Appendices

EC.1. Concavity of $L(\cdot)$ in Proposition 1

In this section, we prove that the maximum single-period profit $L(y)$ is concave in y. This result is one of the components of Proposition 1.

We note $y \in \mathcal{Z}$. By part (a) of Proposition 1 and the definition of $R(y,\mathbf{r})$, it follows that

$$L(y) = R(y,\mathbf{r}_y) = E_{\mathbf{v}}[R(y,\mathbf{r}_y,\mathbf{v})]. \tag{EC.1}$$

Now, we fix the the vector $\mathbf{v}$. From (4),

$$R(y,\mathbf{r}_y,\mathbf{v}) = \max_{k\in\mathcal{Z}^+}\{J(v(1)) + J(v(2)) + \ldots + J(v(k)) - g(y-k)\ \} \ .$$

For each nonnegative integer k, we let

$$\bar{J}(k,\mathbf{v}) = \begin{cases} 0, & \text{if } k=0; \\ J(v(1)) + J(v(2)) + \cdots + J(v(k)), & \text{if } k \in \{1,2,\ldots\}. \end{cases}$$

Observe that the marginal difference $\bar{J}(k+1,\mathbf{v})$ and $\bar{J}(k,\mathbf{v})$ is $J(v(k))$; the sequence $\{J(v(k))\}$ is decreasing because $J(\cdot)$ is an increasing function and $\{v(k)\}$ is a decreasing sequence. Therefore, $\bar{J}(k,\mathbf{v})$ is concave in k for every $\mathbf{v}$.

We extend the definition of $\bar{J}(k,\mathbf{v})$ to $\Re^+$ as follows: for $k \in \Re^+$, let

$$\bar{J}(k,\mathbf{v}) = (\lceil k\rceil - k)\cdot \bar{J}(\lfloor k\rfloor, \mathbf{v}) + (k - \lfloor k\rfloor)\cdot \bar{J}(\lceil k\rceil, \mathbf{v}) \ .$$

That is, for any fixed $\mathbf{v}$, $\bar{J}(k,\mathbf{v})$ is a piecewise linear function of k with slope changes only at integers. Let $\tilde{g}(\cdot)$ be a piecewise-linear extension of $g(\cdot)$ to $\Re$. We also define, for $y \in \Re$ and $k \in \Re^+$,

$$\varphi(y,k,\mathbf{v}) \ = \ \bar{J}(k,\mathbf{v}) - \tilde{g}(y-k) \ . \tag{EC.2}$$

Thus, for each integer y and value vector $\mathbf{v}$, we have

$$R(y,\mathbf{r}_y,\mathbf{v}) \ = \ \max_{k\in\mathcal{Z}^+}\varphi(y,k,\mathbf{v}). \tag{EC.3}$$

We claim that for any fixed $\mathbf{v}$, $\varphi(y,k,\mathbf{v})$ is jointly concave in (y,k) in $\Re \times \Re^+$. We know that $\bar{J}(\cdot,\mathbf{v})$ is concave, and $\tilde{g}(\cdot)$ is convex. Since $y-k$ is a linear function of (y,k), equation (EC.2) implies $\varphi(y,k,\mathbf{v})$ is jointly concave in (y,k).

Thus, for any fixed $\mathbf{v}$ and integer y, $\varphi(y,k,\mathbf{v})$ is concave in $k \in \Re^+$. Since $\varphi(y,k,\mathbf{v})$ is piecewise linear in k with slope changes only at integer points, there exists an integer value of k that maximizes $\varphi(y,k,\mathbf{v})$. Thus,

$$\max_{k\in\mathcal{Z}^+} \varphi(y,k,\mathbf{v}) = \max_{k\in\Re^+} \varphi(y,k,\mathbf{v}) \ . \tag{EC.4}$$

Since $\varphi(y,k,\mathbf{v})$ is a jointly concave function in $(y,k) \in \Re \times \Re^+$, $\max_{k\in\Re^+} \varphi(y,k,\mathbf{v})$ is concave with respect to y in $\Re$. Thus, $\max_{k\in\Re^+} \varphi(y,k,\mathbf{v})$ is also concave, in the discrete sense, with respect to y in $\mathcal{Z}$. It follows from (EC.3) and (EC.4) that $R(y,\mathbf{r}_y,\mathbf{v})$ is concave with respect to y in $\mathcal{Z}$. From (EC.1), we conclude $L(y)$ is concave in y.

