

Inventory Management with Auctions and Other Sales Channels: Optimality of (s, S) Policies*

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Abstract

We study periodic-review inventory replenishment problems with fixed ordering costs, and show the optimality of (s, S) inventory replenishment policies. Inventory replenishment is instantaneous, i.e., the lead time is zero. We consider several sales mechanisms, e.g., auction mechanisms, name-your-own-price mechanisms, and multiple heterogeneous sales channels. We prove this result by showing that these models satisfy a condition given by Huh and Janakiraman (2004), which is sufficient for the optimality of (s, S) policies. Thus, this paper shows that the optimality of (s, S) policies extends well beyond the traditional sales environments studied so far in the inventory literature.

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1 Introduction

In this paper, we study periodic-review inventory replenishment problems with fixed ordering costs when replenishment is instantaneous, i.e., the lead time is zero. In the classical literature on inventory theory, products are assumed to be sold at an exogenously fixed price. The demand distribution in each period is exogenously determined; the manager is assumed to have no control over demand. In contrast, in many business environments, managers routinely make decisions that affect demand and sales quantities by controlling some “sales levers” such as pricing, advertising, and product availability among sales channels. (We call any decision variable that affects the demand distribution a *sales lever*.) The most popular form of demand control studied in the inventory literature is dynamic pricing, where the seller dynamically posts a price that influences the demand distribution. (We refer to this as a *posted price channel*.) In this paper, we consider controlling the following sales levers: (i) the reserve price in periodic auctions, (ii) the minimum acceptable price in “name-your-own-price” channels, and (iii) product allocation and prices in multiple sales channels. We show that the optimal replenishment policy is of the (s, S) type in the above-mentioned models. Thus, this paper demonstrates that the optimality of this simple and popular replenishment policy extends well beyond the traditional sales environments studied so far in the inventory literature.

We briefly discuss the possible rationales for dynamic demand control (e.g., dynamic pricing) for replenishable and non-perishable products. For a stationary demand environment and a sufficiently long planning horizon, the optimal demand control is static over time *when inventory is replenished every period*. This is because the marginal value of a unit of inventory held at the end of a period does not change through time. However, when there is a fixed ordering cost, it is not optimal to order every period; in fact, the time between two successive orders is not a constant and can potentially be large. Therefore, demand control will optimally be dynamic in such models. In this paper, we study the optimal inventory replenishment problem with a fixed ordering cost and dynamic demand control in a variety of examples which we motivate below.

First, consider a seller who has a fixed cost for replenishing inventory and sells through an online auction channel like eBay. Each week, she auctions her inventory using a suitably determined reserve price. Any unsold inventory is carried over to the next week. The seller’s optimal control problem

is to maximize her expected profit over a planning horizon. In each week, she has to decide whether to place a replenishment order or not, and, if so, how much to order; also, she has to determine the reserve price for that week's auction depending on the available inventory. This example motivates section 4.

Our second motivating example is the “name-your-own price” mechanism of *priceline.com*. Although this company currently sells only perishable products like air-tickets and hotel rooms, we believe it is quite plausible for sellers of certain slow-moving, non-perishable products with high ordering costs to adopt this mechanism. See Farwell (2001) for a discussion. Buyers arrive sequentially, and the manager needs to decide the minimum price at which an order will be accepted. This minimum price dynamically changes over time depending on the inventory availability. This example motivates section 5 on bid-price control.

Third, several firms sell the same product in several bricks-and-mortar stores as well as one or more Internet channels. For certain products, replenishment of inventory is fairly infrequent due to the economics involved; however when there is an inventory imbalance across stores caused by uneven sales, inventory can be transferred among the stores, if necessary. We study such a firm in section 6.2. Our fourth example is closely related to the previous example. Many companies maintain a centralized pool of inventory to serve different sales channels, each of which is dedicated to a particular customer segment, e.g., individual home consumer, non-profit organizations such as universities, and for-profit businesses. We consider such a firm in section 6.3, allowing the possibility of dynamic channel-specific pricing.

In many cases, the frequency with which prices can be changed is higher than that at which inventory can be replenished, especially when ordering costs are high or when replenishment schedules are governed by several other constraints. For example, in some retail stores, prices can be updated frequently using electronic shelf labeling systems. This motivates our analysis in section 7.

In sections 4 through 7, we show that (s, S) inventory replenishment policies are optimal for models motivated by the examples described above. In the classical approach, proving the optimality of (s, S) policies is based on a dynamic programming formulation. Its value function exhibits a certain structure, which is shown to be preserved by an inductive argument. When there is no

sales lever, this structure is K -convexity, a concept due to Scarf (1960). Nearly half a century later, Chen and Simchi-Levi (2004a,b) have extended it to “symmetric K -convexity” to establish the optimality of (s, S) policies when the demand distribution in each period is a function of the price set by the manufacturer in that period. A handful of papers show the same result for related models with dynamically posted prices, e.g., Chen et al. (2006); Feng and Chen (2004). A recent paper by Huh and Janakiraman (2004) provides an alternate approach for proving optimality results in stochastic inventory models with multiple sales levers. They have identified a sufficient condition (Condition 1 in Section 3), for the optimality of (s, S) -type replenishment policies. They show that this condition is satisfied by existing models in the literature that combine pricing and inventory decisions. While Huh and Janakiraman (2004) have only considered a single sales channel with dynamically posted prices, we show in this paper that Condition 1 is applicable to more general settings.

The remainder of the paper is organized as follows. In section 2, we present a review of the literature in this area. Section 3 contains the basic model and the main result of Huh and Janakiraman (2004) since their result is essential to this paper. In section 4, we develop an inventory replenishment model in which sales are conducted through auctions. Section 5 contains a model where customers arrive and make price offers which can be accepted or rejected by the seller. Section 6 describes models with multiple sales channels. In section 6.1, we study a model with sequential sales channels, where the first channel is a posted price channel, and the second channel is an auction channel. In sections 6.2 and 6.3, we analyze models in which sales are conducted through multiple channels simultaneously. Section 7 contains a modification of the dynamic pricing model in which the seller has the capability of dynamically changing the price between two successive ordering epochs. We conclude in section 8.

2 Literature Survey

Combined Pricing-Inventory Control. Recently, there has been a growing interest in periodic-review inventory models in which demand is stochastic and influenced by the pricing decision, i.e., demand $D(p, \epsilon)$ is a function of price p , where ϵ is a stochastic component. When the fixed ordering cost of inventory replenishment is zero and excess demand is backordered, Federgruen

and Heching (1999) showed the optimality of the base stock inventory policy. Establishing the structure of optimal inventory policies is more challenging when the fixed ordering cost is strictly positive. Under the assumption of complete backlogging, Chen and Simchi-Levi (2004a,b) showed the optimality of (s, S) inventory policies for finite and infinite horizon models under different assumptions. They used the term (s, S, p) policy, which refers to an (s, S) inventory policy where the price p depends on y , the inventory level after ordering. Under the assumption of lost sales, Chen et al. (2006) showed a similar result using a finite horizon model. For stationary systems, Huh and Janakiraman (2004) used an alternate proof technique to obtain a few extensions: (i) the joint concavity of the expected single-period profit in Chen and Simchi-Levi (2004b) can be replaced by the weaker assumption of joint quasi-concavity, and (ii) the lost sales result of Chen et al. (2006) can be extended to the infinite horizon discounted profit criterion. For more references, please see Huh and Janakiraman (2004).

Auctions and Other Sales Channels. Single-period auctions have been well-studied in the economics literature. We refer readers to a textbook by Krishna (2002) for details. There are several papers that have addressed maximizing the seller's revenue when a finite quantity is sold over a time horizon. Some of them use auctions exclusively (Segev et al. (2001); Lavi and Nisan (2004); Vulcano et al. (2002); Gallien (2006)) while others use a combination of auctions and posted prices (Etzion et al. (2006); Caldentey and Vulcano (2004)). For an excellent review of management science research on online auctions, see Pinker et al. (2003).

The only paper to our knowledge that considers inventory replenishment when the sales channel is not a posted price channel is Van Ryzin and Vulcano (2004). They proved the optimality of a base-stock policy with periodic auctions when there is *no fixed cost* for inventory replenishment. When a positive fixed cost is present and an auction is run every period, there is no known result on the structure of the optimal inventory policies. Furthermore, the optimal inventory replenishment problem has never been studied in the presence of multiple sales channels. In this paper, we address the inventory replenishment problem with a fixed ordering cost and more general types of sales channels.

3 Inventory Replenishment and Sales Mechanisms: A General Model

The main result of this paper is the optimality of (s, S) -type inventory policies in systems with a variety of sales channels. In this section, we review Condition 1 of Huh and Janakiraman (2004) and its implication on the optimality of (s, S) policies. In subsequent sections, we demonstrate that the systems we study satisfy Condition 1.

The basic model is the following. Let c and K be the variable cost and fixed ordering cost of inventory replenishment. Without loss of generality, we assume $c = 0$ by appropriately transforming the holding and back-order cost function and assuming that all inventory at the end of the horizon can be salvaged at c dollars per unit. We assume instantaneous inventory replenishment, i.e., the replenishment lead time is zero. Let x and y be the inventory levels before and after possible replenishment in a period, respectively. Let \mathbf{d} be the vector of sales levers (e.g., price), and $D(\mathbf{d}, \epsilon)$ be the stochastic demand, where ϵ is a random variable. (We denote vectors by a bold-face.) In this paper, the inventory levels (x and y) and demand $D(\mathbf{d}, \epsilon)$ are assumed to be *integers*. When demand exceeds inventory, excess demand is backlogged. The single-period expected profit, excluding inventory replenishment cost, is denoted by $\pi(y, \mathbf{d})$, which depends on both y and \mathbf{d} .

The following sequence of events takes place in each period:

- (1) The available inventory level x at the beginning of a period is observed.
- (2) A replenishment decision is made to raise the inventory to $y \geq x$. A fixed ordering cost of K is incurred if $y > x$.
- (3) A sales lever \mathbf{d} is chosen. Demand is a random variable $D(\mathbf{d}, \epsilon)$ whose distribution is determined by the choice of \mathbf{d} . In each of the models we study, the sales lever and the dependence of demand on it are further described in detail.
- (4) Demand is realized and generates revenue. A holding and back-order cost $g(y - D(\mathbf{d}, \epsilon))$ is incurred. The expected single-period profit $\pi(y, \mathbf{d})$ is the difference between the expected revenue and $E[g(y - D(\mathbf{d}, \epsilon))]$.

An inventory replenishment policy is said to be an (s, S) policy if there exist two numbers s and S such that the inventory level is raised to S whenever $x \leq s$ and no order is placed otherwise. If these two numbers depend on the period index t , then we call this an (s_t, S_t) policy.

