theorem 1.** If any of the following conditions holds, then $D$ satisfies Assumption 1.

(a) The demand $D$ in each period (either discrete or continuous) is bounded, that is, $\bar{M} < \infty$.

(b) The demand $D$ in each period (either discrete or continuous) has an increasing failure rate (IFR) distribution.

(c) $D$ has a finite variance and the distribution $F$ of $D$ has a density function $f$ and a failure rate function $r(t)$ of $F$ that does not decrease to zero faster than $1/t$; that is,

$$\lim_{t \to \infty} t \cdot r(t) = \infty,$$

where for any $t \geq 0$, $r(t) = f(t)/(1 - F(t))$.

**Proof:** Let us prove part (a) first. Since $D = \sum_{n=1}^{r+1} D_n$, if the demand in each period has a bounded support, so is $D$. It is easy to verify that

$$E[D - t | D > t] \leq (\bar{M} - t) \text{ for all } t \leq \bar{M}.$$

Dividing both sides by $t$ and taking the limit as $t$ approaches $\bar{M}$ gives the desired result.

To prove part (b), if the demand in each period has an IFR distribution, it follows from Corollary 1.B.20 on page 23 in Shaked and Shanthikumar (1994) that $D$ also has an IFR distribution. Then, it follows from Section 1.B.1 of Shaked and Shanthikumar (1994) that for any $s \geq 0$,

$$P \{ D - t_2 > s \mid D > t_2 \} \leq P \{ D - t_1 > s \mid D > t_1 \},$$

which implies that the mean residual life $m_D(t)$ is a decreasing function in $t$, giving us the desired result.
To prove part (c), we can assume without loss of generality that $F(x) < 1$ for all $x \geq 0$. For any $x$, let $\bar{F}(x) = 1 - F(x)$. It then follows from the definition of $m_D(t)$ that

$$m_D(t) = \frac{\bar{F}(u)du}{tF(t)}.$$ 

Since $E[D^2] < \infty$, it follows that $E[D] = \int_0^\infty \bar{F}(u)du < \infty$. Therefore,

$$\lim_{t \to \infty} \int_t^\infty \bar{F}(u)du = 0.$$ 

Moreover, we have that $E[D^2] = \int_0^\infty 2u\bar{F}(u)du$. Since $D$ has a finite second moment, it follows that

$$\lim_{t \to \infty} t\bar{F}(t) = 0,$$

implying that both the numerator and the denominator in the expression for $m_D(t)/t$ converge to zero at $t$ increases to infinity. Since $D$ is assumed to be a continuous random variable, we can apply L'Hospital's Rule to conclude that

$$\lim_{t \to \infty} \frac{m_D(t)}{t} = \lim_{t \to \infty} \frac{-\bar{F}(t)}{tf(t)} = \lim_{t \to \infty} \frac{1}{t \cdot t^\theta - 1} = 0,$$

which is the desired result.

\section*{EC.2. Proof of Proposition 1}

\textbf{Proposition 1.} For any $\theta > 1$, $b \geq 0$, $h \geq 0$, and $\nu > 0$, if $D$ has a distribution function $F_\theta$, then

$$\lim_{t \to \infty} \frac{m_D(t)}{t} = \frac{1}{\theta - 1} \quad \text{and} \quad \lim_{b \to \infty} \frac{C^{B*}(h, \nu b)}{C^{B*}(h, b)} = \nu^{1/\theta}.$$ 

\textbf{Proof:} It is easy to verify that $1 - F_\theta(x) = 1/(1 + x)^\theta$. It follows that $E[D] = \int_0^\infty 1 - F_\theta(x)dx = 1/(\theta - 1)$. Then, using the fact that $m_D(t) = \int_t^\infty \mathbb{P}\{D > z\}dz / \mathbb{P}\{D > t\}$, we can also show that $m_D(t) = (1 + t)/(\theta - 1)$, which proves the first part of Proposition 1.