EC.2. Proof of (8) in Lemma 1

In this section, we provide the proof of claim (8). We fix any $\mathbf{v}$, and show

$$R(y^2,\mathbf{r},\mathbf{v}) - R(y^1,\tilde{\mathbf{r}},\mathbf{v}) \ \le \ R(y^2,\mathbf{r}_{y^2},\mathbf{v}) - R(y^1,\mathbf{r}_{y^1},\mathbf{v}) \ .$$

From the definition of $\tilde{\mathbf{r}}$, we get $\tilde{r}(i) \le v^*_{y^1}(i)$ for $i \le y^1$, and $\tilde{r}(i) = v^*_{y^1}(i)$ for $i > y^1$. Thus, $\kappa(\mathbf{r}_{y^1},\mathbf{v}) \le \kappa(\tilde{\mathbf{r}},\mathbf{v})$. We consider two disjoint cases.

<u>Case A:</u> $\kappa(\mathbf{r}_{y^1},\mathbf{v}) = \kappa(\tilde{\mathbf{r}},\mathbf{v})$.

First, we compare $R(y^1,\mathbf{r}_{y^1},\mathbf{v})$ and $R(y^1,\tilde{\mathbf{r}},\mathbf{v})$. Now, $I[v(i) \ge v^*_{y^1}(i)] = I[v(i) \ge \tilde{r}(i)]$ for each i, by the definition of κ. Then,

$$\begin{aligned} &[J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \ge v^*_{y^1}(i)] \\ &= [J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \ge \tilde{r}(i)], \end{aligned}$$

implying $R(y^1,\mathbf{r}_{y^1},\mathbf{v}) = R(y^1,\tilde{\mathbf{r}},\mathbf{v})$ by (4). Also, we compare $R(y^2,\mathbf{r}_{y^2},\mathbf{v})$ and $R(y^2,\mathbf{r},\mathbf{v})$. Observe from equation (4) that $\mathbf{r}_{y^2}$ maximizes $R(y^2,\mathbf{r},\mathbf{v})$ for each $\mathbf{v}$. Therefore, we obtain $R(y^2,\mathbf{r}_{y^2},\mathbf{v}) \ \ge \ R(y^2,\mathbf{r},\mathbf{v})$. Thus, combining these two results, we obtain claim (8).

Case B: $\kappa(\mathbf{r}_{y^1},\mathbf{v}) < \kappa(\tilde{\mathbf{r}},\mathbf{v})$.

Since $\tilde{r}(i) = v^*_{y^1}(i)$ holds for $i > y^1$, by the definition of $\tilde{\mathbf{r}}$, it follows that

$$\kappa(\mathbf{r}_{y^1},\mathbf{v}) \ < \ \kappa(\tilde{\mathbf{r}},\mathbf{v}) \ \leq \ y^1.$$

From equation (4), we observe that if the following pair of inequalities holds for each positive integer i, then (8) holds.

$$\left[J(v(i)) + \Delta g(y^2 - i + 1)\right] \cdot I[v(i) \geq v^*_{y^2}(i)] \ \geq \ \left[J(v(i)) + \Delta g(y^1 - i + 1)\right] \cdot I[v(i) \geq v^*_{y^1}(i)] \quad \text{(EC.5)}$$

and

$$\left[J(v(i)) + \Delta g(y^1 - i + 1)\right] \cdot I[v(i) \geq \tilde{r}(i)] \ \geq \ \left[J(v(i)) + \Delta g(y^2 - i + 1)\right] \cdot I[v(i) \geq r(i)]. \quad \text{(EC.6)}$$

We first show (EC.5). For each i,

$$\begin{aligned}
&[J(v(i)) + \Delta g(y^2 - i + 1)] \cdot I[v(i) \geq v^*_{y^2}(i)] \\
&= [J(v(i)) + \Delta g(y^2 - i + 1)]^+ \\
&\geq [J(v(i)) + \Delta g(y^1 - i + 1)]^+ \\
&= [J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq v^*_{y^1}(i)].
\end{aligned}$$

The equalities above follow from the definition of $v^*_{y^2}$ and $v^*_{y^1}$. The inequality follows from the convexity of g.