We state Condition 1, which ensures that the problem has sufficient structure in terms of the expected single-period profit π . Let $Q(y) := \max_{\mathbf{d}} \pi(y, \mathbf{d})$, and let y^* be a maximizer of $Q(\cdot)$.

Condition 1.

- (a) $Q(y)$ is quasi-concave, i.e., $Q(y)$ is increasing¹ for $y \leq y^*$, and decreasing for $y \geq y^*$, and
- (b) for any y^1 and y^2 satisfying $y^* \leq y^1 < y^2$ and any \mathbf{d}^2 , there exists

$$\mathbf{d}^1 \in \{ \mathbf{d} \mid \pi(y^1, \mathbf{d}) \geq \pi(y^2, \mathbf{d}^2) \}$$

such that for any ϵ ,

$$y^1 - D(\mathbf{d}^1, \epsilon) \leq \max\{ y^2 - D(\mathbf{d}^2, \epsilon), y^* \}. \quad (1)$$

Observe that Condition 1 is stated in terms of a single-period profit function. We also state a stronger version of Condition 1.

Condition 1*. *Condition 1 is satisfied with the following added requirement:*

- (c) for any y^1 and y^2 satisfying $y^* \geq y^1 > y^2$, and any \mathbf{d}^2 , there exists \mathbf{d}^1 satisfying $\pi(y^1, \mathbf{d}^1) \geq \pi(y^2, \mathbf{d}^2)$ such that for any ϵ ,

$$y^1 - D(\mathbf{d}^1, \epsilon) \geq y^2 - D(\mathbf{d}^2, \epsilon) .$$

By definition, Condition 1* implies Condition 1. We now state one of the main results of Huh and Janakiraman (2004):

Theorem 1 (Huh and Janakiraman (2004)). *Suppose Condition 1 holds.*

- (a) *If $K > 0$, (s, S) replenishment policies are optimal with the infinite horizon discounted-profit model. If $K = 0$, the optimal replenishment policy is myopic and of the base-stock type with the finite horizon or the infinite horizon discounted-profit model.*

¹In this paper, we use *increasing* (decreasing) to mean *non-decreasing* (non-increasing).

(b) In addition, if Condition 1* also holds, then an (s_t, S_t) policy is optimal for a finite-horizon model with $K > 0$.

Theorem 1 simplifies the task of proving the optimality of base-stock and (s, S) policies. It is sufficient to focus on a single period and verify Condition 1. Thus, the analysis of a multi-period dynamic program is reduced to examining a simpler single-period problem.

4 Periodic Auctions with Inventory Replenishment and Fixed Costs

In this section, we study systems in which inventory is sold through an auction in every period. The reserve prices used in each period are dynamically chosen based on the inventory levels. In section 4.1, we review the so-called modified second price auction for a single period. In section 4.2, we restrict our attention to this type of auction, and show that Condition 1 is satisfied. In section 4.3, we extend this result to the case where backorders are not allowed. In section 4.4, we discuss an important issue associated with determining the optimal auction to use in each period.

4.1 A Modified Second-Price Auction

Let us now consider a single-period multi-unit auction in which each customer bids for at most a single unit. Suppose there are y units available to be auctioned. If k is the number of units sold in the auction, then the ending inventory is $y - k$ units. Let $S(\cdot)$ be the salvage value obtained as a function of the ending inventory $y - k$. The seller's problem is to design an auction maximizing the expected revenue and the salvage value.

Suppose customer n has the realized value of v_n , and submits a bid of b_n . We assume that customers' values are nonnegative, and have an identical and independent distribution with CDF F and PDF f . We let $\mathbf{v} = (v_1, v_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$ be the *value vector* and the *bid vector*, respectively. Let $b(i)$ be the i 'th highest bid, and $v(i)$ be the i 'th highest value. Ties are broken arbitrarily. When there is no ambiguity, we also use \mathbf{v} and \mathbf{b} to denote $(v(1), v(2), \dots)$ and $(b(1), b(2), \dots)$, respectively.

We use a second price auction with an appropriately chosen vector of non-decreasing reserve prices, which Vulcano et al. (2002) refer to as the *modified second price auction*. In this auction, the seller selects the reserve price vector $\mathbf{r} = (r(1), r(2), \dots)$ where $r(1) \leq r(2) \leq \dots$, and the i 'th unit will be sold if and only if $b(i) \geq r(i)$. Given a particular bid vector \mathbf{b} , the number of sold units is given by $\kappa(\mathbf{r}, \mathbf{b}) = \max\{i : b(i) \geq r(i)\}$. The highest $\kappa(\mathbf{r}, \mathbf{b})$ bidders receive a unit, and pay a uniform price of $\max\{r(\kappa(\mathbf{r}, \mathbf{b})), b(\kappa(\mathbf{r}, \mathbf{b}) + 1)\}$. It is well known that a dominant strategy for buyers is to bid their own values in this auction.

From Maskin and Riley (1989) and Vulcano et al. (2002), the seller's optimization problem is to set the reserve price vector to maximize the expected profit, i.e.,

$$\begin{aligned} \max_{\mathbf{r}} \quad & E_{\mathbf{v}} \left[\sum_{i=1}^y \{J(v(i)) - \Delta S(y - i)\} \cdot I[v(i) \geq r(i)] \right] \\ \text{s. t.} \quad & r(1) \leq r(2) \leq \dots \leq r(y) \end{aligned} \tag{2}$$

where I is a binary indicator function, $\Delta S(x) = S(x + 1) - S(x)$ is the marginal salvage value, and $J(v) = v - 1/\rho(v)$ is the *virtual value* function where $\rho(v) = f(v)/(1 - F(v))$ is the associated failure rate function. We assume, as in Maskin and Riley (1989) and Vulcano et al. (2002), that the virtual value function J is strictly increasing.

4.2 Periodic Auction Sales Channel: Backorders Allowed

In this section, we consider a model in which inventory is sold using an auction in each period. Inventory is reviewed every period and replenished, if necessary. We consider the class of modified second-price auctions with reserve price vectors.

The number of customers and the value vectors are independently and identically distributed across time periods, and each buyer submits his bid only once in the period he arrives. If a customer fails to procure a unit in the period he arrives, he leaves the system. These assumptions are also used by Van Ryzin and Vulcano (2004).

In the *periodic auction* model, the following sequence of events takes place in each period.

- (1) The available inventory level x at the beginning of a period is observed.

- (2) An ordering decision is made to raise the inventory to $y \geq x$. A fixed ordering cost of K is incurred if $y > x$.
- (3) A reserve price vector $\mathbf{r} = (r(1), r(2), \dots)$ is chosen and announced to the bidders, where $r(1) \leq r(2) \leq \dots$.
- (4) Customers arrive and submit their bids. The bid vector $\mathbf{b} = (b(1), b(2), \dots)$ is sorted in a descending order.
- (5) The allocation and payment decisions are made according to the modified second-price auction mechanism. A holding and back-order cost of $g(y - k)$ is incurred, where k is the number of units sold in this period.

By the incentive compatibility property, customers' bids are the same as their values, i.e., $b(i) = v(i)$. Thus, the number of units sold equals

$$\kappa(\mathbf{r}, \mathbf{v}) = \sum_{i=1}^{\infty} I[v(i) \geq r(i)], \quad (3)$$

where I is the indicator function. We suppose $g(\cdot)$ is nonnegative and convex in \mathcal{Z} , and satisfies $g(0) = 0$. Furthermore, we assume that the holding cost is linear, i.e., there exists $h \geq 0$ such that $g(z) = h \cdot z$ for $z \geq 0$. The back-order cost does not have to be linear. Let $\Delta g(z) = g(z) - g(z-1)$.

We note that our model allows the seller to sell more units than the inventory level. This would occur when buyers' bids are sufficiently high. (We analyze the case where backorders are not allowed in section 4.3.)

Let $R(y, \mathbf{r})$ denote the seller's expected single-period profit when the inventory level after replenishment is y and the reserve price vector is \mathbf{r} . The value of $R(y, \mathbf{r})$ includes the holding and back-order cost $g(\cdot)$, but excludes the fixed ordering cost K . By the revenue equivalence theorem, we know that $R(y, \mathbf{r}) = E_{\mathbf{v}} [R(y, \mathbf{r}, \mathbf{v})]$, where

$$R(y, \mathbf{r}, \mathbf{v}) = -g(y) + \sum_{i=1}^{\infty} [J(v(i)) + \Delta g(y - i + 1)] \cdot I[v(i) \geq r(i)]. \quad (4)$$

We recall our assumption that the virtual value function $J(\cdot)$ is a strictly increasing function. It follows that $J(v(i))$ is decreasing in i . Let

$$v^* = J^{-1}(0), \text{ and } v_y^*(i) = J^{-1}(-\Delta g(y - i + 1)). \quad (5)$$

We remark that if the starting inventory is y , then $v_y^*(i)$ is the least value above which it is profitable, in this single period, to sell the i 'th unit to a customer with that value. Let

$$\mathbf{r}_y = (v_y^*(1), v_y^*(2), \dots). \quad (6)$$

Then, the components of \mathbf{r}_y are increasing since $J(\cdot)$ is an increasing function and $g(\cdot)$ is convex. Observe from equation (4) that \mathbf{r}_y maximizes $R(y, \mathbf{r}, \mathbf{v})$ for each \mathbf{v} ; that is, \mathbf{r}_y is the optimal reserve price vector for a single-period problem.

We will now show through Proposition 2 and Lemma 3 that Condition 1 holds in this model. Let $L(y)$ be the maximum possible expected profit in a single period, i.e.,

$$L(y) = \max_{\mathbf{r}} R(y, \mathbf{r}).$$

Let y^* be the largest maximizer of $L(y)$. Let $\Delta L(y) = L(y+1) - L(y)$.

Proposition 2. *In the periodic auction model, the following statements are true:*

- (a) $L(y) = R(y, \mathbf{r}_y)$.
- (b) $L(y)$ is concave.
- (c) $\Delta L(y) \leq 0$ for $y \geq y^*$, and $\Delta L(y) \geq 0$ for $y < y^*$.
- (d) $y^* \geq 0$.