To establish the second part, note that by definition $S^{B*}(h, b) = F_\theta^{-1}(b/(b + h))$, which implies that $S^{B*}(h, b) = \left(\frac{b+h}{h}\right)^{1/\theta} - 1$. Then, we have that

$$E\left[\left(D - S^{B*}(h, b)\right)^+\right] = \mathbb{P}\{D > S^{B*}(h, b)\} \cdot E\left[D - S^{B*}(h, b) \mid D > S^{B*}(h, b)\right].$$
\[ \frac{h}{b+h}m_D(S^g(h,b)) = \frac{h(1+S^g(h,b))}{(b+h)(\theta-1)}, \]

where the last equality follows from the formula for \( m_D(\cdot) \). Thus,

\[ E\left[ (S^g(h,b) - D)^+ \right] = E\left[ S^g(h,b) - D \right] + E\left[ (D - S^g(h,b))^+ \right] \]

\[ = S^g(h,b) - \frac{1}{\theta-1} + \frac{h(1+S^g(h,b))}{(\theta-1)(b+h)}, \]

and therefore,

\[ C^g(h,b) = hE\left[ (S^g(h,b) - D)^+ \right] + bE\left[ (D - S^g(h,b))^+ \right] \]

\[ = h \left( S^g(h,b) - \frac{1}{\theta-1} \right) + \frac{h(1+S^g(h,b))}{\theta-1} = \frac{h\theta S^g(h,b)}{\theta-1}. \]

Thus,

\[ \lim_{b \to \infty} \frac{C^g(h,\nu b)}{C^g(h,b)} = \lim_{b \to \infty} \frac{S^g(h,\nu b)}{S^g(h,b)} = \lim_{b \to \infty} \left( \frac{\nu b + h}{h} \right)^{1/\theta} - 1 = \nu^{1/\theta}, \]

which is the desired result.

**EC.3. Proof of Lemma 3**

**Lemma 3.** For every \( S \) and any starting state in period 1, the sequence of the expected cost per period over the interval \([1,T]\) given by

\[ \sum_{t=1}^{T} E[ h \cdot (X_t^{L,S} - D_t)^+ + b \cdot (D_t - X_t^{L,S})^+] \]

converges to a limit that is independent of the starting state, as \( T \to \infty \).

**Proof:** Let \( M = \sup \{ x : P(D \leq x) = 0 \} \) denote the lowest possible single period demand. Huh et al. (2006) show the convergence of the stochastic process \( \{X_t^{L,S}\} \) for all \( S > M \cdot (\tau + 1) \). This implies the result of the lemma for all such \( S \). Next, we discuss the case of \( S \leq M \cdot (\tau + 1) \). We will show that

\[ \lim_{T \to \infty} \frac{\sum_{t=1}^{T} E[ h \cdot (X_t^{L,S} - D_t)^+ + b \cdot (D_t - X_t^{L,S})^+] }{T} \]
exists and equals $b \cdot (\mu - S/(\tau + 1))$ for all $S \leq M \cdot (\tau + 1)$, where $\mu = E[D]$. We will first show that the limsup of this sequence is bounded above by this quantity, and then the liminf is bounded below by this quantity.

**Upper Bound.** Let $IP_1$ denote the inventory position at the beginning of period 1. Consider the following policy $\pi$ whose on-hand inventory process and lost sales process will be denoted by $\{X_{t,\pi}^L\}$ and $\{LOST_{t,\pi}\}$, respectively. If $IP_1 > S$, $\pi$ mimics the order-up-to $S$ policy until the first period in which the inventory position falls below $S$ before the ordering opportunity (and therefore, reaches $S$ after ordering). Let us call this period as $T$. It is easy to verify that $E[T] < \infty$ if $E[D_t] > 0$. For all $t \in \{1, \ldots, T\}$, $X_{t,\pi}^L = X_{t}^{L,S}$. For all $t \in \{1, \ldots, T - 1\}$, $LOST_{t,\pi}^L = LOST_{t}^{L,S}$.

The policy $\pi$ deviates from the order-up-to $S$ policy in the following sense from period $T$ onwards.

We introduce a standard modification of introducing the sales decision to the inventory system; in addition to the inventory replenishment decision, the manager can determine the sales quantity, which is bounded above by, but can be strictly less than, the minimum of the on-hand inventory level and realized demand. If the manager does not satisfy demand to the maximum extent possible in a period, then both a lost sales penalty and a holding cost are incurred. Since cost parameters are stationary over time, it is easy to show that under the order-up-to-$S$ policy, the introduction of this sales lever does not decrease the $T$-horizon cost, for any $T \geq 1$. Let the policy $\pi$ order-up-to $S$ each period, as usual, but we define the sales decision of this policy from period period $T$ onwards as follows: do not sell any unit in the interval $[T, T + \tau - 1]$, and sell exactly $S/(\tau + 1)$ units in each period of the interval $[T + \tau, \infty)$.

