Electronic Companion—“Inventory Management with Auctions and Other Sales Channels: Optimality of (s, S) Policies” by Woonghee Tim Huh and Ganesh Janakiraman, Management Science, DOI 10.1287/mnsc.1070.0767.
Appendices

EC.1. Concavity of $L(\cdot)$ in Proposition 1

In this section, we prove that the maximum single-period profit $L(y)$ is concave in $y$. This result is one of the components of Proposition 1.

We note $y \in \mathbb{Z}$. By part (a) of Proposition 1 and the definition of $R(y,r)$, it follows that

$$L(y) = R(y,r_y) = E_v[R(y,r_y,v)].$$

(EC.1)

Now, we fix the the vector $v$. From (4),

$$R(y,r_y,v) = \max_{k \in \mathbb{Z}^+} \left\{ J(v(1)) + J(v(2)) + \ldots + J(v(k)) - g(y-k) \right\}.$$

For each nonnegative integer $k$, we let

$$\bar{J}(k,v) = \begin{cases} 
0, & \text{if } k = 0; \\
J(v(1)) + J(v(2)) + \ldots + J(v(k)), & \text{if } k \in \{1,2,\ldots\}.
\end{cases}$$

Observe that the marginal difference $\bar{J}(k+1,v)$ and $\bar{J}(k,v)$ is $J(v(k))$; the sequence $\{J(v(k))\}$ is decreasing because $J(\cdot)$ is an increasing function and $\{v(k)\}$ is a decreasing sequence. Therefore, $\bar{J}(k,v)$ is concave in $k$ for every $v$.

We extend the definition of $\bar{J}(k,v)$ to $\mathbb{R}^+$ as follows: for $k \in \mathbb{R}^+$, let

$$\bar{J}(k,v) = ([k] - k) \cdot \bar{J}([k],v) + (k - [k]) \cdot \bar{J}([k],v).$$

That is, for any fixed $v$, $\bar{J}(k,v)$ is a piecewise linear function of $k$ with slope changes only at integers. Let $\tilde{g}(\cdot)$ be a piecewise-linear extension of $g(\cdot)$ to $\mathbb{R}$. We also define, for $y \in \mathbb{R}$ and $k \in \mathbb{R}^+$,

$$\varphi(y,k,v) = \bar{J}(k,v) - \tilde{g}(y-k).$$

(EC.2)

Thus, for each integer $y$ and value vector $v$, we have

$$R(y,r_y,v) = \max_{k \in \mathbb{Z}^+} \varphi(y,k,v).$$

(EC.3)
We claim that for any fixed \( v, \varphi(y, k, v) \) is jointly concave in \((y, k)\) in \(\mathbb{R} \times \mathbb{R}^+\). We know that \( \bar{J}(\cdot, v) \) is concave, and \( \bar{g}(\cdot) \) is convex. Since \( y - k \) is a linear function of \((y, k)\), equation (EC.2) implies \( \varphi(y, k, v) \) is jointly concave in \((y, k)\).

Thus, for any fixed \( v \) and integer \( y \), \( \varphi(y, k, v) \) is concave in \( k \in \mathbb{R}^+ \). Since \( \varphi(y, k, v) \) is piecewise linear in \( k \) with slope changes only at integer points, there exists an integer value of \( k \) that maximizes \( \varphi(y, k, v) \). Thus,

\[
\max_{k \in \mathbb{Z}^+} \varphi(y, k, v) = \max_{k \in \mathbb{R}^+} \varphi(y, k, v). \tag{EC.4}
\]

Since \( \varphi(y, k, v) \) is a jointly concave function in \((y, k) \in \mathbb{R} \times \mathbb{R}^+\), \( \max_{k \in \mathbb{R}^+} \varphi(y, k, v) \) is concave with respect to \( y \) in \(\mathbb{R}\). Thus, \( \max_{k \in \mathbb{R}^+} \varphi(y, k, v) \) is also concave, in the discrete sense, with respect to \( y \) in \(\mathbb{Z}\). It follows from (EC.3) and (EC.4) that \( R(y, \tilde{r}, v) \) is concave with respect to \( y \) in \(\mathbb{Z}\). From (EC.1), we conclude \( L(y) \) is concave in \( y \).