Proof. Since \mathbf{r}_y maximizes $R(y, \mathbf{r}, \mathbf{v})$ for each \mathbf{v} from equation (4), it follows clearly that \mathbf{r}_y maximizes $R(y, \mathbf{r})$, proving (a). The proof of (b) is based on the following property

$$R(y, \mathbf{r}_y, \mathbf{v}) = \max_{k \in \mathcal{Z}^+} \{J(v(1)) + J(v(2)) + \dots + J(v(k)) - g(y - k)\}$$

and is given in appendix A.1. Statement (c) immediately follows from (b). To prove (d), consider the definition (4) of $R(y, \mathbf{r}, \mathbf{v})$:

$$\begin{aligned} R(0, \mathbf{r}, \mathbf{v}) &= -g(0) + \sum_{i=1}^{\infty} [J(v(i)) + \Delta g(1-i)] \cdot I[v(i) \geq r(i)], \quad \text{and} \\ R(-1, \mathbf{r}, \mathbf{v}) &= -g(-1) + \sum_{i=1}^{\infty} [J(v(i)) + \Delta g(-i)] \cdot I[v(i) \geq r(i)]. \end{aligned}$$

For any integer z , we have $g(z) \geq 0 = g(0)$. By the convexity of $g(\cdot)$, we have $\Delta g(-i+1) \geq \Delta g(-i)$. It follows that $R(0, \mathbf{r}, \mathbf{v}) \geq R(-1, \mathbf{r}, \mathbf{v})$ for any \mathbf{r} and \mathbf{v} . Thus,

$$R(-1, \mathbf{r}_{-1}) = E_{\mathbf{v}}[R(-1, \mathbf{r}_{-1}, \mathbf{v})] \leq E_{\mathbf{v}}[R(0, \mathbf{r}_{-1}, \mathbf{v})] = R(0, \mathbf{r}_{-1}) \leq R(0, \mathbf{r}_0).$$

Therefore, $\Delta L(-1) = L(0) - L(-1) \geq 0$ from (a). Since y^* is the largest maximizer of $L(\cdot)$, we get $y^* \geq 0$. \square

From (5) and (6), we recall $\mathbf{r}_y = (v_y^*(1), v_y^*(2), \dots)$ where $v_y^*(i) = J^{-1}(-\Delta g(y - i + 1))$.

Lemma 3. *For $y^2 > y^*$, consider a system \mathcal{A} with y^2 units of inventory after replenishment and the reserve price vector $\mathbf{r} = (r(1), r(2), \dots)$. For $y^1 \in [y^*, y^2]$, let $\tilde{\mathcal{A}}$ be another system with y^1 units of inventory after replenishment and the reserve price vector $\tilde{\mathbf{r}} = (\tilde{r}(1), \tilde{r}(2), \dots)$, where*

$$\tilde{r}(i) = \begin{cases} \min\{v_{y^1}^*(i), r(i)\}, & \text{if } 1 \leq i \leq y^1 \\ v_{y^1}^*(i), & \text{if } i \geq y^1 + 1. \end{cases}$$

For a given realization of \mathbf{v} , let $z^1(\mathbf{v})$ and $z^2(\mathbf{v})$ denote the ending inventories in systems $\tilde{\mathcal{A}}$ and \mathcal{A} , respectively. Then, the single-period expected profits satisfy $R(y^1, \tilde{\mathbf{r}}) \geq R(y^2, \mathbf{r})$, and

$$z^1(\mathbf{v}) \leq \max\{0, z^2(\mathbf{v})\} \leq \max\{y^*, z^2(\mathbf{v})\}. \quad (7)$$

Proof. Since \mathbf{r} is a valid reserve price vector, $r(i)$ is increasing in i . Since $v_y^*(i)$ is increasing in i for any fixed y , it follows that the components of $\tilde{\mathbf{r}}$ are increasing.

For any given \mathbf{v} , we recall that $\kappa(\tilde{\mathbf{r}}, \mathbf{v})$ and $\kappa(\mathbf{r}, \mathbf{v})$ are the number of units sold in $\tilde{\mathcal{A}}$ and \mathcal{A} , respectively. We show (7) by considering two disjoint cases. In the first case, suppose $\kappa(\mathbf{r}, \mathbf{v}) \leq y^1$, i.e., $z^1(\mathbf{v}) \geq 0$. By the construction of \tilde{r} , we have $\tilde{r}(i) \leq r(i)$ for each $i = 1, 2, \dots, y^1$, implying $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \geq \kappa(\mathbf{r}, \mathbf{v})$. Thus,

$$z^1(\mathbf{v}) = y^1 - \kappa(\tilde{\mathbf{r}}, \mathbf{v}) \leq y^2 - \kappa(\mathbf{r}, \mathbf{v}) = z^2(\mathbf{v})$$

Thus, we obtain (7) in this case. In the second case, we have $\kappa(\mathbf{r}, \mathbf{v}) > y^1$. It follows $v(y^1) \geq r(y^1) \geq \tilde{r}(y^1)$. Therefore, we have $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \geq y^1$, and $z^1(\mathbf{v}) \leq 0$. Now (7) follows from the fact that $y^* \geq 0$ (Proposition 2).

Next, we proceed to show $R(y^1, \tilde{\mathbf{r}}) \geq R(y^2, \mathbf{r})$. We recall that $\kappa(\mathbf{r}_{y^1}, \mathbf{v})$ and $\kappa(\mathbf{r}_{y^2}, \mathbf{v})$ are the number of units sold when the reserve price vectors are \mathbf{r}_{y^1} and \mathbf{r}_{y^2} , respectively. We claim that for any \mathbf{v} ,

$$R(y^2, \mathbf{r}, \mathbf{v}) - R(y^1, \tilde{\mathbf{r}}, \mathbf{v}) \leq R(y^2, \mathbf{r}_{y^2}, \mathbf{v}) - R(y^1, \mathbf{r}_{y^1}, \mathbf{v}). \quad (8)$$

The proof of this claim, given in appendix A.2, is not straightforward and is based on analyzing multiple cases. Then, it follows that

$$\begin{aligned} R(y^2, \mathbf{r}) - R(y^1, \tilde{\mathbf{r}}) &= E_{\mathbf{v}} [R(y^2, \mathbf{r}, \mathbf{v}) - R(y^1, \tilde{\mathbf{r}}, \mathbf{v})] \\ &\leq E_{\mathbf{v}} [R(y^2, \mathbf{r}_{y^2}, \mathbf{v}) - R(y^1, \mathbf{r}_{y^1}, \mathbf{v})] \\ &= L(y^2) - L(y^1), \end{aligned}$$

where the last equality comes from Proposition 2. Since $y_2 > y_1 \geq y^*$, we have $L(y^1) \geq L(y^2)$, from which we obtain $R(y^1, \tilde{\mathbf{r}}) \geq R(y^2, \mathbf{r})$. \square

Lemma 3 establishes that Condition 1 holds for the periodic auction sales channel model presented in this section. Thus we obtain the conclusion of Theorem 1 (a), in particular the optimality of (s, S) policies for the infinite-horizon discounted-profit model.

Theorem 4. *The optimal inventory replenishment policy for the infinite-horizon discounted-profit model is of the (s, S) type under the following assumptions: (i) the class of auctions under consideration is the class of modified second-price auctions with increasing reserve price vectors, and (ii) the number of customers and the value vectors are independently and identically distributed across time periods.*

Unfortunately, we are unable to verify Condition 1* for this model, which prevents us from establishing the finite-horizon analogue of Theorem 4.

4.3 Periodic Auction Sales Channel: Backorders Not Allowed

In section 4.2, the seller was allowed to sell more units than the inventory level and backlog excess demand. In this section, we assume that the seller cannot sell any more units than the inventory level after replenishment in each period.

When back-orders are not allowed, the analysis of section 4.2 carries over by setting an arbitrarily high cost for back-orders, i.e., $g(z) = \infty$ for $z < 0$. (It is useful to define $\Delta g(z) = -\infty$ for $z \leq 0$.) Thus, the seller does not want to backlog any demand, i.e., $r(i) = \infty$ for any $i > y$, where y is the inventory level after replenishment. Thus, (s, S) replenishment policies are guaranteed to be optimal when backorders are not allowed. Moreover, in this case, a slightly stronger version of Lemma 3 holds. This result will be useful later in the analysis of multiple channels in sections 6.2 and 6.3.

Lemma 5. *Under the assumptions and definitions in Lemma 3, if no back-orders are allowed, the conclusions of Lemma 3 hold. Moreover, we have*

$$z^1(\mathbf{v}) \leq z^2(\mathbf{v}) .$$

Proof. Since back-orders are not allowed, the reserve price vector $\tilde{\mathbf{r}} = (\tilde{r}(1), \tilde{r}(2), \dots)$ in Lemma 3 is given by

$$\tilde{r}(i) = \begin{cases} \min\{v_{y^1}^*(i), r(i)\}, & \text{if } 1 \leq i \leq y^1 \\ \infty, & \text{if } i \geq y^1 + 1. \end{cases}$$

We consider two separate cases. If $\kappa(\mathbf{r}, \mathbf{v}) \leq y^1$, then the definition of $\tilde{\mathbf{r}}$ implies that $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \geq \kappa(\mathbf{r}, \mathbf{v})$. Thus, we have $y^1 - \kappa(\tilde{\mathbf{r}}, \mathbf{v}) \leq y^2 - \kappa(\mathbf{r}, \mathbf{v})$, and the required inequality holds. Otherwise, if $\kappa(\mathbf{r}, \mathbf{v}) > y^1$, it follows $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) = y^1$. Since back-orders are not allowed, we must have $z^1(\mathbf{v}) = 0 \leq z^2(\mathbf{v})$. \square

In this section, we have so far used a non-decreasing reserve price vector. We now comment on the modified second-price auction with a uniform reserve price. Let us suppose that the seller is restricted to set the same reserve price for each of the y units, and does not sell more than y units. Consider Lemma 5. Suppose that for some r , the reserve price vector \mathbf{r} satisfies $r(i) = r$ for $i \leq y^2$, and $r(i) = \infty$ for $i > y^2$. Let $\tilde{r} = \min\{J^{-1}(-h), r\}$. Then, since $v_{y^1}^*(i) = J^{-1}(-h)$ for each $i \leq y^1$, the construction of $\tilde{\mathbf{r}}$ satisfies $\tilde{r}(i) = \tilde{r}$ for $i \leq y^1$, and $\tilde{r}(i) = \infty$ for $i > y^1$. In other words, if \mathbf{r} represents a uniform reserve price, then $\tilde{\mathbf{r}}$ also represents a uniform reserve price. Thus, when the reserve price vector is restricted to a uniform price, Condition 1 still holds, as in Lemma 5.

We remark that our analysis also holds for the modified *first-price* auction (pay-your-bid auction) as in Van Ryzin and Vulcano (2004).

4.4 Optimal Auctions and Non-Concave Salvage Value Functions

In this section, we revisit the single-period auction discussed in Section 4.1, and explain why we restrict our attention to the modified second-price auction as opposed to an optimal auction. The reader may skip this section without affecting the understanding of the remainder of the paper.

When the salvage value function $S(\cdot)$ is concave, Vulcano et al. (2002) have shown that the modified second price auction is indeed an optimal auction. However, with a fixed ordering cost K , the profit-to-go function from the dynamic program is typically not concave. (Even in the traditional inventory models where demands are exogenous, the profit-to-go function fails to be concave.) Consequently, we are interested in the design of an optimal auction in the case where $S(\cdot)$ is not necessarily concave.