Now, we show (EC.6) by considering two subcases.

- *Subcase B1:* $i \in \{1, 2, \ldots, y^1\}$.

Since $i \leq y^1$ and $\Delta g(z) = h$ for all $z > 0$, observe from the definition of $v^*_{y^1}$ that $v^*_{y^1}(i) = J^{-1}(-h)$ holds. In this case, $\tilde{r}(i)$ is either $r(i)$ or $v^*_{y^1}(i)$.

If $\tilde{r}(i) = v^*_{y^1}(i)$, then

$$\begin{aligned}
&[J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)] \\
&= [J(v(i)) + h] \cdot I[v(i) \geq v^*_{y^1}(i)] \qquad \text{(since } \tilde{r}(i) = v^*_{y^1}(i) \text{ and } y^1 - i + 1 > 0\text{)}
\end{aligned}$$

$$= [J(v(i)) + h]^+ \qquad \text{(since } v^*_{y^1}(i) = J^{-1}(-h)\text{)}$$

$$\geq [J(v(i)) + h] \cdot I[v(i) \geq r(i)] \qquad \text{(by a property of } [\cdot]^+\text{)}$$

$$= \left[J(v(i)) + \Delta g(y^2 - i + 1)\right] \cdot I[v(i) \geq r(i)] \qquad \text{(since } y^2 - i + 1 > 0\text{)} .$$

Otherwise, $\tilde{r}(i) = r(i)$ holds. Then,

$$[J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)] = \left[J(v(i)) + \Delta g(y^2 - i + 1)\right] \cdot I[v(i) \geq r(i)] \; ,$$

since $\Delta g(y^1 - i + 1) = h = \Delta g(y^2 - i + 1)$.

- *Subcase B2: $i > y^1$.*

Since $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \leq y^1$ holds in Case B, it follows $i > \kappa(\tilde{\mathbf{r}}, \mathbf{v})$. Thus, $I[v(i) \geq \tilde{r}(i)] = 0$, implying

$$[J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)] = 0 \; .$$

Furthermore, since $i > y^1 > \kappa(\mathbf{r}_{y^1}, \mathbf{v})$, we get $i \geq \kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1$. Thus, $v(i) \leq v(\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1)$.

$$\begin{aligned} J(v(i)) + \Delta g(y^2 - i + 1) &\leq J(v(i)) + h \\ &\leq J(v(\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1)) + h \\ &\leq 0. \end{aligned}$$

The first inequality follows from the convexity of g whose derivative is bounded above by h. The second inequality follows from the monotonicity of J. For the last inequality, observe that the virtual value from the $(\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1)$'th "highest" customer is smaller than $-h$, the holding cost savings from selling that unit.

Thus, we complete the proof of (EC.6) for both subcases B1 and B2. This completes the proof of (8) in Lemma 1.

EC.3. Proof of Proposition 2

Let $\tilde{\mathcal{D}}$ be the compact interval $[d_L, d_U]$. Thus, $\mathcal{D}$ is the set of integers in $\tilde{\mathcal{D}}$. For any fixed integer y and $d \in \mathcal{D}$, define $\pi(y, d) = \pi^1(d) + \pi^2(y - d)$, where

$$\pi^1(d) = p(d) \cdot d \; , \qquad \text{and}$$

$$\pi^2(r) = E[L_A(r-\epsilon_P)] \ .$$

Now, for any integer y and real $d \in \tilde{\mathcal{D}}$, define

$$\tilde{\pi}^1(d) = (1-\lambda)\cdot\tilde{\pi}^1(\lfloor d \rfloor) \ + \ \lambda\cdot\tilde{\pi}^1(\lceil d \rceil)$$