**EC.2. Proof of (8) in Lemma 1**

In this section, we provide the proof of claim (8). We fix any \( v \), and show

\[
R(y^2, r, v) - R(y^1, \tilde{r}, v) \leq R(y^2, r_{y^2}, v) - R(y^1, r_{y^1}, v). \tag{4}
\]

From the definition of \( \tilde{r} \), we get \( \tilde{r}(i) \leq v_{y^1}^*(i) \) for \( i \leq y^1 \), and \( \tilde{r}(i) = v_{y^1}^*(i) \) for \( i > y^1 \). Thus, \( \kappa(r_{y^1}, v) \leq \kappa(\tilde{r}, v) \). We consider two disjoint cases.

---

**Case A:** \( \kappa(r_{y^1}, v) = \kappa(\tilde{r}, v) \).

First, we compare \( R(y^1, r_{y^1}, v) \) and \( R(y^1, \tilde{r}, v) \). Now, \( I[v(i) \geq v_{y^1}^*(i)] = I[v(i) \geq \tilde{r}(i)] \) for each \( i \), by the definition of \( \kappa \). Then,

\[
[J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq v_{y^1}^*(i)]
= [J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)],
\]

implying \( R(y^1, r_{y^1}, v) = R(y^1, \tilde{r}, v) \) by (4). Also, we compare \( R(y^2, r_{y^2}, v) \) and \( R(y^2, r, v) \). Observe from equation (4) that \( r_{y^2} \) maximizes \( R(y^2, r, v) \) for each \( v \). Therefore, we obtain \( R(y^2, r_{y^2}, v) \geq R(y^2, r, v) \). Thus, combining these two results, we obtain claim (8).
CASE B: $\kappa(\bar{r}_{y^1}, v) < \kappa(\bar{r}, v)$.

Since $\bar{r}(i) = v_{y^1}^*(i)$ holds for $i > y^1$, by the definition of $\bar{r}$, it follows that

$$\kappa(\bar{r}_{y^1}, v) < \kappa(\bar{r}, v) \leq y^1.$$

From equation (4), we observe that if the following pair of inequalities holds for each positive integer $i$, then (8) holds.

$$\left[ J(v(i)) + \Delta g(y^2 - i + 1) \cdot I[v(i) \geq v_{y^2}^*(i)] \right] \geq \left[ J(v(i)) + \Delta g(y^1 - i + 1) \right] \cdot I[v(i) \geq v_{y^1}^*(i)] \quad \text{(EC.5)}$$

and

$$\left[ J(v(i)) + \Delta g(y^1 - i + 1) \cdot I[v(i) \geq \bar{r}(i)] \right] \geq \left[ J(v(i)) + \Delta g(y^2 - i + 1) \right] \cdot I[v(i) \geq r(i)]. \quad \text{(EC.6)}$$

We first show (EC.5). For each $i$,

$$\left[ J(v(i)) + \Delta g(y^2 - i + 1) \cdot I[v(i) \geq v_{y^2}^*(i)] \right] = \left[ J(v(i)) + \Delta g(y^2 - i + 1) \right]^+ \geq \left[ J(v(i)) + \Delta g(y^1 - i + 1) \right]^+$$

$$= \left[ J(v(i)) + \Delta g(y^1 - i + 1) \right] \cdot I[v(i) \geq v_{y^1}^*(i)].$$

The equalities above follow from the definition of $v_{y^2}^*$ and $v_{y^1}^*$. The inequality follows from the convexity of $g$.

Now, we show (EC.6) by considering two subcases.

- **Subcase B1: $i \in \{1, 2, \ldots, y^1\}$**.

Since $i \leq y^1$ and $\Delta g(z) = h$ for all $z > 0$, observe from the definition of $v_{y^1}^*$ that $v_{y^1}^*(i) = J^{-1}(-h)$ holds. In this case, $\bar{r}(i)$ is either $r(i)$ or $v_{y^1}^*(i)$.

If $\bar{r}(i) = v_{y^1}^*(i)$, then

$$\left[ J(v(i)) + \Delta g(y^1 - i + 1) \cdot I[v(i) \geq \bar{r}(i)] \right] = [J(v(i)) + h] \cdot I[v(i) \geq v_{y^1}^*(i)] \quad \text{(since } \bar{r}(i) = v_{y^1}^*(i) \text{ and } y^1 - i + 1 > 0)$$
= [J(v(i)) + h^+] (since v^*_i(i) = J^{-1}(-h))

≥ [J(v(i)) + h] · I[v(i) ≥ r(i)] (by a property of [·]^+)

= [J(v(i)) + Δg(y^2 - i + 1)] · I[v(i) ≥ r(i)] (since y^2 - i + 1 > 0).