Maskin and Riley (1989) have shown that the expected revenue of the seller is determined by the allocation, a result known as the *revenue equivalence theorem*. Let $\dot{q}_n(\mathbf{v})$ be the binary variable indicating whether a unit is allocated to customer n given a value vector \mathbf{v} . Recall $J(v) = v - 1/\rho(v)$ is the *virtual value* function where $\rho(v) = f(v)/(1 - F(v))$ is the associated failure rate function. Then, the expected seller's revenue is given by

$$E_{\mathbf{v}} \left[\sum_n J(v_n) \cdot \dot{q}_n(\mathbf{v}) \right], \quad (9)$$

provided that (i) the bidder with $v_n = 0$ has an expected equilibrium profit of 0, and (ii) $\dot{q}_n(\mathbf{v}) \leq \dot{q}_n(\mathbf{v}')$ whenever \mathbf{v} and \mathbf{v}' are identical except $v_n < v'_n$.

Let $q_i(\mathbf{v})$ be the indicator that the customer with the i 'th highest value is awarded a unit. Recall that $\Delta S(x) = S(x + 1) - S(x)$ is the marginal salvage value. Given that y units are available for auction, it can be shown using (9) that the seller's expected profit is

$$E_{\mathbf{v}} \left[\sum_{i=1}^y \{J(v(i)) - \Delta S(y - i)\} \cdot q_i(\mathbf{v}) \right]$$

plus an additive constant, $S(y)$. Assume, as in Maskin and Riley (1989) and Vulcano et al. (2002), that the virtual value function J is an increasing function. Therefore, if we sell k units to the bidders

with the k highest values, the expected marginal profit from the i 'th unit is $J(v(i)) - \Delta S(y - i)$. If S is zero as in Maskin and Riley (1989), or concave as in Vulcano et al. (2002), the expected marginal profit is decreasing in i . Therefore, the modified second price auction with the reserve price vector $(J^{-1}(\Delta S(y - 1)), J^{-1}(\Delta S(y - 2)), \dots)$ is an optimal mechanism.

However, if $S(\cdot)$ is not concave, then $J^{-1}(\Delta S(y - i))$ may not be increasing in i , and it is not clear if there is any reserve price vector with which the modified second-price auction is an optimal mechanism. This is because it is possible that the i 'th unit will be sold even if the marginal profit $J(v(i)) - \Delta S(y - i)$ is negative since the marginal profit from subsequent units will be substantially high. In other words, a marginal loss from one unit may be offset by the marginal profits from selling the subsequent units. To our knowledge, the optimal auction with a non-concave salvage value function (or equivalently, a non-convex production cost function) is not known. This is an interesting research question in its own right. Also, it appears that even if such an optimal auction can be found, it would be a complex one. Consequently, we have restricted our attention to the modified second-price auction of Vulcano et al. (2002).

5 Inventory Replenishment with Bid-Price Control

Consider a *bid-price control* model, also known as a name-your-own-price model. Suppose that in each period, the seller observes the available inventory x , and has an option of raising it to any level $y \geq x$; a set up cost of K is incurred if y is strictly greater than x . In each period, there is at most one customer arrival, and the arrival probability is q . (This could be treated as a discrete-time approximation of a Poisson arrival model.) The customer submits a bid of v for a single unit, and we assume that the customer bids are independently and identically distributed as $F(\cdot)$. The seller decides whether or not to accept this bid. Since the bids are independent, it is optimal to accept the bid if it exceeds a threshold that depends on the inventory level after ordering. An accepted order is back-ordered if inventory is not available. If z is the inventory level at the end of the period, a holding and back-order cost of $g(z)$ is incurred, where $g(z) = h \cdot [z]^+ + b \cdot [z]^-$ for some nonnegative constants h and b . This model is motivated by the name-your-own-price practice of priceline.com.

We will now demonstrate that this model satisfies Condition 1*.

Lemma 6. *In the bid-price control model described in this section, Condition 1* is satisfied, with the stronger property $y^1 - D(\mathbf{d}^1, \epsilon) \leq y^2 - D(\mathbf{d}^2, \epsilon)$ in lieu of (1). Thus, the conclusions of Theorem 1 (a) and (b) hold.*

Proof. When the inventory level is y , the maximum expected single-period profit, excluding the ordering cost, is given by

$$Q(y) = -g(y) + \max_r (E[v|v \geq r] + \Delta g(y)) \cdot (1 - F(r)).$$

We claim that $Q(\cdot)$ is quasi-concave. To see this, let

$$\begin{aligned} \phi_1 &= \max_r (E[v|v \geq r] - b) \cdot (1 - F(r)) \quad \text{and} \\ \phi_2 &= \max_r (E[v|v \geq r] + h) \cdot (1 - F(r)). \end{aligned}$$

Note that both ϕ_1 and ϕ_2 are independent of y . It follows that

$$Q(y) = \begin{cases} b \cdot y + \phi_1 & \text{if } y \leq 0, \\ -h \cdot y + \phi_2 & \text{if } y \geq 1. \end{cases}$$

Thus, Q is increasing for $y \leq 0$, and decreasing for $y \geq 1$. Thus, Q is quasi-concave on the set of integers, proving the claim.

Furthermore, suppose $y^* \leq y^1 < y^2$. Consider two systems $\tilde{\mathcal{A}}$ and \mathcal{A} , in which the inventory levels after replenishment are y^1 and y^2 , respectively. For the $\tilde{\mathcal{A}}$ system, let \tilde{r} be the maximizer in the expression of ϕ_1 if $y^1 \leq 0$, and the maximizer in ϕ_2 if $y^1 \geq 1$. Thus, $\tilde{\mathcal{A}}$ system achieves the maximum expected single-period profit at an inventory level of y^1 , which exceeds the maximum expected profit of \mathcal{A} (by the quasi-concavity of Q). Furthermore, the ending inventory of the $\tilde{\mathcal{A}}$ system is at most the ending inventory of the \mathcal{A} -system since the demand is at most one.

We apply a similar argument when $y^* \geq y^1 > y^2$ to show that the ending inventory of $\tilde{\mathcal{A}}$ is greater than or equal to the ending inventory in \mathcal{A} , and $\tilde{\mathcal{A}}$ achieves a higher expected profit. \square

The above lemma can be extended to allow for Markov-modulated arrival probabilities.

6 Inventory Replenishment and Allocation in Multiple Sales Channels

In this section, we study the optimal inventory policy when there are multiple sales channels. We present a formal definition of a sales channel. A *sales channel* $\mathcal{M} = (D, \Lambda)$ consists of a demand function, D , and a revenue function, Λ :

- $D(\mathbf{d}, \epsilon)$ is stochastic demand as a function of the sales lever \mathbf{d} , where ϵ is a random variable.
- $\Lambda(\mathbf{d})$ is the expected single-period revenue when the sales lever is \mathbf{d} .

We provide a few examples of sales channels. The first example is the dynamic pricing channel with *additive demand* which is common in the literature. Here, the sales lever is the expected demand d , which belongs to $\{d_L, d_L + 1, \dots, d_U\}$, where d_L and d_U are integers. Then,

$$\begin{aligned} D(d, \epsilon) &= d + \epsilon \quad \text{and} \\ \Lambda(d) &= p(d) \cdot d, \end{aligned}$$

where ϵ is an integer random variable with $E[\epsilon] = 0$, and $p(d)$ is the price corresponding to d .

The second example is the auction model of section 4, where the sales lever is the reserve price vector. Let $\mathbf{V}(\epsilon) = (V(1, \epsilon), V(2, \epsilon), \dots)$ be the sorted value vector parameterized by a random variable ϵ , and let \mathbf{r} be a reserve price vector. Here,

$$\begin{aligned} D(\mathbf{r}, \epsilon) &= \kappa(\mathbf{r}, \mathbf{V}(\epsilon)) \quad \text{and} \\ \Lambda(\mathbf{r}) &= E \left[\sum_{i=1}^{D(\mathbf{r}, \epsilon)} J(V(i, \epsilon)) \right], \end{aligned}$$

where $\kappa(\cdot, \cdot)$ is the number of units sold as defined in (3).

In the next three subsections, we will present three different models of systems with multiple sales channels. These models differ with respect to whether inventory is shared or pre-allocated among channels, and whether the channels are operated simultaneously or sequentially.

6.1 Posted Pricing and Auctioning: A Sequential Model

In this section, we consider a sequential dual sales channel system in which the first channel is a posted price channel and the second is an auction channel. In each period, inventory is first allocated to the posted price channel, and then the remaining inventory is sent to the auction channel. One may interpret the posted price channel as a primary channel and the auction channel as a secondary channel. This section is motivated by recent papers on online auctions such as Budish and Takeyama (2001), Matthews (2003), Etzion et al. (2006), Van Ryzin and Vulcano (2004), and Caldentey and Vulcano (2004).

In the *posted pricing and auction dual channel* model, the following sequence of events takes place in each period:

- (1) The available inventory level x at the beginning of a period is observed.
- (2) A replenishment decision is made to raise the inventory to $y \geq x$. A fixed ordering cost of K is incurred if $y > x$.
- (3) The sales lever d (“demand level”) for the posted price channel is chosen, where $d \in \mathcal{D}$ where $\mathcal{D} = \{d_L, d_{L+1}, \dots, d_U\}$ for some integers d_L and d_U . Then, demand $D(d, \epsilon_P) = d + \epsilon_P$ is realized, where ϵ_P is an integer random variable with $E[\epsilon_P] = 0$. We require $\epsilon_P \geq -d_L$ with probability one so that $D(d, \epsilon_P)$ is always nonnegative. Let $d_P = D(d, \epsilon_P)$ be the realized posted price channel demand. (We call this model the *discrete additive demand model*.)
- (4) The reserve price vector $\mathbf{s} = (s(y-d_P-1), s(y-d_P-2), \dots, s(0), s(-1), \dots)$ for the auction channel is chosen, where the components of \mathbf{s} are nondecreasing. Thus, $s(y-d_P-1)$ is the reserve price for the first unit sold through the auction channel, $s(y-d_P-2)$ is the reserve price for the second unit, and so on. Then, the customer value vector $\mathbf{v} = (v(1), v(2), v(3), \dots)$ is realized. The number of units sold through the auction equals

$$k(\mathbf{s}, \mathbf{v}, y, d_P) = \sum_{i=d_P+1}^{\infty} I[v(i-d_P) \geq s(y-i)].$$

- (5) Holding and back-order costs are charged based on the ending inventory level $y - D(d, \epsilon_P) - k(\mathbf{s}, \mathbf{v}, y, d_P)$. We assume a linear holding cost and a convex back-order cost function, as in section 4.