(We claim that the above policy $\pi$ is well-defined. We need to demonstrate that it is possible to sell exactly $S/(\tau + 1)$ units in period $T + \tau$ onwards. First, the demand in each period exceeds $M$, which, by assumption, exceeds $S/(\tau + 1)$. Second, observe that the inventory position at the beginning of period $T$ is $S$ by definition. By construction, $\pi$ does not sell any units in the interval $[T, T + \tau - 1]$. This implies that $X_{T+\tau}^{L,\pi} = S$. Since the demand in every period exceeds $S/(\tau + 1)$ and we have $S$ units on hand at the beginning of period $T + \tau$, it is possible to sell exactly $S/(\tau + 1)$
units in each period of the interval $[\bar{T} + \tau, \bar{T} + 2\tau]$. Moreover, this also implies that $X_t^{L,\pi} = S/(\tau + 1)$ for $t = \bar{T} + 2\tau$.

Now, we prove the claim for the interval $[\bar{T} + 2\tau + 1, \infty)$. Notice that the quantity ordered in any period $t$, $t > \bar{T} + \tau$, is the amount sold in the previous period. This implies that exactly $S/(\tau + 1)$ units are received at the beginning of period $\bar{T} + 2\tau + 1$, thereby implying the availability of exactly $S/(\tau + 1)$ units for sale at the beginning of that period. This implies that it is feasible to sell exactly $S/(\tau + 1)$ units in that period also. From this period onwards, the inventory on hand at the beginning of every period and the sales in every period are both exactly equal to $S/(\tau + 1)$, thus proving the claim about the sales quantities. Since all available units are sold, no holding costs are incurred in these periods.)

Based on the above claim, the following facts can easily be verified for all $t \geq \bar{T} + 2\tau + 1$: (i) $X_t^{L,\pi} = S/(\tau + 1)$, (ii) the ending inventory in period $t$ is zero and so, the holding cost incurred in that period is zero, (iii) the expected lost sales cost in period $t$ is $b \cdot (\mu - S/(\tau + 1))$ and (iv) the cost incurred in the interval $[1, t]$ by the order-up-to $S$ policy is smaller than the cost incurred by $\pi$ in that interval for every sample path of demands.

Fact (iv) implies that

$$
\lim_{T \to \infty} \sup_{t=1}^{T} \frac{\sum_{i=1}^{T} E[h \cdot (X_{t}^{L,S} - D_t)^{+} + b \cdot (D_t - X_{t}^{L,S})^{+}]}{T} \leq \lim_{T \to \infty} \sup_{t=1}^{T} \frac{E[\text{cost incurred by } \pi \text{ in } [1, T]]}{T}.
$$

Facts (ii) and (iii) above establish that

$$
\lim_{T \to \infty} \frac{E[\text{cost incurred by } \pi \text{ in } [1, T]]}{T}
$$

exists and equals $b \cdot (\mu - S/(\tau + 1))$. Thus, we have proved that

$$
\lim_{T \to \infty} \sup_{t=1}^{T} \frac{\sum_{i=1}^{T} E[h \cdot (X_{t}^{L,S} - D_t)^{+} + b \cdot (D_t - X_{t}^{L,S})^{+}]}{T} \leq b \cdot (\mu - S/(\tau + 1)).
$$

Lower Bound. We will now show that $b \cdot (\mu - S/(\tau + 1))$ is a lower bound on the lim inf of the average expected cost of the order-up-to $S$ policy. By the definitions of $\bar{T}$ and the order-up-to $S$
policy, we know that for all $t \geq T$, the inventory position at the beginning of a period is exactly $S$. This means that the maximum number of units that can be sold in the interval $[t, t+\tau]$ is $S$. The expected demand in this interval is $\mu \cdot (\tau + 1)$. So, $b \cdot (\mu \cdot (\tau + 1) - S)$ is a lower bound on the expected lost sales penalty costs incurred in the interval $[t, t+\tau]$ for any $t \geq T$. Recall that $T$ has a finite expectation. Therefore,

$$
\lim_{T \to \infty} \inf \sum_{t=1}^{T} E \left[ \frac{h \cdot (X_{t}^{L,S} - D_{t})^+ + b \cdot (D_{t} - X_{t}^{L,S})^+}{T} \right] \geq b \cdot (\mu - S/(\tau + 1)) .
$$

Thus, combining the upper and lower bounds, we have shown that

$$
\lim_{T \to \infty} \sum_{t=1}^{T} E \left[ \frac{h \cdot (X_{t}^{L,S} - D_{t})^+ + b \cdot (D_{t} - X_{t}^{L,S})^+}{T} \right]
$$

exists and equals $b \cdot (\mu - S/(\tau + 1))$ when $S \leq M \cdot (\tau + 1)$.