Otherwise, r̃(i) = r(i) holds. Then,

[J(v(i)) + Δg(y^1 - i + 1)] · I[v(i) ≥ r(i)] = [J(v(i)) + Δg(y^2 - i + 1)] · I[v(i) ≥ r(i)],

since Δg(y^1 - i + 1) = h = Δg(y^2 - i + 1).

• Subcase B2: i > y^1.

Since κ(̃r, v) ≤ y^1 holds in Case B, it follows i > κ(̃r, v). Thus, I[v(i) ≥ r̃(i)] = 0, implying

[J(v(i)) + Δg(y^1 - i + 1)] · I[v(i) ≥ r̃(i)] = 0.

Furthermore, since i > y^1 > κ(r_y^1, v), we get i ≥ κ(r_y^1, v) + 1. Thus, v(i) ≤ v(κ(r_y^1, v) + 1).

J(v(i)) + Δg(y^2 - i + 1) ≤ J(v(i)) + h
≤ J(v(κ(r_y^1, v) + 1)) + h
≤ 0.

The first inequality follows from the convexity of g whose derivative is bounded above by h. The second inequality follows from the monotonicity of J. For the last inequality, observe that the virtual value from the (κ(r_y^1, v) + 1)th “highest” customer is smaller than −h, the holding cost savings from selling that unit.

Thus, we complete the proof of (EC.6) for both subcases B1 and B2. This completes the proof of (8) in Lemma 1.

**EC.3. Proof of Proposition 2**

Let ̃D be the compact interval [d_L, d_U]. Thus, D is the set of integers in ̃D. For any fixed integer y and d ∈ D, define π(y, d) = π^1(d) + π^2(y - d), where

π^1(d) = p(d) · d ,

and
\[ \pi^2(r) = E[L_A(r - \epsilon_P)]. \]

Now, for any integer \( y \) and real \( d \in \tilde{D} \), define
\[ \tilde{\pi}^1(d) = (1 - \lambda) \cdot \tilde{\pi}^1([d]) + \lambda \cdot \tilde{\pi}^1([d]) \]
where \( \lambda = d - [d] \). Similarly, for any real \( r \), define
\[ \tilde{\pi}^2(r) = (1 - \lambda) \cdot \tilde{\pi}^2([r]) + \lambda \cdot \tilde{\pi}^2([r]) \]
where \( \lambda = r - [r] \). For any real \( y \) and \( d \in \tilde{D} \), let
\[ \tilde{\pi}(y, d) = \tilde{\pi}^1(d) + \tilde{\pi}^2(y - d). \]

Since the expected revenue from the posted price channel, \( \pi^1(d) = p(d) \cdot d \), is assumed to be concave with respect to \( d \in D \), its linear interpolation \( \tilde{\pi}^1(d) \) is also concave with respect to \( d \in \tilde{D} \).

The concavity of \( L_A(t) \), the maximum expected single-period profit from the auction channel, is proved in Proposition 1 of section 4. Thus, \( \tilde{\pi}(y, d) \) is jointly concave with respect to \( y \) and \( d \). Thus, \( \max_{d \in \tilde{D}} \tilde{\pi}(y, d) \) is concave with respect to \( y \).

Moreover, from the construction of \( \tilde{\pi} \), if \( y \) is an integer, then \( \tilde{\pi}(y, d) \) is a piece-wise linear interpolation of \( \pi(y, d) \) with respect to \( d \). Thus, for fixed integer \( y \), the single-dimensional function \( \tilde{\pi}(y, \cdot) \) has at least one integer maximizer, i.e.,
\[ Q(y) = \max_{d \in \tilde{D}} \pi(y, d) = \max_{d \in \tilde{D}} \tilde{\pi}(y, d). \]

The conclusions of the last two paragraphs together imply that \( Q(y) \) is concave with respect to \( y \).

**EC.4. Proof of Theorem 4**

We need the following lemma about the optimal allocation problem which is useful in studying the multiple channel problem.