The first d_P units are sold through the posted price channel. For $i > d_P$, $s(y - i)$ acts as the reserve price for the $(i - d_P)$ 'th unit in the auction channel, i.e., the $(i - d_P)$ 'th is sold if and only if $v(i - d_P) \geq s(y - i)$. We assume a separate stream of customers for each sales channel. Within the posted price channel, the demand distribution in each period depends only on the sales level d of that period. Similarly, in the auction channel, the number of bidders and their valuations are independent and identically distributed over time.

The expected revenue from the posted price channel is $p(d) \cdot d$, which we assume to be concave in d . Let $\tau = y - d_P$ be the inventory level between steps (3) and (4), i.e., immediately after demand from the posted price channel is satisfied. Let $R_A(\tau, \mathbf{s})$ be the expected single-period profit from the auction channel, similar to R in section 4. The value of $R_A(\tau, \mathbf{s})$ includes the holding and back-order cost. Let $L_A(\tau)$ be the maximum expected single-period profit from the auction channel, i.e., $L_A(\tau) = \max_{\mathbf{s}} R_A(\tau, \mathbf{s})$. This $L_A(\cdot)$ corresponds to $L(\cdot)$ in section 4. The maximum value of $L_A(\cdot)$ is achieved when the reserve price vector \mathbf{s} satisfies $s(y - i) = v_y^*(i)$, i.e.,

$$(s(y - 1), s(y - 2), \dots) = (v_y^*(1), v_y^*(2), \dots) = \mathbf{r}_y,$$

where we recall $v_y^*(i) = J^{-1}(-\Delta g(y - i + 1))$. It follows that the maximum total expected profit from both channels in a single period is

$$Q(y) = \max_{d \in \mathcal{D}} p(d) \cdot d + E[L_A(y - d - \epsilon_P)].$$

Proposition 7. *$Q(y)$ is a concave function.*

Proof. The proof is based on constructing a continuous-space interpolation of the single period profit function with respect to y and d . The details are available in Appendix A.3. \square

Theorem 8. *The sequential list-price and auction channel model described in this section satisfies Condition 1, and the conclusion of Theorem 1 (a) holds.*

Proof. We have already shown the concavity of Q . Let τ^* be the maximizer of $E[L_A(\tau - \epsilon_P)]$ over τ , and let d^* be the maximizer of $p(d) \cdot d$ over d . It is straightforward to observe that $Q(\cdot)$ is maximized at $y^* = d^* + \tau^*$.

Suppose y^1 and y^2 satisfy $y^* \leq y^1 < y^2$. Consider two systems $\tilde{\mathcal{A}}$ and \mathcal{A} , in which the inventory levels after replenishment are y^1 and y^2 , respectively. Let (d, \mathbf{s}) be the pair of posted price sales lever and reserve price vector used in \mathcal{A} . We show how to choose $(\tilde{d}, \tilde{\mathbf{s}})$ for the $\tilde{\mathcal{A}}$ system to satisfy the second part of Condition 1.

Let $\tau^2 = y^2 - d$. Set $\tau^1 = \min\{y^1 - d^*, \tau^2\}$ and $\tilde{d} = y^1 - \tau^1$. We consider two disjoint cases. In the first case, we assume $\tau^1 = \tau^2$. Then, we must have $d^* \leq \tilde{d} \leq d$, which implies $p(\tilde{d}) \cdot \tilde{d} \geq p(d) \cdot d$ by the concavity of revenue from the posted price channel. We set $\tilde{\mathbf{s}} = \mathbf{s}$, and it follows that

$$E[R_A(\tau^1 - \epsilon_P, \tilde{\mathbf{s}})] = E[R_A(\tau^2 - \epsilon_P, \mathbf{s})].$$

Furthermore, the ending inventories of $\tilde{\mathcal{A}}$ and \mathcal{A} systems are the same.

In the second case, we have $\tau^1 = y^1 - d^* < \tau^2$, and $\tilde{d} = d^*$. Thus, $p(\tilde{d}) \cdot \tilde{d} = p(d^*) \cdot d^* \geq p(d) \cdot d$. Furthermore, we have $\tau^* \leq \tau^1 < \tau^2$. Then, we set $\tilde{\mathbf{s}} = (\tilde{s}(y^1 - 1), \tilde{s}(y^1 - 2), \dots)$ such that

$$\tilde{s}(y^1 - i) = \begin{cases} \min \{ J^{-1}(-\Delta g(y^1 - i)), s(y^1 - i + (\tau^2 - \tau^1)) \}, & \text{if } i \leq y^1; \\ J^{-1}(-\Delta g(y^1 - i)), & \text{if } i > y^1. \end{cases}$$

Following an argument similar to the proof of Lemma 3, it can be shown that (i) $\tilde{\mathcal{A}}$ has an expected single-period profit no less than \mathcal{A} , and (ii) for every realization of ϵ_P , the ending inventory of $\tilde{\mathcal{A}}$ is either negative or at most the ending inventory of \mathcal{A} . \square

REMARK. Consider a special case of the model presented here, when $d_L = d_U$, that is, the sales lever d in the posted price channel is pre-determined. Here, the first channel is a primary channel representing a long-term supply contract at a fixed price, and the auction channel is a secondary channel.

6.2 Multiple Sales Channels with Inventory Allocation

Recently, there has been some research on the simultaneous sales of a product through multiple sales channels, e.g., Etzion et al. (2006); Caldentey and Vulcano (2004). These papers model demand such that customers arrive in a single stream, observe all channels, and decide which channel to purchase from. However, there are also situations where the customers may be aware of only one sales channel, or prefer only one channel because of reasons such as geographic proximity.

For example, an Amazon Marketplace customer may not be aware that the same product is sold through another sales channel, or may not be comfortable purchasing through that channel. In this section, we study such a multiple sales channel model with independent demands. Furthermore, we assume that the existing aggregate inventory is pre-allocated among multiple sales channels at the beginning of each period. Several firms sell the same product in several bricks-and-mortar stores as well as one or more Internet channels, and while the replenishment of inventory might be infrequent, inventory can be redistributed among the channels more frequently, if necessary. (The case of inventory allocation after demand realization is discussed in the next section 6.3.)

We suppose that there are M sales channels indexed by $m = 1, 2, \dots, M$. In each period, the inventory level after possible replenishment is rationed among the M sales channels, and the seller sets the sales lever for each channel. The demand distribution faced by each channel is determined by its sales lever, and independent of one another. The following sequence of events takes place in each period. After the available inventory level x at the beginning of a period is observed, the manager raises the inventory to $y \geq x$ units, possibly incurring a fixed ordering cost of K if $y > x$. Then, the inventory is allocated among sales channels. We denote by y_m the amount of inventory allocated to channel m ; thus, $\sum_m y_m = y$. For each channel m , a sales lever \mathbf{d}_m is chosen. The expected single-period revenue from channel m is $\Lambda_m(\mathbf{d}_m)$. Then, demand $D_m(\mathbf{d}_m, \epsilon_m)$ is realized from for each channel, and a holding cost of $H(z^h)$ and back-order cost of $B(z^b)$ are incurred, where $z_m = y_m - D_m(\mathbf{d}_m, \epsilon_m)$ and

$$z^h = \sum_m [z_m]^+ \quad \text{and} \quad z^b = \sum_m [z_m]^-.$$

We assume that both the holding cost and the back-order cost are linear, i.e., $H(z^h) = h \cdot z^h$ and $B(z^b) = b \cdot z^b$ for some nonnegative constants h and b . In the next period, inventory, if any, should first be allocated towards back-orders. The net inventory of the entire system at the beginning of the next period is $z = z^h - z^b$. *Note that inventory can be shared among channels before demand is realized, but not after it is realized.*

Let $g(z) = h \cdot [z]^+ + b \cdot [z]^-$. Then, $H(z^h) + B(z^b) = \sum_m g(z_m)$. If we charge the holding and back-order cost $g(z_m)$ to sales channel m , the expected profit of sales channel m is given by

$$\pi_m(y_m, \mathbf{d}_m) = \Lambda_m(\mathbf{d}_m) - E[g(y_m - D_m(\mathbf{d}_m, \epsilon_m))]. \quad (10)$$

The seller's expected profit is the sum of expected profits in all sales channels. Let $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_M)$. For fixed y and \mathbf{d} , the seller's maximum system-wide expected profit is

$$\pi(y, \mathbf{d}) = \max \left\{ \sum_m \pi_m(y_m, \mathbf{d}_m) \mid \sum_m y_m = y \right\}.$$

We proceed by assuming that the distribution $D_m(\mathbf{d}_m, \epsilon_m)$ is discrete, and its support is a set of consecutive integers for each m .

To state our results in the remainder of section 6, it is useful to formalize the following notions.

Condition 2. *Condition 1 is satisfied where (1) is replaced by $y^1 - D(\mathbf{d}^1, \epsilon) \leq y^2 - D(\mathbf{d}^2, \epsilon)$.*

Condition 2*. *Condition 1* is satisfied where (1) is replaced by $y^1 - D(\mathbf{d}^1, \epsilon) \leq y^2 - D(\mathbf{d}^2, \epsilon)$.*

It is easy to observe that Condition 2* implies Condition 1* and Condition 2, each of which implies Condition 1. We remark that the periodic auction channel without back orders (section 4.3) and the bid-price control model (section 5) satisfy Condition 2 and 2*, respectively; see Lemmas 5 and 6.

Theorem 9. *Consider the multiple sales channel model with inventory allocation. Suppose that the support of demand in each channel is a set of consecutive integers. If each channel satisfies Condition 2, then the multiple sales channel model also satisfies Condition 2, implying Theorem 1 (a). If each channel satisfies Condition 2*, then the multiple sales channel model also satisfies Condition 2*, implying Theorem 1 (a) and (b).*

Proof. The complete proof can be found in Appendix A.4. We sketch the main ideas here. Suppose $y^* \leq y^1 < y^2$. For any choice of inventory allocation for the y^2 -system, we choose the allocation of the y^1 system such that the inventory level for each channel is either the same as the corresponding channel in the y^2 system, or lies in the interval bounded below by the optimal inventory level for the channel and above by the inventory level allocated by the y^2 -system. Choose the sales lever for each channel according Condition 2 or Condition 2*. Condition 2 implies Condition 1, implying the conclusion of Theorem 1 (a); Condition 2* implies Condition 1*, implying the conclusion of Theorem 1 (a) and (b). Then, the required results can be verified. \square

We remark that these sales channels need not be identical. For example, one sales channel may be a channel with dynamically posted prices such as the discrete additive demand model. Another sales channel may be the periodic auction channel described in section 4.3.

6.3 Multiple Posted Price Channels with Inventory Sharing

In the multiple channel model of section 6.2, the inventory at the beginning of each period is allocated among sales channels before demand is realized, and the underage cost is incurred if demand in a channel exceeds inventory allocated to that channel. In this section, we consider a model in which inventory is shared among channels, and the underage cost is incurred only if the total demand in all channels exceeds the inventory level. We focus on posted price channels.