EC.4. Proof of Lemma 4

**Lemma 4.** Assume the starting state (in period 1) is such that there are $S/(\tau + 1)$ units on hand and $S/(\tau + 1)$ units due to be delivered in each of the periods 2, . . . , $\tau$. Then, the sequence of the distributions of the random variables $\{X_{t}^{L,S}\}$ converges.

**Proof:** As mentioned earlier, Huh et al. (2006) show the convergence of the stochastic process $\{X_{t}^{L,S}\}$ for all $S > M \cdot (\tau + 1)$ independent of the starting state vector. For any $S \leq M \cdot (\tau + 1)$, it is easy to verify that

$$X_{t}^{L,S} = S/(\tau + 1) \forall t$$

because $D_{t} \geq M \forall t$. This implies the result with $X_{\infty}^{L,S}$ being the deterministic quantity $S/(\tau + 1)$ for all $S \leq M \cdot (\tau + 1)$.

EC.5. Proof of Theorem 4(a)

**Theorem 4 (a).** For any $h \geq 0$ and $b \geq 0$, the best order-up-to level in the lost sales system $L(h, b)$ is bounded above by the best order-up-to level in the backorder system $B(h, b + \tau h)$ with a backorder penalty cost parameter of $b + \tau h$, that is, $S^{L*}(h, b) \leq S^{B*}(h, b + \tau h)$. 

Proof: Since the long-run average cost $C_{L,S}^c(h,b)$ is convex in $S$ (see Janakiraman and Roundy (2004)), it suffices to show that, for any $\epsilon > 0$,

$$C_{L,S}^{c,+}(h,b) - C_{L,S}^c(h,b) \geq C_{B,S}^{c,+}(h,b + \tau h) - C_{B,S}^c(h,b + \tau h). \quad \text{(EC.1)}$$

Since $X_{L,S}^c = S - \sum_{t=1}^{t-1} D_t + \sum_{t=1}^{t-1} (D_t - X_{L,S}^c)^+$, it follows that

$$E [X_{L,S}^{c}] = S - \tau E[D] + \tau E[(D - X_{L,S}^c)^+] ,$$

where $D$ denotes the demand in a single period. The above result implies that

$$C_{L,S}^c(h,b) = hE[(X_{L,S}^{c} - D)^+] + bE[(D - X_{L,S}^{c})^+]$$
$$= hE[X_{L,S}^{c} - D] + (b + h)E[(D - X_{L,S}^{c})^+]$$
$$= hS - (\tau + 1)hE[D] + (b + (\tau + 1)h)E[(D - X_{L,S}^{c})^+] .$$

Also, observe that

$$C_{B,S}^{c}(h,b + \tau h) = hS - (\tau + 1)hE[D] + (b + (\tau + 1)h)E[(D - X_{L,S}^{c})^+] .$$

Thus, showing (EC.1) is equivalent to showing the following claim:

$$E [(D - X_{L,S}^{c})^+] - E [(D - X_{L,S}^{c})^+] \geq E [(D - X_{L,S}^{c})^+] - E [(D - X_{L,S}^{c})^+] .$$

Since $X_{L,S}^{c} \sim_d S - \sum_{t}^{\tau} D_t$, this claim is equivalent to the following claim:

$$E [(D - X_{L,S}^{c})^+] - E [(D - X_{L,S}^{c})^+] \geq E [(D - (X_{L,S}^{c} + \epsilon))^+] - E [(D - X_{L,S}^{c})^+] .$$

We know from Lemma 1 of Janakiraman and Roundy (2004) that for every sample path of demands $X_{L,S}^{c} \leq X_{L,S}^c + \epsilon$ holds in every period $t$; this implies that

$$X_{L,S}^{c,+} \leq_{st} X_{L,S}^c + \epsilon ,$$

where $\leq_{st}$ refers to first order stochastic dominance. It follows that

$$E [(D - X_{L,S}^{c} + \epsilon)^+] \geq E [(D - (X_{L,S}^{c} + \epsilon))^+] .$$
Thus, to establish the last claim above, it is sufficient to prove that

$$
E\left[(D - (X_{\infty}^{L,S} + \epsilon))^+\right] - E\left[(D - X_{\infty}^{L,S})^+\right] \geq E\left[(D - (X_{\infty}^{B,S} + \epsilon))^+\right] - E\left[(D - X_{\infty}^{B,S})^+\right].
$$

This inequality follows from the convexity of the function $E[(D - x)^+]$ with respect to $x$ and the fact that $X_{\infty}^{B,S}$ is stochastically smaller than $X_{\infty}^{L,S}$ (from Corollary 1).

