

(s, S) **Optimality in Joint Inventory-Pricing Control:**
An Alternate Approach
 ONLINE APPENDICES

Woonghee Tim Huh, Columbia University
 Ganesh Janakiraman, New York University

A A Special Case: Zero Fixed Cost

In this section, we prove the optimality of base-stock list-price policies in the absence of the fixed ordering cost. This result is similar to Federgruen and Heching (1999).

Theorem A.1. *If Condition 1 holds and $K = 0$, then, the optimal ordering policy for the finite horizon problem or the infinite horizon discounted profit problem has the following structure. If $x_t < y^*$, then order up to y^* , and set the sales lever to \mathbf{d}^* . Otherwise, do not order.*

Proof. By Theorem 6.1 of Porteus (2002), it is easy to verify that (y^*, \mathbf{d}^*) is the optimal action pair if $x_t < y^*$. If $x_t \geq y^*$, it is optimal not to order by Corollary 3.3. \square

Remark. We make the following two remarks on Theorem A.1: (a) If $x_t \leq y^*$, then the myopic policy is optimal; (b) Theorem A.1 does not specify the optimal sales lever \mathbf{d}_t in period t for $x_t > y^*$. This computation can be efficiently performed as follows.

Suppose the horizon T is finite or the profit is discounted, i.e., $\gamma < 1$. Federgruen and Heching (1999) give the following dynamic programming formulation. Recall $U_t(x_t)$ is the optimal value function as a function of the beginning inventory x_t in period t , i.e., $U_t(x_t) = \max_{y_t \geq x_t} V_t(y_t)$. Then,

$$U_t(x_t) = \max_{y_t \geq x_t, \mathbf{d}_t} \pi(y_t, \mathbf{d}_t) + \gamma E[U_{t+1}(\psi(y_t - D_t(\mathbf{d}_t, \epsilon)))]$$

for $t \leq T$, and $U_{T+1}(\cdot) = 0$. This dynamic program is well-defined when T is finite.

Once the starting inventory level x_t falls below y^* , the expected profit in each subsequent period is $Q(y^*)$. Thus, the expected profit (discounted to period t) from period t to T is

$$Q(y^*)(1 + \gamma + \gamma^2 + \dots + \gamma^{T-t}) = Q(y^*) \cdot (1 - \gamma^{T-t+1}) / (1 - \gamma).$$

As a result, we simplify the dynamic programming recursion as follows: for any $t \leq T$,

$$U_t(x_t) = \begin{cases} \max_{\mathbf{d}_t} \pi(x_t, \mathbf{d}_t) + \gamma E[U_{t+1}(\psi(x_t - D_t(\mathbf{d}_t, \epsilon)))] & \text{for } x_t > y^*, \\ Q(y^*) \cdot (1 - \gamma^{T-t+1}) / (1 - \gamma), & \text{for } x_t \leq y^*, \end{cases}$$

and $U_{T+1}(\cdot) = 0$. It is well-defined when T is finite. For the infinite-horizon discounted-profit problem, we let $T \rightarrow \infty$ and substitute U_t with its pointwise limiting function.

We point out the following important differences between the results in this section and the corresponding results in Federgruen and Heching (1999) under the backordering assumption. They require the concavity of the expected single-period profit whereas we require Condition 1, which, for example, is satisfied if π is quasi-concave (see Section 4.1). Their sales lever is single dimensional whereas ours is multi-dimensional. However, our simplification relies upon the stationarity of all the parameters of the system, whereas their finite-horizon results do not. For the lost sales case, Chen et al. (2006) prove this result under some assumptions which imply Condition 1, as we have shown in Section 5. It should be noted that subject to the validity of Condition 1, our result holds for both the backordering and the lost sales cases.

B The Chen et al. (2006) Model: Proof of Proposition 5.1

We introduce the following notation and definitions used in Chen et al. (2006). Let $z := y - d(p)$ be the “riskless” leftover inventory at the end of a period. Let $G(z, p)$ be the expected single-period profit without considering the fixed ordering cost, i.e., $G(z, p) = \pi(z + d(p), p)$.² Let $P(y)$ be the optimal price when the starting inventory level after ordering is y , i.e., $P(y) := \arg \max_p \pi(y, p)$. Thus, $\pi(y, P(y)) = Q(y)$. Let $Z(y) := y - d(P(y))$. It follows that

$$G(Z(y), P(y)) = Q(y) = \pi(y, P(y)). \quad (\text{B.1})$$

Also, let $p(z) := \arg \max_p G(z, p)$. We denote the maximizer of $G(z, p(z))$ by Z . Then, the single-period optimal stocking quantity satisfies $y^* = Z + d(p(Z))$.