Suppose there are M sales channels, each of which is a list-pricing model with additive discrete demand, i.e., demand in channel m is given by $D_m(d_m, \epsilon_m) = d_m + \epsilon_m$. The expected single-period revenue from channel m is $\Lambda_m(d_m)$ as a function of the sales lever d_m , which is assumed to be a concave function of d_m . The sequence of events is similar to section 6.2 with a few differences. First, the total inventory level y is *not* allocated among sales channels. Second, the holding and back-order cost is given by $g(z) = h \cdot [z]^+ + b \cdot [z]^-$ where $z = y - \sum_m D_m(d_m, \epsilon_m)$. We refer to this model as the *multiple posted price channel model with inventory sharing*.

Let $\mathbf{d} = (d_1, d_2, \dots, d_M)$. The expected profit in a single period, as a function of the after-replenishment inventory level y , is given by

$$\begin{aligned} \pi(y, \mathbf{d}) &= \sum_m \Lambda_m(d_m) - E[g(y - \sum_m D_m(d_m, \epsilon_m))] \\ &= \sum_m \Lambda_m(d_m) - h \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^+ - b \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^- , \end{aligned}$$

where the expectation is taken over $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_M)$. Thus, it is straightforward to show that π is jointly concave in (y, \mathbf{d}) . It follows $\max_{\mathbf{d}} \pi(y, \mathbf{d})$ is concave in y . Let y^* be its maximizer.

Theorem 10. *The multiple posted price channel model with inventory sharing satisfies Condition 1*, and the conclusions of Theorem 1 (a) and (b) hold.*

Proof. See Appendix A.5. □

7 Dynamic Pricing with Limited Ordering Opportunities

Suppose now that inventory can be replenished in a posted price channel every $T \geq 2$ periods. For example, inventory replenishment is allowed at the beginning of every week, whereas prices can be changed every day. Alternatively, we assume that inventory is replenished in every period, and a single period is divided into T sub-periods, in which prices are set and sales take place. We assume additive demands here.

The following sequence of events takes place in each period, where each period is defined by an ordering opportunity:

- (1) The available inventory level x at the beginning of a period is observed.
- (2) A replenishment decision is made to raise the inventory to $y \geq x$. A fixed ordering cost of K is incurred if $y > x$.
- (3) For each sub-period $t = 1, 2, \dots, T$, let y_t be the inventory level at the beginning of sub-period t . A sales level d_t is chosen, and demand is a random variable given by $D_t(d_t, \epsilon_t) = d_t + \epsilon_t$. An appropriate revenue is realized, and the holding and back-order cost is incurred. We assume full backlogging, and $\alpha \in [0, 1]$ is the discount factor per sub-period.

We suppose that the expected single sub-period profit $\pi_t(y_t, d_t)$, the difference between the revenue and the holding and back-order cost, is jointly concave. In the next lemma, we verify that Condition 1 holds for each period. (Note that we show this result for each *period*, not each sub-period.)

Lemma 11. *In the posted price model with limited ordering opportunities described in this section, Condition 1 is satisfied, and the conclusion of Theorem 1 (a) holds.*

Proof. See Appendix A.6. □

8 Conclusions and Extensions

Inventory managers have long adopted (s, S) policies for inventory replenishment when there is a non-trivial cost or effort associated with placing an order. Inventory researchers have been interested in demonstrating that this class of policies is optimal under certain conditions since the seminal work of Scarf (1960). For about five decades, research in this area was limited to models in which demands cannot be influenced by managerial decisions. In the last few years, starting with Chen and Simchi-Levi (2004a,b), however, progress has been made in establishing the optimality of (s, S) policies in models where demands are influenced by price, which is a control variable. In this paper, we have expanded the scope of sales environments in which (s, S) policies can be shown to be optimal by studying two increasingly popular settings: (a) auctions conducted on a periodic basis and (b) simultaneous sales through multiple channels, for example, physical retail outlets and online retail sites or separate channels for different categories of customers. In addition, we have shown this result for name-your-own-price channels and channels in which prices can be changed more frequently than inventory can be replenished.

We now highlight the modeling assumptions that we needed to prove our main results and also discuss their impact. In our study of periodically conducted auctions, we assumed that a new pool of customers becomes available in each period; in particular, we assume that customers who have participated in an auction unsuccessfully in one period do not participate in subsequent auctions. Similarly, when studying multiple sales channels operating simultaneously, we assumed that each channel has a dedicated customer pool, i.e., customers cannot choose between channels. While this assumption holds in cases where the different channels are made exclusively available to different customer segments, for example household customers and business customers, there are many situations where it does not hold; for example, price-sensitive customers and bargain hunters choose between physical retail outlets and retail websites. In the presence of such customer behavior, the combination of sales levers chosen in these channels will influence the distribution of demand among these channels. It is an open question whether (s, S) policies will continue to be optimal in such an environment. While the primary focus of this paper is the replenishment strategy, we believe that there is a value in understanding how bidding data in the auction case or sales and price data in the multiple channels case can be used to update pricing strategies. Finally,

it is important to develop computational approaches to determine the optimal (s, S) parameters and pricing strategies for the models studied here.

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A Appendix

A.1 Concavity of $L(\cdot)$ in Proposition 2

In this section, we prove that the maximum single-period profit $L(y)$ is concave in y . This result is one of the components of Proposition 2.

We note $y \in \mathcal{Z}$. By part (a) of Proposition 2 and the definition of $R(y, \mathbf{r})$, it follows that

$$L(y) = R(y, \mathbf{r}_y) = E_{\mathbf{v}}[R(y, \mathbf{r}_y, \mathbf{v})]. \quad (11)$$

Now, we fix the the vector \mathbf{v} . From (4),

$$R(y, \mathbf{r}_y, \mathbf{v}) = \max_{k \in \mathcal{Z}^+} \{J(v(1)) + J(v(2)) + \dots + J(v(k)) - g(y - k)\}.$$

For each nonnegative integer k , we let

$$\bar{J}(k, \mathbf{v}) = \begin{cases} 0, & \text{if } k = 0; \\ J(v(1)) + J(v(2)) + \dots + J(v(k)), & \text{if } k \in \{1, 2, \dots\}. \end{cases}$$

Observe that the marginal difference $\bar{J}(k+1, \mathbf{v})$ and $\bar{J}(k, \mathbf{v})$ is $J(v(k))$; the sequence $\{J(v(k))\}$ is decreasing because $J(\cdot)$ is an increasing function and $\{v(k)\}$ is a decreasing sequence. Therefore, $\bar{J}(k, \mathbf{v})$ is concave in k for every \mathbf{v} .

We extend the definition of $\bar{J}(k, \mathbf{v})$ to \mathfrak{R}^+ as follows: for $k \in \mathfrak{R}^+$, let

$$\bar{J}(k, \mathbf{v}) = ([k] - k) \cdot \bar{J}([k], \mathbf{v}) + (k - [k]) \cdot \bar{J}([k], \mathbf{v}).$$

That is, for any fixed \mathbf{v} , $\bar{J}(k, \mathbf{v})$ is a piecewise linear function of k with slope changes only at integers. Let $\tilde{g}(\cdot)$ be a piecewise-linear extension of $g(\cdot)$ to \mathfrak{R} . We also define, for $y \in \mathfrak{R}$ and $k \in \mathfrak{R}^+$,

$$\varphi(y, k, \mathbf{v}) = \bar{J}(k, \mathbf{v}) - \tilde{g}(y - k). \quad (12)$$

Thus, for each integer y and value vector \mathbf{v} , we have

$$R(y, \mathbf{r}_y, \mathbf{v}) = \max_{k \in \mathcal{Z}^+} \varphi(y, k, \mathbf{v}). \quad (13)$$

We claim that for any fixed \mathbf{v} , $\varphi(y, k, \mathbf{v})$ is jointly concave in (y, k) in $\mathfrak{R} \times \mathfrak{R}^+$. We know that $\bar{J}(\cdot, \mathbf{v})$ is concave, and $\tilde{g}(\cdot)$ is convex. Since $y - k$ is a linear function of (y, k) , equation (12) implies $\varphi(y, k, \mathbf{v})$ is jointly concave in (y, k) .

Thus, for any fixed \mathbf{v} and integer y , $\varphi(y, k, \mathbf{v})$ is concave in $k \in \mathfrak{R}^+$. Since $\varphi(y, k, \mathbf{v})$ is piecewise linear in k with slope changes only at integer points, there exists an integer value of k that maximizes $\varphi(y, k, \mathbf{v})$. Thus,

$$\max_{k \in \mathcal{Z}^+} \varphi(y, k, \mathbf{v}) = \max_{k \in \mathfrak{R}^+} \varphi(y, k, \mathbf{v}). \quad (14)$$

Since $\varphi(y, k, \mathbf{v})$ is a jointly concave function in $(y, k) \in \mathfrak{R} \times \mathfrak{R}^+$, $\max_{k \in \mathfrak{R}^+} \varphi(y, k, \mathbf{v})$ is concave with respect to y in \mathfrak{R} . Thus, $\max_{k \in \mathfrak{R}^+} \varphi(y, k, \mathbf{v})$ is also concave, in the discrete sense, with respect to y in \mathcal{Z} . It follows from (13) and (14) that $R(y, \mathbf{r}_y, \mathbf{v})$ is concave with respect to y in \mathcal{Z} . From (11), we conclude $L(y)$ is concave in y .

A.2 Proof of (8) in Lemma 3

In this section, we provide the proof of claim (8). We fix any \mathbf{v} , and show

$$R(y^2, \mathbf{r}, \mathbf{v}) - R(y^1, \tilde{\mathbf{r}}, \mathbf{v}) \leq R(y^2, \mathbf{r}_{y^2}, \mathbf{v}) - R(y^1, \mathbf{r}_{y^1}, \mathbf{v}).$$

From the definition of $\tilde{\mathbf{r}}$, we get $\tilde{r}(i) \leq v_{y^1}^*(i)$ for $i \leq y^1$, and $\tilde{r}(i) = v_{y^1}^*(i)$ for $i > y^1$. Thus, $\kappa(\mathbf{r}_{y^1}, \mathbf{v}) \leq \kappa(\tilde{\mathbf{r}}, \mathbf{v})$. We consider two disjoint cases.

CASE A: $\kappa(\mathbf{r}_{y^1}, \mathbf{v}) = \kappa(\tilde{\mathbf{r}}, \mathbf{v})$.