**EC.6. Proof of Theorem 4(b)**

**Theorem 4**(b). For any $h \geq 0$ and $b \geq 0$, the best order-up-to level in the lost sales system $L(h,b)$ is bounded below by the best order-up-to level in the backorder system $B(2h(\tau + 1), b - h(\tau + 1))$ with a holding cost parameter of $2h(\tau + 1)$ and a backorder penalty cost parameter of $b - h(\tau + 1)$, that is, $S^*(h,b) \geq S^*(2h(\tau + 1), b - h(\tau + 1))$.

**Proof:** Let $A(S)$ and $A(S + \epsilon)$ denote two lost sales inventory systems that use order-up-to policies with parameters $S$ and $S + \epsilon$, respectively. Let the starting state of $A(S)$ (resp., $A(S + \epsilon)$) be such that it has $S$ (resp., $S + \epsilon$) units on hand and none on order. Let $X_t^{L,S}$ and $X_t^{L,S+\epsilon}$ denote the inventory on hand at the beginning of period $t$ in $A(S)$ and $A(S + \epsilon)$, respectively. Let $LOST_t^{L,S}$ and $LOST_t^{L,S+\epsilon}$ denote the amounts of lost sales in period $t$ in the two systems, respectively. That is,

$$LOST_t^{L,S} = (D_t - X_t^{L,S})^+ \quad \text{and} \quad LOST_t^{L,S+\epsilon} = (D_t - X_t^{L,S+\epsilon})^+.$$ 

Consider a third lost sales inventory system $A(S + \epsilon)$ which has the same starting state as $A(S + \epsilon)$ and has the following characteristics, which is operated in parallel to $A(S)$ and $A(S + \epsilon)$. That is, each system is experiencing the same sample path of demands. In the $A(S + \epsilon)$ system, each order raises the inventory position to $S + \epsilon$. Furthermore, in $A(S + \epsilon)$, we assume that for each period in the intervals $[(\tau + 1) + 1, 2 \cdot (\tau + 1)], [3 \cdot (\tau + 1) + 1, 4 \cdot (\tau + 1)], [5 \cdot (\tau + 1) + 1, 6 \cdot (\tau + 1)]$ etc., it does not make all its inventory on hand available for sale. Specifically, the amount of sales in any of these time periods cannot exceed the demand nor the amount of inventory on hand nor
the amount of inventory on hand in the parallel system $\mathcal{A}(S)$. That is, the sales in $\mathcal{A}(S + \epsilon)$ in period $t$ are given by

$$\min\{D_t, X_t^{L,S}, \bar{X}_t^{L,S+\epsilon}\}.$$ 

In all other periods – periods within the intervals $[1, \tau + 1]$, $[2 \cdot (\tau + 1) + 1, 3 \cdot (\tau + 1)]$, $[4 \cdot (\tau + 1) + 1, 5 \cdot (\tau + 1)]$ etc. – the system $\mathcal{A}(S + \epsilon)$ behaves exactly like a lost sales inventory system, and the sales in period $t$ are given by $\min\{D_t, X_t^{L,S+\epsilon}\}$, where $X_t^{L,S+\epsilon}$ denotes the inventory on hand at the beginning of period $t$ in $\mathcal{A}(S + \epsilon)$.

We now make the following claims:

(a) $X_t^{L,S} \leq X_t^{L,S+\epsilon}$ for every $t$,

(b) $\sum_{u=1}^{t} \text{LOST}_u^{L,S} \geq \sum_{u=1}^{t} \text{LOST}_u^{L,S+\epsilon} \geq \sum_{u=1}^{t} \text{LOST}_u^{L,S+\epsilon}$ for every $t$, and

(c) $\sum_{u=2k(\tau+1)+1}^{2(k+1)(\tau+1)} [\text{LOST}_u^{L,S} \leq \text{LOST}_u^{L,S+\epsilon}] \geq \epsilon \cdot 1[\sum_{u=2k(\tau+1)+1}^{2(k+1)(\tau+1)} (D_u - X_u^{L,S}) > \epsilon]$ for every $k \geq 0$.