Chen et al. (2006) give the following list of properties based on Assumption 3. They are related to the expected single-period profit function.

Fact 1 (Equation (6)).

$$\frac{\partial^2 G(z, p)}{\partial p^2} < 0.$$

Fact 2 (Lemma 1 and Its Corollary). $p(z)$ is continuous on $[0, +\infty)$. Furthermore, $p(z)$ is increasing on $[0, \infty)$.

Fact 3 (Lemma 3). $Z(y)$ is nonnegative and increasing on $[0, +\infty)$.

Fact 4 (Theorem 1). $P(y) \leq p(z)$ for $y > y^*$, and $P(y) \geq p(z)$ for $y < y^*$ where $y = z + d(p(z))$. Furthermore, $G(Z(y), P(y))$ is unimodal on $[0, +\infty)$ and y^* is its maximizer.

Proof of Proposition 5.1

Fact 4 shows the quasi-concavity of $Q(y) = G(Z(y), P(y))$, implying Condition 1 (a).

CASE $y^* \leq y^1 < y^2$. Suppose that a pair of y^1 and y^2 satisfy $y^* \leq y^1 < y^2$, and p^2 is given. Let $z^2 := y^2 - d(p^2)$. By (B.1), Fact 4, and the definition of $P(y)$,

$$\pi(y^1, P(y^1)) = G(Z(y^1), P(y^1)) \geq G(Z(y^2), P(y^2)) = \pi(y^2, P(y^2)) \geq \pi(y^2, p^2).$$

²In this section, p corresponds to the sales lever \mathbf{d} used in earlier sections.

Thus, if $Z(y^1) \leq z^2$ holds, $p^1 = P(y^1)$ satisfies Condition 2 (b), i.e., (2). As a result, we proceed by supposing

$$Z(y^1) > z^2. \quad (\text{B.2})$$

We consider the following two cases: $z^2 + d(p(z^2)) \leq y^1$ and $z^2 + d(p(z^2)) > y^1$.

Case $z^2 + d(p(z^2)) \leq y^1$: Choose p^3 such that $d(p^3) = y^1 - z^2$. (This is possible because $d(\cdot)$ is continuous and $y^1 - z^2$ is bounded below by $d(p(z^2))$ and bounded above by $y^2 - z^2$, which is the same as $d(p^2)$.) Since $z^2 + d(p^3) = y^1$ and $z^2 + d(p^2) = y^2$, it follows $y^1 - d(p^3) = z^2 = y^2 - d(p^2)$. We also obtain that

$$z^2 + d(p(z^2)) \leq y^1 < y^2,$$

which is equivalent to

$$d(p(z^2)) \leq d(p^3) < d(p^2).$$

Therefore we obtain $p(z^2) \geq p^3 > p^2$. Now, the definition of $p(z^2) = \arg \max_p G(z^2, p)$ and Fact 1 imply that $G(z^2, p)$ is increasing in p for $p \leq p(z^2)$. Thus,

$$\pi(y^1, p^3) = G(z^2, p^3) \geq G(z^2, p^2) = \pi(y^2, p^2).$$

Thus, Condition 2 (b) is satisfied with p^3 .

Case $z^2 + d(p(z^2)) > y^1$: If $Z(y^1) \leq y^*$, then Condition 2 (b) holds with $p^1 = P(y^1)$. We proceed by assuming otherwise, i.e., $Z(y^1) > y^*$.

Let $y' := z^2 + d(p(z^2)) > y^1$. Then, $y' > y^1 \geq Z(y^1) > y^*$. By Fact 4, we obtain $P(y') \leq p(z^2)$, which implies

$$z^2 = y' - d(p(z^2)) \geq y' - d(P(y')) = Z(y').$$

Thus, by (B.2), $Z(y^1) > z^2 \geq Z(y')$, which contradicts the monotonicity of Z (Fact 3) since $y^1 < y'$. Thus, this case cannot happen.