First, we compare $R(y^1, \mathbf{r}_{y^1}, \mathbf{v})$ and $R(y^1, \tilde{\mathbf{r}}, \mathbf{v})$. Now, $I[v(i) \geq v_{y^1}^*(i)] = I[v(i) \geq \tilde{r}(i)]$ for each i , by the definition of κ . Then,

$$\begin{aligned} & [J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq v_{y^1}^*(i)] \\ &= [J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)], \end{aligned}$$

implying $R(y^1, \mathbf{r}_{y^1}, \mathbf{v}) = R(y^1, \tilde{\mathbf{r}}, \mathbf{v})$ by (4). Also, we compare $R(y^2, \mathbf{r}_{y^2}, \mathbf{v})$ and $R(y^2, \mathbf{r}, \mathbf{v})$. Observe from equation (4) that \mathbf{r}_{y^2} maximizes $R(y^2, \mathbf{r}, \mathbf{v})$ for each \mathbf{v} . Therefore, we obtain $R(y^2, \mathbf{r}_{y^2}, \mathbf{v}) \geq R(y^2, \mathbf{r}, \mathbf{v})$. Thus, combining these two results, we obtain claim (8).

CASE B: $\kappa(\mathbf{r}_{y^1}, \mathbf{v}) < \kappa(\tilde{\mathbf{r}}, \mathbf{v})$.

Since $\tilde{r}(i) = v_{y^1}^*(i)$ holds for $i > y^1$, by the definition of $\tilde{\mathbf{r}}$, it follows that

$$\kappa(\mathbf{r}_{y^1}, \mathbf{v}) < \kappa(\tilde{\mathbf{r}}, \mathbf{v}) \leq y^1.$$

From equation (4), we observe that if the following pair of inequalities holds for each positive integer i , then (8) holds.

$$[J(v(i)) + \Delta g(y^2 - i + 1)] \cdot I[v(i) \geq v_{y^2}^*(i)] \geq [J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq v_{y^1}^*(i)] \quad (15)$$

and

$$[J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)] \geq [J(v(i)) + \Delta g(y^2 - i + 1)] \cdot I[v(i) \geq r(i)]. \quad (16)$$

We first show (15). For each i ,

$$\begin{aligned} & [J(v(i)) + \Delta g(y^2 - i + 1)] \cdot I[v(i) \geq v_{y^2}^*(i)] \\ &= [J(v(i)) + \Delta g(y^2 - i + 1)]^+ \\ &\geq [J(v(i)) + \Delta g(y^1 - i + 1)]^+ \\ &= [J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq v_{y^1}^*(i)]. \end{aligned}$$

The equalities above follow from the definition of $v_{y^2}^*$ and $v_{y^1}^*$. The inequality follows from the convexity of g .

Now, we show (16) by considering two subcases.

- *Subcase B1:* $i \in \{1, 2, \dots, y^1\}$.

Since $i \leq y^1$ and $\Delta g(z) = h$ for all $z > 0$, observe from the definition of $v_{y^1}^*$ that $v_{y^1}^*(i) = J^{-1}(-h)$ holds. In this case, $\tilde{r}(i)$ is either $r(i)$ or $v_{y^1}^*(i)$.

If $\tilde{r}(i) = v_{y^1}^*(i)$, then

$$\begin{aligned} & [J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)] \\ &= [J(v(i)) + h] \cdot I[v(i) \geq v_{y^1}^*(i)] \quad (\text{since } \tilde{r}(i) = v_{y^1}^*(i) \text{ and } y^1 - i + 1 > 0) \\ &= [J(v(i)) + h]^+ \quad (\text{since } v_{y^1}^*(i) = J^{-1}(-h)) \\ &\geq [J(v(i)) + h] \cdot I[v(i) \geq r(i)] \quad (\text{by a property of } [\cdot]^+) \\ &= [J(v(i)) + \Delta g(y^2 - i + 1)] \cdot I[v(i) \geq r(i)] \quad (\text{since } y^2 - i + 1 > 0). \end{aligned}$$

Otherwise, $\tilde{r}(i) = r(i)$ holds. Then,

$$[J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)] = [J(v(i)) + \Delta g(y^2 - i + 1)] \cdot I[v(i) \geq r(i)],$$

since $\Delta g(y^1 - i + 1) = h = \Delta g(y^2 - i + 1)$.

- *Subcase B2: $i > y^1$.*

Since $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \leq y^1$ holds in Case B, it follows $i > \kappa(\tilde{\mathbf{r}}, \mathbf{v})$. Thus, $I[v(i) \geq \tilde{r}(i)] = 0$, implying

$$[J(v(i)) + \Delta g(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)] = 0.$$

Furthermore, since $i > y^1 > \kappa(\mathbf{r}_{y^1}, \mathbf{v})$, we get $i \geq \kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1$. Thus, $v(i) \leq v(\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1)$.

$$\begin{aligned} J(v(i)) + \Delta g(y^2 - i + 1) &\leq J(v(i)) + h \\ &\leq J(v(\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1)) + h \\ &\leq 0. \end{aligned}$$

The first inequality follows from the convexity of g whose derivative is bounded above by h . The second inequality follows from the monotonicity of J . For the last inequality, observe that the virtual value from the $(\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1)$ 'th "highest" customer is smaller than $-h$, the holding cost savings from selling that unit.

Thus, we complete the proof of (16) for both subcases B1 and B2. This completes the proof of (8) in Lemma 3.

A.3 Proof of Proposition 7

Let $\tilde{\mathcal{D}}$ be the compact interval $[d_L, d_U]$. Thus, \mathcal{D} is the set of integers in $\tilde{\mathcal{D}}$. For any fixed integer y and $d \in \mathcal{D}$, define $\pi(y, d) = \pi^1(d) + \pi^2(y - d)$, where

$$\begin{aligned} \pi^1(d) &= p(d) \cdot d, & \text{and} \\ \pi^2(r) &= E[L_A(r - \epsilon_P)]. \end{aligned}$$

Now, for any integer y and real $d \in \tilde{\mathcal{D}}$, define

$$\tilde{\pi}^1(d) = (1 - \lambda) \cdot \tilde{\pi}^1(\lfloor d \rfloor) + \lambda \cdot \tilde{\pi}^1(\lceil d \rceil)$$

where $\lambda = d - \lfloor d \rfloor$. Similarly, for any real r , define

$$\tilde{\pi}^2(r) = (1 - \lambda) \cdot \tilde{\pi}^2(\lfloor r \rfloor) + \lambda \cdot \tilde{\pi}^2(\lceil r \rceil)$$

where $\lambda = r - \lfloor r \rfloor$. For any real y and $d \in \tilde{\mathcal{D}}$, let

$$\tilde{\pi}(y, d) = \tilde{\pi}^1(d) + \tilde{\pi}^2(y - d).$$

Since the expected revenue from the posted price channel, $\pi^1(d) = p(d) \cdot d$, is assumed to be concave with respect to $d \in \mathcal{D}$, its linear interpolation $\tilde{\pi}^1(d)$ is also concave with respect to $d \in \tilde{\mathcal{D}}$. The concavity of $L_A(t)$, the maximum expected single-period profit from the auction channel, is proved in Proposition 2 of section 4. Thus, $\tilde{\pi}(y, d)$ is jointly concave with respect to y and d . Thus, $\max_{d \in \tilde{\mathcal{D}}} \tilde{\pi}(y, d)$ is concave with respect to y .

Moreover, from the construction of $\tilde{\pi}$, if y is an *integer*, then $\tilde{\pi}(y, d)$ is a piece-wise linear interpolation of $\pi(y, d)$ with respect to d . Thus, for fixed integer y , the single-dimensional function $\tilde{\pi}(y, \cdot)$ has at least one integer maximizer, i.e.,

$$Q(y) = \max_{d \in \mathcal{D}} \pi(y, d) = \max_{d \in \tilde{\mathcal{D}}} \tilde{\pi}(y, d).$$

The conclusions of the last two paragraphs together imply that $Q(y)$ is concave with respect to y .

A.4 Proof of Theorem 9

We need the following lemma about the optimal allocation problem which is useful in studying the multiple channel problem.

Lemma 12. *For each $i = 1, 2, \dots, I$, let $f_i(\cdot)$ be a quasi-concave function defined on a set of consecutive integers. Let s_i^* be a maximizer of $f_i(\cdot)$. Then, $f(s) = \max\{\sum_i f_i(s_i) \mid \sum_i s_i = s\}$ is quasi-concave, and achieves its maximum at $\sum_i s_i^*$.*

Proof. First, we provide the proof assuming that the domain of f_i is the set of all integers. Since s_i^* is the maximizer of $f_i(\cdot)$, we have $f_i(s_i) \leq f_i(s_i + 1)$ for $s_i < s_i^*$, and $f_i(s_i) \geq f_i(s_i + 1)$ for $s_i \geq s_i^*$. Let $s^* = \sum_i s_i^*$.

Suppose $s < s^*$. Then, we claim that there exist s_1, s_2, \dots, s_I such that $s = \sum_i s_i$ and $f(s) = \sum_i f_i(s_i)$ satisfying $s_i \leq s_i^*$ for each i . To see this claim, suppose that there exists j such that

$s_j > s_j^*$ and $s_k < s_k^*$. Then, by decreasing s_j by 1 and increasing s_k by 1, we weakly increase the objective function. By repeating this process, we prove the claim.

Furthermore, there exists i' such that $s_{i'} < s_{i'}^*$. Then,

$$f(s) = \sum_i f_i(s_i) = f_{i'}(s_{i'}) + \sum_{i \neq i'} f_i(s_i) \leq f_{i'}(s_{i'} + 1) + \sum_{i \neq i'} f_i(s_i) \leq f(s + 1).$$

Similarly, it can be argued that $s > s^*$ implies $f(s) \leq f(s - 1)$.

If the domain of f_i is a subset of all integers, extend f_i by defining $f_i(s_i) = -\infty$ for each s_i outside the domain. \square

Proof of Theorem 9. Let $Q_m(y_m) = \max_{\mathbf{d}_m} \pi_m(y_m, \mathbf{d}_m)$, and $Q(y) = \max_{\mathbf{d}} \pi(y, \mathbf{d})$. Since each sales channel m satisfies Condition 2, $Q_m(\cdot)$ is quasi-concave. Let y_m^* be a maximizer of Q_m . Then, by Lemma 12, $Q(\cdot)$ is also quasi-concave, and achieves its maximum at $y^* = \sum_m y_m^*$.

Consider two systems $\tilde{\mathcal{A}}$ and \mathcal{A} , in which the inventory levels after replenishment are y^1 and y^2 , respectively. Assume $y^* \leq y^1 < y^2$. Suppose that in the \mathcal{A} system, the seller chooses an allocation of $\mathbf{y} = (y_1, \dots, y_M)$ where $y^2 = \sum_m y_m$, and the sales lever vector of $\mathbf{d} = (\mathbf{d}_1, \dots, \mathbf{d}_M)$. For the $\tilde{\mathcal{A}}$ system, we specify the allocation vector $\tilde{\mathbf{y}} = (\tilde{y}_1, \dots, \tilde{y}_M)$ and the sales lever vector $\tilde{\mathbf{d}} = (\tilde{\mathbf{d}}_1, \dots, \tilde{\mathbf{d}}_M)$ such that for each channel m , the pair of \tilde{y}_m and $\tilde{\mathbf{d}}_m$ satisfy part (b) of Condition 2.