We will now verify statements (a)-(c). Statement (a) can be proved by induction and using the definition of the sales in period $t$ in $\mathcal{A}(S + \epsilon)$. This immediately implies that $\mathcal{A}(S + \epsilon)$ incurs fewer lost sales than $\mathcal{A}(S)$ in every period, thus implying the first part of (b). Moreover, since $\mathcal{A}(S + \epsilon)$ does not sell all the units it has available, the cumulative lost sales incurred by $\mathcal{A}(S + \epsilon)$ in any interval $[1, t]$ exceeds the corresponding quantity in $\mathcal{A}(S + \epsilon)$. This proves the second part of (b). To show (c), divide the time line into cycles, each of length $(\tau + 1)$ periods. That is, $[1, \tau + 1]$ forms the first cycle, $[(\tau + 1) + 1, 2 \cdot (\tau + 1)]$ forms the second cycle and for any $k \geq 1$, $[(k-1)(\tau+1)+1, k(\tau+1)]$ forms the $k^{th}$ cycle. In every period of every even cycle (that is, $[(2k-1)(\tau+1)+1, 2k(\tau+1)]$), $\mathcal{A}(S + \epsilon)$ sells exactly the same number of units as $\mathcal{A}(S)$, although it might have more units available. This ensures that at the beginning of every odd cycle, $\mathcal{A}(S + \epsilon)$ has exactly $\epsilon$ units more on hand than $\mathcal{A}(S)$. This implies that if $\mathcal{A}(S)$ loses $\epsilon$ or more units of sales in an odd cycle, then $\mathcal{A}(S + \epsilon)$ loses $\epsilon$ fewer units than $\mathcal{A}(S)$. This shows (c).

Let us now consider the following relations:

$$\lim_{T \to \infty} E \left( \frac{\sum_{u=1}^{T} \text{LOST}_u^{L,S+\epsilon} - \text{LOST}_u^{L,S}}{T} \right) = \lim_{k \to \infty} E \left( \frac{\sum_{u=1}^{2k(\tau+1)} \text{LOST}_u^{L,S+\epsilon} - \text{LOST}_u^{L,S}}{2k(\tau+1)} \right),$$

where

$$E$$

is the expected value.
\[
1 \left( \sum_{u=2k(\tau+1)+1}^{(2k+1)(\tau+1)} D_u > S + \epsilon \right) \leq 1 \left( \sum_{u=2k(\tau+1)+1}^{(2k+1)(\tau+1)} (D_u - X_u^{\mathcal{L},S})^+ > \epsilon \right),
\]

where the inequality above follows from the observation that the quantity ordered by \( A(S) \) in period \( u+1 \) is \( \min(X_u^{\mathcal{L},S}, D_u) \) which implies that

\[
S = X_u^{\mathcal{L},S} + \sum_{u=2k(\tau+1)+1}^{(2k+1)(\tau+1)-1} \min(X_u^{\mathcal{L},S}, D_u) \geq \sum_{u=2k(\tau+1)+1}^{(2k+1)(\tau+1)} \min(X_u^{\mathcal{L},S}, D_u).
\]

and from the identity that \((D - x)^+ = D - \min(D, x)\). Combining the above inequalities with Statements (b)-(c) and using the fact that demands are identical and independently distributed, we get

\[
\lim_{T \to \infty} E \left( \frac{1}{T} \sum_{u=1}^{T} (LOST_u^{\mathcal{L},S+\epsilon} - LOST_u^{\mathcal{L},S}) \right) = \lim_{k \to \infty} E \left( \sum_{u=1}^{2k(\tau+1)} \frac{(LOST_u^{\mathcal{L},S+\epsilon} - LOST_u^{\mathcal{L},S})}{2k(\tau+1)} \right) \leq \lim_{k \to \infty} E \left( \sum_{u=1}^{2k(\tau+1)} (D_u - X_u^{\mathcal{L},S})^+ > \epsilon \right) \leq \frac{-\epsilon}{2(\tau+1)} \cdot \lim_{k \to \infty} \sum_{u=1}^{k} \mathcal{P} \left( \sum_{u=2k(\tau+1)+1}^{(2k+1)(\tau+1)} (D_u - X_u^{\mathcal{L},S})^+ > \epsilon \right) \leq \frac{-\epsilon}{2(\tau+1)} \cdot \sum_{u=1}^{k} \mathcal{P} \left( D_u > S + \epsilon \right) = \frac{-\epsilon}{2(\tau+1)} \mathcal{P} \{ D > S + \epsilon \}.
\]

Therefore, we get

\[
\lim_{T \to \infty} \lim_{\epsilon \downarrow 0} E \left( \frac{\sum_{u=1}^{T} (LOST_u^{\mathcal{L},S+\epsilon} - LOST_u^{\mathcal{L},S})}{\epsilon} \right) \leq \frac{-1}{2(\tau+1)} \cdot \mathcal{F}(S).
\]