**Lemma EC.1.** For each \( i = 1, 2, \ldots, I \), let \( f_i(\cdot) \) be a quasi-concave function defined on a set of consecutive integers. Let \( s_i^* \) be a maximizer of \( f_i(\cdot) \). Then, \( f(s) = \max \{ \sum_i f_i(s_i) \mid \sum_i s_i = s \} \) is quasi-concave, and achieves its maximum at \( \sum_i s_i^* \).
Proof. First, we provide the proof assuming that the domain of $f_i$ is the set of all integers. Since $s_i^*$ is the maximizer of $f_i(\cdot)$, we have $f_i(s_i) \leq f_i(s_i + 1)$ for $s_i < s_i^*$, and $f_i(s_i) \geq f_i(s_i + 1)$ for $s_i \geq s_i^*$. Let $s^* = \sum_i s_i^*$.

Suppose $s < s^*$. Then, we claim that there exist $s_1, s_2, \ldots, s_I$ such that $s = \sum_i s_i$ and $f(s) = \sum_i f_i(s_i)$ satisfying $s_i \leq s_i^*$ for each $i$. To see this claim, suppose that there exists $j$ such that $s_j > s_j^*$ and $s_k < s_k^*$. Then, by decreasing $s_j$ by 1 and increasing $s_k$ by 1, we weakly increase the objective function. By repeating this process, we prove the claim.

Furthermore, there exists $i'$ such that $s_{i'} < s_{i'}^*$. Then,

$$f(s) = \sum_i f_i(s_i) = f_i(s_{i'}) + \sum_{i \neq i'} f_i(s_i) \leq f_i(s_{i'} + 1) + \sum_{i \neq i'} f_i(s_i) \leq f(s + 1).$$

Similarly, it can be argued that $s > s^*$ implies $f(s) \leq f(s - 1)$.

If the domain of $f_i$ is a subset of all integers, extend $f_i$ by defining $f_i(s_i) = -\infty$ for each $s_i$ outside the domain. □

Proof. [Proof of Theorem 4] Let $Q_m(y_m) = \max_{d_m} \pi_m(y_m, d_m)$, and $Q(y) = \max_d \pi(y, d)$. Since each sales channel $m$ satisfies Condition 2, $Q_m(\cdot)$ is quasi-concave. Let $y_m^*$ be a maximizer of $Q_m$. Then, by Lemma EC.1, $Q(\cdot)$ is also quasi-concave, and achieves its maximum at $y^* = \sum_m y_m^*$.

Consider two systems $\tilde{A}$ and $A$, in which the inventory levels after replenishment are $y_1$ and $y_2$, respectively. Assume $y^* \leq y_1 < y_2$. Suppose that in the $A$ system, the seller chooses an allocation of $y = (y_1, \ldots, y_M)$ where $y_2 = \sum_m y_m$, and the sales lever vector of $d = (d_1, \ldots, d_M)$. For the $\tilde{A}$ system, we specify the allocation vector $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_M)$ and the sales lever vector $\tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_M)$ such that for each channel $m$, the pair of $\tilde{y}_m$ and $\tilde{d}_m$ satisfy part (b) of Condition 2.

From $\sum_m y_m^* = y^* \leq y_1 < y_2 = \sum_m y_m$, there exists an allocation vector $(\tilde{y}_1, \ldots, \tilde{y}_M)$ satisfying $y_1^* = \sum_m \tilde{y}_m$, and

$$\tilde{y}_m = y_m, \quad \text{if } y_m < y_m^*,$$
$$\tilde{y}_m \in \{y_m^*, y_m^* + 1, \ldots, y_m\}, \quad \text{if } y_m \geq y_m^*.$$

We now construct the sales levers for the $\tilde{A}$ system. If $\tilde{y}_m = y_m$, set $\tilde{d}_m = d_m$, and we get, for every $\epsilon_m$,

$$\tilde{y}_m - D_m(\tilde{d}_m, \epsilon_m) = y_m - D_m(d_m, \epsilon_m),$$

and
\[ \pi_m(\tilde{y}_m, \tilde{d}_m) = \pi_m(y_m, d_m). \]

Otherwise, we have \( y_m^* \leq \tilde{y}_m < y_m \). By Condition 2, there exists \( \tilde{d}_m \) such that, for any \( \epsilon_m \), we have
\[
\begin{align*}
\tilde{y}_m - D_m(\tilde{d}_m, \epsilon_m) &\leq y_m - D_m(d_m, \epsilon_m), \quad \text{and} \\
\pi_m(\tilde{y}_m, \tilde{d}_m) &\geq \pi_m(y_m, d_m)
\end{align*}
\]