From $\sum_m y_m^* = y^* \leq y^1 < y^2 = \sum_m y_m$, there exists an allocation vector $(\tilde{y}_1, \dots, \tilde{y}_M)$ satisfying $y^1 = \sum_m \tilde{y}_m$, and

$$\begin{aligned} \tilde{y}_m &= y_m, & \text{if } y_m < y_m^* \\ \tilde{y}_m &\in \{y_m^*, y_m^* + 1, \dots, y_m\}, & \text{if } y_m \geq y_m^*. \end{aligned}$$

We now construct the sales levers for the $\tilde{\mathcal{A}}$ system. If $\tilde{y}_m = y_m$, set $\tilde{\mathbf{d}}_m = \mathbf{d}_m$, and we get, for every ϵ_m ,

$$\begin{aligned} \tilde{y}_m - D_m(\tilde{\mathbf{d}}_m, \epsilon_m) &= y_m - D_m(\mathbf{d}_m, \epsilon_m), \quad \text{and} \\ \pi_m(\tilde{y}_m, \tilde{\mathbf{d}}_m) &= \pi_m(y_m, \mathbf{d}_m). \end{aligned}$$

Otherwise, we have $y_m^* \leq \tilde{y}_m < y_m$. By Condition 2, there exists $\tilde{\mathbf{d}}_m$ such that, for any ϵ_m , we have

$$\begin{aligned}\tilde{y}_m - D_m(\tilde{\mathbf{d}}_m, \epsilon_m) &\leq y_m - D_m(\mathbf{d}_m, \epsilon_m), \quad \text{and} \\ \pi_m(\tilde{y}_m, \tilde{\mathbf{d}}_m) &\geq \pi_m(y_m, \mathbf{d}_m)\end{aligned}$$

Therefore, for every $\epsilon = (\epsilon_1, \dots, \epsilon_M)$, it follows

$$\begin{aligned}y^1 - \sum_m D_m(\tilde{\mathbf{d}}_m, \epsilon_m) &\leq y^2 - \sum_m D_m(\mathbf{d}_m, \epsilon_m), \quad \text{and} \\ \pi(y^1, \tilde{\mathbf{d}}) &= \sum_m \pi_m(\tilde{y}_m, \tilde{\mathbf{d}}_m) \geq \sum_m \pi_m(y_m, \mathbf{d}_m) = \pi(y^2, \mathbf{d}),\end{aligned}$$

satisfying part (b) of Condition 2 for the multiple sales channel model.

Now consider the case $y^* \geq y^1 > y^2$ under Condition 2*. Here, set $\tilde{y}_m = y_m$ if $y_m > y_m^*$; otherwise, let $y_m \leq \tilde{y}_m \leq y^*$. A similar analysis can be applied. \square

A.5 Proof of Theorem 10

We first prove the result assuming that both y and d_m 's are real-valued. Let $(y^*, \mathbf{d}^*) = \arg \max \pi(y, \mathbf{d})$. From the additivity of the demand function, $d_m^* = \arg \max \Lambda_m(d_m)$ where each Λ_m is concave, and y^* is a minimizer of $g(y|\mathbf{d}^*)$ where

$$g(y|\mathbf{d}) = h \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^+ + b \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^-.$$

Note that $g(y|\mathbf{d})$ depends on y and \mathbf{d} only through $y - \sum_m d_m$ in a convex manner.

The first part of Condition 1* follows from the concavity of $\max_{\mathbf{d}} \pi(y, \mathbf{d})$ in y . We will now prove part (b). Consider any y^1 and y^2 satisfying $y^* \leq y^1 < y^2$, and any $\mathbf{d}^2 = (d_1^2, \dots, d_M^2)$. Recall $\mathbf{d}^* = \arg \max_{\mathbf{d}} \pi(y^*, \mathbf{d})$ and let $\mathbf{d}^\circ = \arg \max_{\mathbf{d}} \{\pi(y^2, \mathbf{d}) \mid \sum_m d_m = \sum_m d_m^2\}$. By an application of the proof of Lemma 12, we can assume, without loss of generality, that we have either (i) $d_m^\circ \leq d_m^*$ for all m , or (ii) $d_m^\circ \geq d_m^*$ for all m . If (i) occurs, set $d_m^1 = d_m^*$ for each m . Thus, $d_m^* = d_m^1 \geq d_m^\circ$ for each m , and $y^* - \sum_m d_m^* \leq y^1 - \sum_m d_m^1 \leq y^2 - \sum_m d_m^\circ$. Otherwise, in case of (ii), define $\lambda = (y^1 - y^*) / (y^2 - y^*)$. Then, set $\mathbf{d}^1 = (d_1^1, d_2^1, \dots, d_M^1)$ such that

$$\begin{aligned}\sum_m d_m^1 &= (1 - \lambda) \sum_m d_m^* + \lambda \sum_m d_m^\circ \\ d_m^1 &\in \{d_m^*, d_m^\circ\} \quad \text{for each } m \neq m', \text{ and} \\ d_{m'}^1 &\in [d_{m'}^*, d_{m'}^\circ],\end{aligned}$$

for some $m' \in \{1, \dots, M\}$. It follows that $y^1 - \sum_m d_m^1$ is a convex combination of $y^* - \sum_m d_m^*$ and $y^2 - \sum_m d_m^c$. Therefore, in both cases, it is straightforward to show

$$\begin{aligned} \pi(y^1, \mathbf{d}^1) &\geq \pi(y^2, \mathbf{d}^c) \geq \pi(y^2, \mathbf{d}^2), \quad \text{and} \\ y^1 - \sum_m D_m(d_m^1, \epsilon_m) &\leq \max \left\{ y^2 - \sum_m D_m(d_m^c, \epsilon_m), y^* - \sum_m D_m(d_m^*, \epsilon_m) \right\}, \end{aligned}$$

where $\mathbf{d}^1 = (d_1^1, \dots, d_M^1)$. Since $\sum_m D_m(d_m^c, \epsilon_m) = \sum_m D_m(d_m^2, \epsilon_m)$, part (b) of Condition 1* is satisfied.

For part (c), apply a similar analysis to the case of $y^* \geq y^1 > y^2$. If $y^2 - \sum_m D_m(d_m^2, \epsilon_m) \leq y^* - \sum_m D_m(d_m^*, \epsilon_m)$, then choose \mathbf{d}^1 similar to the case (ii) above; otherwise, choose \mathbf{d}^1 such that $y^2 - \sum_m D_m(d_m^2, \epsilon_m) = y^1 - \sum_m D_m(d_m^1, \epsilon_m)$, and $d_m^* \geq d_m^1 \geq d_m^2$ for each channel m .

We remark that if y and d_m 's are integer-valued, then probabilistic rounding results in the required conclusions.

A.6 Proof of Lemma 11

For each t , we define $L_t(\cdot)$ recursively as following:

$$L_t(y_t) = \begin{cases} \max_{d_t} \pi_t(y_t, d_t) + \alpha \cdot E[L_{t+1}(y_t - d_t - \epsilon_t)], & \text{if } t < T. \\ \max_{d_t} \pi_t(y_T, d_T), & \text{if } t = T. \end{cases}$$

By the convexity of π , L_t is convex for each t . Let y^* be the maximizer of L_1 .

Suppose $y^* \leq y^1 \leq y^2$. For a fixed sequence of $\epsilon_1, \epsilon_2, \dots, \epsilon_T$, let \mathcal{A}^* be the optimal system starting with the inventory level y^* . Let $d_1^*, d_2^*, \dots, d_T^*$ be the optimal sequence of decisions in \mathcal{A}^* . (Clearly, d_t^* depends on $\epsilon_1, \dots, \epsilon_{t-1}$, but we suppress that dependence to simplify notation.) Let y_t^* be the beginning of sub-period inventory level in \mathcal{A}^* .

Consider two systems $\tilde{\mathcal{A}}$ and \mathcal{A} , and suppose that the inventory levels at the beginning of sub-period $t = 1$ are y^1 and y^2 , respectively. Suppose that for fixed $\epsilon_1, \epsilon_2, \dots, \epsilon_T$, the decisions of \mathcal{A} are given by $d_1^2, d_2^2, \dots, d_T^2$. Let $\lambda \in [0, 1]$ such that $y^1 = \lambda y^* + (1 - \lambda)y^2$. For each t , choose the decision d_t^1 of the $\tilde{\mathcal{A}}$ system such that $d_t^1 = \lambda d_t^* + (1 - \lambda)d_t^2$. Let z_t^* , z_t^1 and z_t^2 be the ending inventories in

sub-period t in systems \mathcal{A}^* , $\tilde{\mathcal{A}}$ and \mathcal{A} , respectively. Thus, if $y_t^1 = \lambda y_t^* + (1 - \lambda)y_t^2$, then

$$\begin{aligned}
z_t^1 &= y_t^1 - (d_t^1 + \epsilon_t) \\
&= [\lambda y_t^* + (1 - \lambda)y_t^2] - [\lambda d_t^* + (1 - \lambda)d_t^2 + \epsilon_t] \\
&= \lambda[y_t^* - d_t^* - \epsilon_t] + (1 - \lambda)[y_t^2 - d_t^2 - \epsilon_t] \\
&= \lambda z_t^* + (1 - \lambda)z_t^2.
\end{aligned}$$

By induction, we show the above result for all t . Since $z_T^* \leq y^*$, it follows $z_T^1 \leq \max\{z_T^2, y^*\}$.

Furthermore, the expected single sub-period profit in t satisfies

$$\pi_t(y_t^1, d_t^1) \geq \lambda \cdot \pi_t(y_t^*, d_t^*) + (1 - \lambda) \cdot \pi_t(y_t^2, d_t^2),$$

where expectation is taken over ϵ_t . Thus, the total expected profit in all T sub-periods satisfies

$$\sum_{t=1}^T \alpha^{t-1} \pi_t(y_t^1, d_t^1) \geq \lambda \cdot \sum_{t=1}^T \alpha^{t-1} \pi_t(y_t^*, d_t^*) + (1 - \lambda) \cdot \sum_{t=1}^T \alpha^{t-1} \pi_t(y_t^2, d_t^2).$$

Therefore, the total expected profit in the $\tilde{\mathcal{A}}$ system (left-hand side) is at least the total expected profit in the \mathcal{A} system (last term on the right-hand side).