Since \( \lim_{\epsilon \downarrow 0} (C_{\mathcal{L},S+\epsilon}(h,b) - C_{\mathcal{L},S}(h,b))/\epsilon \geq 0 \) holds for any \( S \geq S^{\mathcal{L}*}(h,b) = \arg \min_{S \geq 0} C_{\mathcal{L},S}(h,b) \), and we know from (1) that

\[
C_{\mathcal{L},S+\epsilon} - C_{\mathcal{L},S} = h \cdot \epsilon + (b + h \cdot (\tau+1)) \cdot \lim_{T \to \infty} E\left[ \sum_{u=1}^{T} (LOST_u^{\mathcal{L},S+\epsilon} - LOST_u^{\mathcal{L},S}) \right]/T,
\]

it follows, for all \( S \geq S^{\mathcal{L}*}(h,b) = \arg \min_{S \geq 0} C_{\mathcal{L},S}(h,b) \), that

\[
0 \leq h - (b + h \cdot (\tau+1))\mathcal{F}(S)/[2(\tau+1)].
\]
This inequality applied at $S^{\mathcal{L}*}(h, b)$ implies

$$F^{-1}\left(\frac{b - h \cdot (\tau + 1)}{b + h \cdot (\tau + 1)}\right) \leq \arg\min_{S \geq 0} C^{\mathcal{L}, S}(h, b).$$

Thus, we obtain the required result since $(b - h \cdot (\tau + 1))/(b + h \cdot (\tau + 1))$ is the newsvendor fractile of the $S^{\mathcal{B}*}(2h(\tau + 1), b - h(\tau + 1))$ system.

**EC.7. Proof of the Asymptotic Optimality of $\tilde{S}$ (Section 7.1)**

Recall the definition of $\tilde{S}$ from Section 7.1:

$$\tilde{S} = \frac{b}{b + h} \cdot S^{\mathcal{B}*}(h, b, \tau) + \frac{h}{b + h} \cdot S^{\mathcal{B}*}(h, b, 0).$$

The goal of this section is to prove that

$$\lim_{b \to \infty} \frac{C^{\mathcal{L}, \tilde{S}}(h, b)}{C^{\mathcal{L}*}(h, b)} = 1,$$

whenever the demand distribution satisfies Assumption 1 and $P(D = M) = 0$.

We make a useful preliminary observation on the value of $\tilde{S}$. Since $S^{\mathcal{B}*}(h, b, \tau)$ is larger than $S^{\mathcal{B}*}(h, b, 0)$, we obtain the following inequalities:

$$\frac{b}{b + h} S_b \leq \tilde{S} \leq S_b,$$

where $S_b = S^{\mathcal{B}*}(h, b, \tau)$.

From Lemma 5, we obtain

$$C^{\mathcal{B}, \tilde{S}}(h, b/(\tau + 1)) \leq C^{\mathcal{L}, \tilde{S}}(h, b) \leq C^{\mathcal{B}, \tilde{S}}(h, b + \tau h).$$

Thus, by dividing the middle and the rightmost expressions by $C^{\mathcal{L}*}(h, b)$,

$$\frac{C^{\mathcal{L}, \tilde{S}}(h, b)}{C^{\mathcal{L}*}(h, b)} \leq \frac{C^{\mathcal{B}, \tilde{S}}(h, b + \tau h)}{C^{\mathcal{L}*}(h, b)}.$$ 

Also, recall from the proof of Theorem 5:

$$1 \leq \frac{C^{\mathcal{L}, S_b + \tau h}(h, b)}{C^{\mathcal{L}*}(h, b)} \leq \frac{C^{\mathcal{B}, S_b + \tau h}(h, b + \tau h)}{C^{\mathcal{L}*}(h, b)} \leq \frac{C^{\mathcal{B}, S_b + \tau h}(h, b + \tau h)}{C^{\mathcal{B}*}(h, b/(\tau + 1))} \to 1.$$
Therefore, from these two inequalities, it suffices to show
\[ \lim_{b \to \infty} \frac{C_{B,S}(h, b + \tau h)}{C_{B,Sb}(h, b + \tau h)} = 1. \]

However, from Theorem 2, it suffices to show
\[ \lim_{b \to \infty} \frac{C_{B,\tilde{S}}(h, b + \tau h)}{C_{B,Sb}(h, b)} = 1. \]

From the definition of \( S_b \), it is clear that the ratio within this limit is greater than 1 and so the limit itself is also at least 1. We claim that this limit is also at most 1. The two opposite inequalities then imply that the limit equals one. The proof of the claim follows.