where $\lambda = d - \lfloor d \rfloor$. Similarly, for any real r, define

$$\tilde{\pi}^2(r) = (1-\lambda)\cdot\tilde{\pi}^2(\lfloor r \rfloor) \ + \ \lambda\cdot\tilde{\pi}^2(\lceil r \rceil)$$

where $\lambda = r - \lfloor r \rfloor$. For any real y and $d \in \tilde{\mathcal{D}}$, let

$$\tilde{\pi}(y,d) = \tilde{\pi}^1(d) \ + \ \tilde{\pi}^2(y-d) \ .$$

Since the expected revenue from the posted price channel, $\pi^1(d) = p(d)\cdot d$, is assumed to be concave with respect to $d \in \mathcal{D}$, its linear interpolation $\tilde{\pi}^1(d)$ is also concave with respect to $d \in \tilde{\mathcal{D}}$. The concavity of $L_A(t)$, the maximum expected single-period profit from the auction channel, is proved in Proposition 1 of section 4. Thus, $\tilde{\pi}(y,d)$ is jointly concave with respect to y and d. Thus, $\max_{d\in\tilde{\mathcal{D}}}\tilde{\pi}(y,d)$ is concave with respect to y.

Moreover, from the construction of $\tilde{\pi}$, if y is an *integer*, then $\tilde{\pi}(y,d)$ is a piece-wise linear interpolation of $\pi(y,d)$ with respect to d. Thus, for fixed integer y, the single-dimensional function $\tilde{\pi}(y,\cdot)$ has at least one integer maximizer, i.e.,

$$Q(y) \ = \ \max_{d\in\mathcal{D}}\pi(y,d) \ = \ \max_{d\in\tilde{\mathcal{D}}}\tilde{\pi}(y,d) \ .$$

The conclusions of the last two paragraphs together imply that $Q(y)$ is concave with respect to y.

EC.4. Proof of Theorem 4

We need the following lemma about the optimal allocation problem which is useful in studying the multiple channel problem.

LEMMA EC.1. *For each $i = 1,2,\ldots,I$, let $f_i(\cdot)$ be a quasi-concave function defined on a set of consecutive integers. Let s_i^* be a maximizer of $f_i(\cdot)$. Then, $f(s) = \max\{\sum_i f_i(s_i) \mid \sum_i s_i = s\}$ is quasi-concave, and achieves its maximum at $\sum_i s_i^*$.*

Proof. First, we provide the proof assuming that the domain of f_i is the set of all integers. Since s_i^* is the maximizer of $f_i(\cdot)$, we have $f_i(s_i) \leq f_i(s_i+1)$ for $s_i < s_i^*$, and $f_i(s_i) \geq f_i(s_i+1)$ for $s_i \geq s_i^*$. Let $s^* = \sum_i s_i^*$.

Suppose $s < s^*$. Then, we claim that there exist $s_1, s_2, \ldots, s_I$ such that $s = \sum_i s_i$ and $f(s) = \sum_i f_i(s_i)$ satisfying $s_i \leq s_i^*$ for each i. To see this claim, suppose that there exists j such that $s_j > s_j^*$ and $s_k < s_k^*$. Then, by decreasing s_j by 1 and increasing s_k by 1, we weakly increase the objective function. By repeating this process, we prove the claim.

Furthermore, there exists i' such that $s_{i'} < s_{i'}^*$. Then,

$$f(s) \;=\; \sum_i f_i(s_i) \;=\; f_{i'}(s_{i'}) \;+\; \sum_{i \neq i'} f_i(s_i) \;\leq\; f_{i'}(s_{i'}+1) \;+\; \sum_{i \neq i'} f_i(s_i) \;\leq\; f(s+1).$$

Similarly, it can be argued that $s > s^*$ implies $f(s) \leq f(s-1)$.