Therefore, for every \( \epsilon = (\epsilon_1, \ldots, \epsilon_M) \), it follows
\[
\begin{align*}
y^1 - \sum_m D_m(\tilde{d}_m, \epsilon_m) &\leq y^2 - \sum_m D_m(d_m, \epsilon_m), \quad \text{and} \\
\pi(y^1, \tilde{d}) = \sum_m \pi_m(\tilde{y}_m, \tilde{d}_m) &\geq \sum_m \pi_m(y_m, d_m) = \pi(y^2, d),
\end{align*}
\]
satisfying part (b) of Condition 2 for the multiple sales channel model.

Now consider the case \( y^* \geq y^1 > y^2 \) under Condition 2*. Here, set \( \tilde{y}_m = y_m \) if \( y_m > y_m^* \); otherwise, let \( y_m \leq \tilde{y}_m \leq y^* \). A similar analysis can be applied. \( \square \)

**EC.5. Proof of Theorem 5**

We first prove the result assuming that both \( y \) and \( d_m \)’s are real-valued. Let \((y^*, d^*) = \arg\max \pi(y, d)\). From the additivity of the demand function, \( d_m^* = \arg\max \Lambda_m(d_m) \) where each \( \Lambda_m \) is concave, and \( y^* \) is a minimizer of \( g(y|d^*) \) where
\[
g(y|d) = h \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^+ + b \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^-. \]

Note that \( g(y|d) \) depends on \( y \) and \( d \) only through \( y - \sum_m d_m \) in a convex manner.

The first part of Condition 1* follows from the concavity of \( \max_d \pi(y, d) \) in \( y \). We will now prove part (b). Consider any \( y^1 \) and \( y^2 \) satisfying \( y^* \leq y^1 < y^2 \), and any \( d^2 = (d_1^2, \ldots, d_M^2) \). Recall \( d^* = \arg\max_d \pi(y^*, d) \) and let \( d^o = \arg\max_d \{ \pi(y^2, d) \mid \sum_m d_m = \sum_m d_m^2 \} \). By an application of the proof of Lemma EC.1, we can assume, without loss of generality, that we have either (i) \( d_i^o \leq d_m^* \) for all \( m \), or (ii) \( d_i^o \geq d_m^* \) for all \( m \). If (i) occurs, set \( d_1^1 = d_m^* \) for each \( m \). Thus, \( d_m^* = d_i^1 \geq d_i^o \) for each \( m \), and \( y^* - \sum_m d_m^* \leq y^1 - \sum_m d_1^1 \leq y^2 - \sum_m d_m^o \). Otherwise, in case of (ii), define \( \lambda = (y^1 - y^*)/(y^2 - y^*) \). Then, set \( d^1 = (d_1^1, d_2^1, \ldots, d_M^1) \) such that
\[
\sum_m d_m^1 = (1 - \lambda) \sum_m d_m^* + \lambda \sum_m d_m^o.
\]
\[ d^1_m \in \{d^*_m, d^\circ_m\} \quad \text{for each } m \neq m', \quad \text{and} \]
\[ d^0_{m'} \in [d^*_{m'}, d^\circ_{m'}] , \]
for some \( m' \in \{1, \ldots, M\} \). It follows that \( y^1 - \sum_m d^1_m \) is a convex combination of \( y^* - \sum_m d^*_m \) and \( y^2 - \sum_m d^\circ_m \). Therefore, in both cases, it is straightforward to show
\[
\pi(y^1, d^1) \geq \pi(y^2, d^*) \geq \pi(y^2, d^2) , \quad \text{and} \]
\[
y^1 - \sum_m D_m(d^1_m, \epsilon_m) \leq \max \left\{ y^2 - \sum_m D_m(d^\circ_m, \epsilon_m), \ y^* - \sum_m D_m(d^*_m, \epsilon_m) \right\} ,
\]
where \( d^1 = (d^1_1, \ldots, d^1_M) \). Since \( \sum_m D_m(d^\circ_m, \epsilon_m) = \sum_m D_m(d^2_m, \epsilon_m) \), part (b) of Condition 1* is satisfied.