Let us rewrite the ratio within the limit as
\[ \frac{C_{B,\tilde{S}}(h, b + \tau h)}{C_{B,Sb}(h, b)} = \left[ \frac{h \cdot E[(\tilde{S} - D)^+]}{h \cdot E[(S_b - D)^+]} \right] \cdot \left[ \frac{1 + \frac{(b+\tau h) \cdot E[(D - \tilde{S})^+]}{h \cdot E[(S - D)^+]}}{1 + \frac{b \cdot E[(D - S_b)^+]}{h \cdot E[(S_b - D)^+]}} \right]. \]

Since \( \tilde{S} \leq S_b \), the ratio in the first set of square brackets is at most 1. We will now show that the limit of the ratio in the second set of square brackets is 1.

By part (a) of Theorem 2, we know that the denominator of the ratio within the second set of square brackets converges to 1 as \( b \) approaches \( \infty \). So, it only remains to show that the numerator also converges to 1, that is, the ratio \( \frac{(b+\tau h) \cdot E[(D - \tilde{S})^+]}{h \cdot E[(S - D)^+]} \) converges to zero. Since \( \tilde{S} \geq \left( \frac{b}{b + h} \right) \cdot S_b \), this ratio is bounded above by
\[ \frac{(b + \tau h) \cdot E[(D - (\frac{b}{b + h}) \cdot S_b)^+]}{h \cdot E[((\frac{b}{b + h}) \cdot S_b - D)^+]} \cdot \frac{1}{1 + \frac{b \cdot E[(D - S_b)^+]}{h \cdot E[(S_b - D)^+]}}. \]

Note that since \( \frac{b}{b + \tau h} \) approaches 1 as \( b \to \infty \), it is sufficient to show that the following limit is zero:
\[ \lim_{b \to \infty} \frac{b \cdot E[(D - (\frac{b}{b + h}) \cdot S_b)^+]}{h \cdot E[((\frac{b}{b + h}) \cdot S_b - D)^+]} = 0. \]

Case: Bounded \( D \). When \( D \) is bounded but not deterministic, the denominator of this ratio converges to a strictly positive number. We will show that the numerator converges to zero. Recall \( \overline{M} = \sup\{x : F(x) < 1\} \). Then,
\[ b \cdot E[(D - (\frac{b}{b + h}) \cdot S_b)^+] \]
\[
\begin{align*}
&\leq b \cdot P[D > S_b] \cdot [M - \left(\frac{b}{b+h}\right) \cdot S_b] + b \cdot P[\left(\frac{b}{b+h}\right) \cdot S_b < D \leq S_b] \cdot [S_b - \left(\frac{b}{b+h}\right) \cdot S_b] \\
&\leq \frac{bh}{b+h} \cdot [M - \left(\frac{b}{b+h}\right) \cdot S_b] + b \cdot P[\left(\frac{b}{b+h}\right) \cdot S_b < D \leq S_b] \cdot \frac{h}{b+h} \cdot S_b.
\end{align*}
\]

As \(b \to \infty\), we have \(bh/(b+h) \to h < \infty\) and \(S_b \to M < \infty\). Also, the probability \(P[\left(\frac{b}{b+h}\right) \cdot S_b < D \leq S_b] \leq P[\left(\frac{b}{b+h}\right) \cdot S_b < D] \to 0\) since \(\sup_{x < M} F(x) < 1\). Thus, we conclude that \(b \cdot E[(D - \left(\frac{b}{b+h}\right) \cdot S_b)^+] \to 0\), as required.

**Case: Unbounded** \(D\). Here, we make use of the following two inequalities, which are obtained from algebraic manipulations:

\[
E[(D - \left(\frac{b}{b+h}\right) \cdot S_b)^+] \leq \left(\frac{h}{b+h}\right) \cdot E[D] + \left(\frac{b}{b+h}\right) \cdot E[(D - S_b)^+] \quad \text{and} \quad E[\left(\left(\frac{b}{b+h}\right) \cdot S_b - D\right)^+] \geq \left(\frac{b}{b+h}\right) \cdot E[(S_b - D)^+] - \left(\frac{h}{b+h}\right) \cdot E[D].
\]

Thus, an upper bound on the limit above can be written as

\[
\lim_{b \to \infty} \frac{E[D] + \frac{b}{b+h} E[(S_b - D)^+]}{1 - \frac{h}{b} E[(S_b - D)^+]},
\]

which is zero by an application of Theorem 2(a) and the fact that \(E[(S_b - D)^+] \to \infty\) as \(b \to \infty\).

This completes the proof of the claim that

\[
\lim_{b \to \infty} \frac{C^{\mathcal{E},S}(h,b)}{C^{\mathcal{E},*}(h,b)} = 1.
\]