If the domain of f_i is a subset of all integers, extend f_i by defining $f_i(s_i) = -\infty$ for each s_i outside the domain. □

Proof. [Proof of Theorem 4] Let $Q_m(y_m) \;=\; \max_{\mathbf{d}_m} \pi_m(y_m, \mathbf{d}_m)$, and $Q(y) \;=\; \max_{\mathbf{d}} \pi(y, \mathbf{d})$. Since each sales channel m satisfies Condition 2, $Q_m(\cdot)$ is quasi-concave. Let y_m^* be a maximizer of Q_m. Then, by Lemma EC.1, $Q(\cdot)$ is also quasi-concave, and achieves its maximum at $y^* = \sum_m y_m^*$.

Consider two systems $\tilde{\mathcal{A}}$ and $\mathcal{A}$, in which the inventory levels after replenishment are y^1 and y^2, respectively. Assume $y^* \leq y^1 < y^2$. Suppose that in the $\mathcal{A}$ system, the seller chooses an allocation of $\mathbf{y} \;=\; (y_1, \ldots, y_M)$ where $y^2 = \sum_m y_m$, and the sales lever vector of $\mathbf{d} \;=\; (\mathbf{d}_1, \ldots, \mathbf{d}_M)$. For the $\tilde{\mathcal{A}}$ system, we specify the allocation vector $\tilde{\mathbf{y}} = (\tilde{y}_1, \ldots, \tilde{y}_M)$ and the sales lever vector $\tilde{\mathbf{d}} \;=\; (\tilde{\mathbf{d}}_1, \ldots, \tilde{\mathbf{d}}_M)$ such that for each channel m, the pair of $\tilde{y}_m$ and $\tilde{\mathbf{d}}_m$ satisfy part (b) of Condition 2.

From $\sum_m y_m^* = y^* \leq y^1 < y^2 = \sum_m y_m$, there exists an allocation vector $(\tilde{y}_1, \ldots, \tilde{y}_M)$ satisfying $y^1 = \sum_m \tilde{y}_m$, and

$$\begin{aligned} \tilde{y}_m \;&=\; y_m, && \text{if } y_m < y_m^* \\ \tilde{y}_m \;&\in\; \{y_m^*, y_m^*+1, \ldots, y_m\}, && \text{if } y_m \geq y_m^*. \end{aligned}$$

We now construct the sales levers for the $\tilde{\mathcal{A}}$ system. If $\tilde{y}_m = y_m$, set $\tilde{\mathbf{d}}_m = \mathbf{d}_m$, and we get, for every ϵ_m,

$$\tilde{y}_m - D_m(\tilde{\mathbf{d}}_m, \epsilon_m) = y_m - D_m(\mathbf{d}_m, \epsilon_m), \quad \text{and}$$

$$\pi_m(\tilde{y}_m, \tilde{\mathbf{d}}_m) = \pi_m(y_m, \mathbf{d}_m).$$

Otherwise, we have $y_m^* \le \tilde{y}_m < y_m$. By Condition 2, there exists $\tilde{\mathbf{d}}_m$ such that, for any ϵ_m, we have

$$\tilde{y}_m - D_m(\tilde{\mathbf{d}}_m, \epsilon_m) \le y_m - D_m(\mathbf{d}_m, \epsilon_m), \quad \text{and}$$

$$\pi_m(\tilde{y}_m, \tilde{\mathbf{d}}_m) \ge \pi_m(y_m, \mathbf{d}_m)$$

Therefore, for every $\epsilon = (\epsilon_1, \ldots, \epsilon_M)$, it follows

$$y^1 - \sum_m D_m(\tilde{\mathbf{d}}_m, \epsilon_m) \le y^2 - \sum_m D_m(\mathbf{d}_m, \epsilon_m), \quad \text{and}$$
$$\pi(y^1, \tilde{\mathbf{d}}) \; = \; \sum_m \pi_m(\tilde{y}_m, \tilde{\mathbf{d}}_m) \ge \sum_m \pi_m(y_m, \mathbf{d}_m) \; = \; \pi(y^2, \mathbf{d}),$$

satisfying part (b) of Condition 2 for the multiple sales channel model.