For part (c), apply a similar analysis to the case of \( y^* \geq y^1 > y^2 \). If \( y^2 - \sum_m D_m(d^2_m, \epsilon_m) \leq y^* - \sum_m D_m(d^*_m, \epsilon_m) \), then choose \( d^1 \) similar to the case (ii) above; otherwise, choose \( d^1 \) such that \( y^2 - \sum_m D_m(d^\circ_m, \epsilon_m) = y^1 - \sum_m D_m(d^1_m, \epsilon_m) \), and \( d^*_m \geq d^\circ_m \geq d^1_m \) for each channel \( m \).

We remark that if \( y \) and \( d_m \)'s are integer-valued, then probabilistic rounding results in the required conclusions.

**EC.6. Proof of Lemma 4**

For each \( t \), we define \( L_t(\cdot) \) recursively as following:
\[
L_t(y_t) = \begin{cases} 
\max_{d_t} \pi_t(y_t, d_t) + \alpha \cdot E[L_{t+1}(y_t - d_t - \epsilon_t)], & \text{if } t < T, \\
\max_{d_t} \pi_t(y_T, d_T), & \text{if } t = T.
\end{cases}
\]
By the convexity of \( \pi \), \( L_t \) is convex for each \( t \). Let \( y^* \) be the maximizer of \( L_1 \).

Suppose \( y^* \leq y^1 \leq y^2 \). For a fixed sequence of \( \epsilon_1, \epsilon_2, \ldots, \epsilon_T \), let \( \mathcal{A}^* \) be the optimal system starting with the inventory level \( y^* \). Let \( d^*_1, d^*_2, \ldots, d^*_T \) be the optimal sequence of decisions in \( \mathcal{A}^* \). (Clearly, \( d^*_t \) depends on \( \epsilon_1, \ldots, \epsilon_t \), but we suppress that dependence to simplify notation.) Let \( y^*_t \) be the beginning of sub-period inventory level in \( \mathcal{A}^* \).

Consider two systems \( \hat{\mathcal{A}} \) and \( \mathcal{A} \), and suppose that the inventory levels at the beginning of sub-period \( t = 1 \) are \( y^1 \) and \( y^2 \), respectively. Suppose that for fixed \( \epsilon_1, \epsilon_2, \ldots, \epsilon_T \), the decisions of \( \mathcal{A} \) are given by \( d^1_1, d^1_2, \ldots, d^1_T \). Let \( \lambda \in [0, 1] \) such that \( y^1 = \lambda y^* + (1 - \lambda)y^2 \). For each \( t \), choose the decision
of the $\tilde{A}$ system such that $d_i^1 = \lambda d_i^* + (1 - \lambda)d_i^2$. Let $z_i^*$, $z_i^1$ and $z_i^2$ be the ending inventories in sub-period $t$ in systems $A^*$, $\tilde{A}$ and $A$, respectively. Thus, if $y_i^1 = \lambda y_i^* + (1 - \lambda)y_i^2$, then

$$z_i^1 = y_i^1 - (d_i^1 + \epsilon_i)$$

$$= [\lambda y_i^* + (1 - \lambda)y_i^2] - [\lambda d_i^* + (1 - \lambda)d_i^2 + \epsilon_i]$$

$$= \lambda[y_i^* - d_i^* - \epsilon_i] + (1 - \lambda)[y_i^2 - d_i^2 - \epsilon_i]$$

$$= \lambda z_i^* + (1 - \lambda)z_i^2.$$

By induction, we show the above result for all $t$. Since $z_T^* \leq y^*$, it follows $z_T^1 \leq \max\{z_T^2, y^*\}$.

Furthermore, the expected single sub-period profit in $t$ satisfies

$$\pi_t(y_i^1, d_i^1) \geq \lambda \cdot \pi_t(y_i^*, d_i^*) + (1 - \lambda) \cdot \pi_t(y_i^2, d_i^2),$$

where expectation is taken over $\epsilon_i$. Thus, the total expected profit in all $T$ sub-periods satisfies

$$\sum_{t=1}^{T} \alpha^{t-1} \pi_t(y_i^1, d_i^1) \geq \lambda \sum_{t=1}^{T} \alpha^{t-1} \pi_t(y_i^*, d_i^*) + (1 - \lambda) \sum_{t=1}^{T} \alpha^{t-1} \pi_t(y_i^2, d_i^2).$$

Therefore, the total expected profit in the $\tilde{A}$ system (left-hand side) is at least the total expected profit in the $A$ system (last term on the right-hand side).