CASE $y^* \geq y^1 > y^2$. Consider a pair y^1 and y^2 such that $y^2 < y^1 \leq y^*$. For any p^2 , let $z^2 := y^2 - d(p^2)$. We will show that there exist p^1 such that (i) $z^1 := y^1 - d(p^1) \geq z^2$, and (ii) $G(z^1, p^1) \geq G(z^2, p^2)$.

Recall $Z(y^1) = y^1 - d(P(y^1))$. If $z^2 \leq Z(y^1)$, then $p^1 = P(y^1)$ satisfies (i). Furthermore, the quasi-concavity of Q implies

$$\pi(y^1, P(y^1)) = Q(y^1) \geq Q(y^2) = \pi(y^2, P(y^2)) \geq \pi(y^2, p^2),$$

which shows (ii). Thus, we proceed by assuming

$$Z(y^1) < z^2. \quad (\text{B.3})$$

Let $y' := z^2 + d(p(z^2))$.

Case $y' \geq y^1$: We choose p^1 such that $y^1 = z^2 + d(p^1)$. (Note that such a p^1 can be chosen because $d(p(\cdot))$ is a continuous function, and $y^1 - z^2$ is bounded below by $d(p^2) = y^2 - z^2$ and above by $d(P(y^1)) = y^1 - Z(y^1)$.) Then, $z^2 = y^1 - d(p^1)$ shows (i). Now, note that

$$y^2 = z^2 + d(p^2) < y^1 = z^2 + d(p^1) \leq y' = z^2 + d(p(z^2))$$

implies $p^2 > p^1 \geq p(z^2)$, and $p = p(z^2)$ maximizes $G(z^2, p)$. Thus, Fact 1 shows $\pi(y^1, p^1) = G(z^2, p^1) \geq G(z^2, p^2) = \pi(y^2, p^2)$, verifying (ii).

Case $y' < y^1$: Recall $z^2 + d(p(z^2)) = y'$. Also, by the monotonicity of Z (Fact 3) and (B.3), we have $Z(y') \leq Z(y^1) < z^2$. Thus,

$$d(p(z^2)) = y' - z^2 < y' - Z(y') = d(P(y')) ,$$

implying $p(z^2) > P(y')$. Then, by Fact 4, $y' > y^*$ must hold. However, it contradicts the assumption $y' < y^1 \leq y^*$. Thus, this case cannot happen.

C The Case of Stochastically Increasing, Additive Demands

All the results we have shown in this paper so far assume that all cost parameters and the demand distributions are stationary. In this section, we extend our results to a special class of non-stationary problems. We use the Additive Demand Model in this section. Due to non-stationarity, we will use the time index as a subscript for all parameters.

Theorem C.1. *In the finite-horizon model, suppose the following conditions hold:*

- (a) *Demand is additive.*
- (b) *$\{y_t^*\}$ is an increasing sequence.³*
- (c) *$K_t \geq \gamma \cdot K_{t+1}$ holds for every t .*
- (d) *Either (i) excess demand is backordered and Assumption 2 holds for every t ; or, (ii) excess demand is lost and Assumption 3 holds for every t .⁴*

Then, an (s_t, S_t) policy is optimal in each period t .

Proof. The proof of this theorem is similar to the stationary case presented in the paper, and we provide a high-level sketch of this proof. In the backordering case, an argument similar to Proposition 4.3 shows modified versions of Condition 1 and Condition 2, where y^* is replaced with y_t^* . In the lost sales case, likewise, Proposition 5.1 can be adapted to show the same result. Then, the proof of Theorem 2.2 can be adjusted to show the required result using the modified versions of Condition 1 and Condition 2. \square

³In models with stationary cost parameters, this assumption usually holds when $\{\epsilon_t\}$ is a sequence of stochastically increasing random variables.

⁴In (i), Assumption 2 now involves π_t, ϕ_t^R and ϕ_t^H . In (ii), Assumption 3 now involves $d_t(p), \epsilon_t, B_t, f_t, F_t, r_t, \pi_t, h_t$ and b_t .

References

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