Now consider the case $y^* \ge y^1 > y^2$ under Condition 2*. Here, set $\tilde{y}_m = y_m$ if $y_m > y_m^*$; otherwise, let $y_m \le \tilde{y}_m \le y^*$. A similar analysis can be applied. □

EC.5. Proof of Theorem 5

We first prove the result assuming that both y and d_m's are real-valued. Let $(y^*, \mathbf{d}^*) = \arg\max \pi(y, \mathbf{d})$. From the additivity of the demand function, $d_m^* = \arg\max \Lambda_m(d_m)$ where each Λ_m is concave, and y^* is a minimizer of $g(y|\mathbf{d}^*)$ where

$$g(y|\mathbf{d}) = h \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^+ + b \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^- \; .$$

Note that $g(y|\mathbf{d})$ depends on y and $\mathbf{d}$ only through $y - \sum_m d_m$ in a convex manner.

The first part of Condition 1* follows from the concavity of $\max_{\mathbf{d}} \pi(y, \mathbf{d})$ in y. We will now prove part (b). Consider any y^1 and y^2 satisfying $y^* \le y^1 < y^2$, and any $\mathbf{d}^2 = (d_1^2, \ldots, d_M^2)$. Recall $\mathbf{d}^* = \arg\max_{\mathbf{d}} \pi(y^*, \mathbf{d})$ and let $\mathbf{d}^\circ = \arg\max_{\mathbf{d}} \{\pi(y^2, \mathbf{d}) \mid \sum_m d_m = \sum_m d_m^2\}$. By an application of the proof of Lemma EC.1, we can assume, without loss of generality, that we have either (i) $d_m^\circ \le d_m^*$ for all m, or (ii) $d_m^\circ \ge d_m^*$ for all m. If (i) occurs, set $d_m^1 = d_m^*$ for each m. Thus, $d_m^* = d_m^1 \ge d_m^\circ$ for each m, and $y^* - \sum_m d_m^* \le y^1 - \sum_m d_m^1 \le y^2 - \sum_m d_m^\circ$. Otherwise, in case of (ii), define $\lambda = (y^1 - y^*)/(y^2 - y^*)$. Then, set $\mathbf{d}^1 = (d_1^1, d_2^1, \ldots, d_M^1)$ such that

$$\sum_m d_m^1 = (1 - \lambda) \sum_m d_m^* + \lambda \sum_m d_m^\circ$$

$$d^1_m \in \{d^*_m, d^\circ_m\} \qquad \text{for each } m \neq m', \text{ and}$$

$$d^1_{m'} \in [d^*_{m'}, d^\circ_{m'}] \;,$$

for some $m' \in \{1,\dots,M\}$. It follows that $y^1 - \sum_m d^1_m$ is a convex combination of $y^* - \sum_m d^*_m$ and $y^2 - \sum_m d^\circ_m$. Therefore, in both cases, it is straightforward to show

$$\pi(y^1, \mathbf{d}^1) \geq \pi(y^2, \mathbf{d}^\circ) \;\geq\; \pi(y^2, \mathbf{d}^2) \;, \qquad \text{and}$$

$$y^1 - \sum_m D_m(d^1_m, \epsilon_m) \leq \max\left\{ y^2 - \sum_m D_m(d^\circ_m, \epsilon_m),\; y^* - \sum_m D_m(d^*_m, \epsilon_m) \right\},$$

where $\mathbf{d}^1 = (d^1_1, \dots, d^1_M)$. Since $\sum_m D_m(d^\circ_m, \epsilon_m) = \sum_m D_m(d^2_m, \epsilon_m)$, part (b) of Condition 1^* is satisfied.

For part (c), apply a similar analysis to the case of $y^* \geq y^1 > y^2$. If $y^2 - \sum_m D_m(d^2_m, \epsilon_m) \leq y^* - \sum_m D_m(d^*_m, \epsilon_m)$, then choose $\mathbf{d}^1$ similar to the case (ii) above; otherwise, choose $\mathbf{d}^1$ such that $y^2 - \sum_m D_m(d^2_m, \epsilon_m) = y^1 - \sum_m D_m(d^1_m, \epsilon_m)$, and $d^*_m \geq d^1_m \geq d^2_m$ for each channel m.

We remark that if y and d_m's are integer-valued, then probabilistic rounding results in the required conclusions.

EC.6. Proof of Lemma 4

For each t, we define $L_t(\cdot)$ recursively as following:

$$L_t(y_t) = \begin{cases} \max_{d_t} \pi_t(y_t, d_t) + \alpha \cdot E[L_{t+1}(y_t - d_t - \epsilon_t)], & \text{if } t < T. \\ \max_{d_t} \pi_t(y_T, d_T), & \text{if } t = T. \end{cases}$$

By the convexity of π, L_t is convex for each t. Let y^* be the maximizer of L_1.

Suppose $y^* \leq y^1 \leq y^2$. For a fixed sequence of $\epsilon_1, \epsilon_2, \dots, \epsilon_T$, let $\mathcal{A}^*$ be the optimal system starting with the inventory level y^*. Let $d^*_1, d^*_2, \dots, d^*_T$ be the optimal sequence of decisions in $\mathcal{A}^*$. (Clearly, d^*_t depends on $\epsilon_1, \dots, \epsilon_{t-1}$, but we suppress that dependence to simplify notation.) Let y^*_t be the beginning of sub-period inventory level in $\mathcal{A}^*$.

Consider two systems $\tilde{\mathcal{A}}$ and $\mathcal{A}$, and suppose that the inventory levels at the beginning of sub-period $t = 1$ are y^1 and y^2, respectively. Suppose that for fixed $\epsilon_1, \epsilon_2, \dots, \epsilon_T$, the decisions of $\mathcal{A}$ are given by $d^2_1, d^2_2, \dots, d^2_T$. Let $\lambda \in [0,1]$ such that $y^1 = \lambda y^* + (1-\lambda) y^2$. For each t, choose the decision

d_t^1 of the $\tilde{\mathcal{A}}$ system such that $d_t^1 = \lambda d_t^* + (1-\lambda) d_t^2$. Let z_t^*, z_t^1 and z_t^2 be the ending inventories in sub-period t in systems $\mathcal{A}^*$, $\tilde{\mathcal{A}}$ and $\mathcal{A}$, respectively. Thus, if $y_t^1 = \lambda y_t^* + (1-\lambda) y_t^2$, then

$$\begin{aligned} z_t^1 &= y_t^1 - (d_t^1 + \epsilon_t) \\ &= [\lambda y_t^* + (1-\lambda) y_t^2] - [\lambda d_t^* + (1-\lambda) d_t^2 + \epsilon_t] \\ &= \lambda [y_t^* - d_t^* - \epsilon_t] + (1-\lambda)[y_t^2 - d_t^2 - \epsilon_t] \\ &= \lambda z_t^* + (1-\lambda) z_t^2. \end{aligned}$$

By induction, we show the above result for all t. Since $z_T^* \le y^*$, it follows $z_T^1 \le \max\{z_T^2, y^*\}$.

Furthermore, the expected single sub-period profit in t satisfies

$$\pi_t(y_t^1, d_t^1) \ge \lambda \cdot \pi_t(y_t^*, d_t^*) + (1-\lambda) \cdot \pi_t(y_t^2, d_t^2),$$

where expectation is taken over ϵ_t. Thus, the total expected profit in all T sub-periods satisfies

$$\sum_{t=1}^{T} \alpha^{t-1} \pi_t(y_t^1, d_t^1) \ge \lambda \cdot \sum_{t=1}^{T} \alpha^{t-1} \pi_t(y_t^*, d_t^*) + (1-\lambda) \cdot \sum_{t=1}^{T} \alpha^{t-1} \pi_t(y_t^2, d_t^2).$$

Therefore, the total expected profit in the $\tilde{\mathcal{A}}$ system (left-hand side) is at least the total expected profit in the $\mathcal{A}$ system (last term on the right-hand side).