Numerical Study for Diffusion in Material Mixtures: 
The Treatment of Mixed Material Cells

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In the modeling of hydrodynamics on an Eulerian mesh, the interface between different material types can be advected in a manner such that a cell on this mesh may contain multiple material types. Material properties such as conductivity for pure materials are well known and tabulated for use in computation, but these mixtures will have material properties which may not be well represented by these empirical values or even some averaging of them. This project focused on the treatment of these mixed cells by approximating the sub-cell structure from gradients in the material volume fraction. Using information from this “interface reconstruction”, the sub-cell structure was used to improve approximations of directional conductivity. Finally, this method was compared to a simple averaging of material properties in the mixed cells.

I. INTRODUCTION

The heat equation, a specific case of the diffusion equation, is one of the most elementary partial differential equations. Given in differential form:

\[ c_p \rho \frac{\partial T}{\partial t} = -\nabla \cdot \vec{F} = \nabla \cdot (k \nabla T) \]  

(1)

where \( c_p \) is the thermal heat capacity, \( \rho \) is the mass density and \( k \) is the thermal conductivity.

For our interests, we will be focusing on the role of thermal conductivity in this equation, so for the rest of this discussion we will take the product of heat capacity and density to be unity for all materials in question. As such the equation now only involves one coefficient which we will continue to refer to as \( k \), the conductivity (which is generally called the thermal diffusivity).

For the simple case of \( k \) being a constant across the domain of interest, with reasonable boundary and initial conditions, an analytic solution is easily obtained through separation of variables. In 2D, for the case of Dirichlet boundaries, \( \partial \Omega = 0 \), and initially constant temperature of \( u_0 \), we have:

\[ u(x,t) = \sum_{m,n} A_{mn} \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{L} y\right)e^{-k \left(\frac{m^2+n^2}{L^2}\right) t} \]  

(2)

\[ A_{mn} = \int\int u_0 \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{L} y\right) dxdy \]  

(3)

Note higher order terms decay exponentially faster, so after a short time only the lowest order solutions play a significant role.

We can consider the discrete form of the heat equation that we will numerically solve. If \( k \) is constant on the domain we can pull it out of the derivative. Looking first at the RHS, we can consider the Laplacian, which we know can be discretized in various forms. Using the second order centered differencing scheme, we approximate:

\[ \nabla \cdot (k \nabla T) \approx k \nabla^2 T \approx k \frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} \]  

(4)

Consideration of the time derivative is more involved. An explicit scheme would take the terms on the RHS to be from the previous time, \( t^n \), while an implicit scheme takes these terms to be from the next step, \( t^{n+1} \). Explicit solvers are simple, but are fundamentally constrained by the CFL limit for stability. We will use an implicit time step:

\[ \frac{T_i^{n+1} - T_i^n}{\Delta t} = k \frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{(\Delta x)^2} \]  

(5)

and we solve this expression for \( T^{n+1} \).
Solving this equation numerically shows the expected behavior which can be compared to the analytic solution. Note that this method has first order accuracy in time but second order in space. This is shown easily by plotting error from the solution calculated with increasing resolution in time or space on a log-log plot. An example solution is shown in Figure 1.

FIG. 1: Solutions to the heat equation with boundaries fixed at zeros and initially constant temperature on the domain.

II. MATERIAL MIXTURES

So far we have discussed the case of domains where the materials type is constant. If material properties are not constant in space, then $k$ cannot simply be pulled out of the derivative. This makes solving the heat equation much more challenging. If we consider the case of two materials joined at a sharp interface, then we can have a discontinuity in the material properties in the domain. In the differential equation, this discontinuity would result in a singularity which is clearly not physical.

While there are certain tricks one can play via variable transformation and other methods, an analytic solution to this problem is difficult and very specific. From physical intuition we can ballpark what the solution will look like. Let's take the 1D case of two materials initially at a uniform temperature, cooled from both ends. If the left material has a higher conductivity than the right, we would expect heat to leave faster through the left side than the right. As such, the left material would cool quickly, resulting in more rapid cooling of the right material as well. This is illustrated in Figure 2. We know that where the two materials contact they should be at the same temperature, and that the flux into the left material from the interface must equal the flux out of the right material on that side. Therefore both temperature and thermal flux are continuous across the interface. This basic idea is a good sanity check, but for quantitative analysis we will need to do much better.

FIG. 2: Comparing the expected solutions between the pure material solution (blue) and the mixed material where the left half has twice the conductivity as the right.

We would like to note at this point that for this project we focused on thermal diffusion on an Eulerian mesh, a mesh that is fixed in the “lab” frame of reference. This is in contrast to a Lagrangian mesh, which is a mesh that moves with the material. As we are dealing Eulerian meshes, it is likely that if there is advection in the simulation, cells with different material types will mix. This will result in cells containing multiple values for conductivity. For bookkeeping of materials on a sub-cell scale, the fraction of each material in the cell is recorded, known as the “material volume fraction.” This volume fraction is known for each material type in each cell. We will from here on refer to “pure” cells as having a volume fraction of 1 for a specific material, and “mixed” cells as having multiple material types. Note that all information pertaining to the structure of that material in the cell is lost in using a volume fraction. The material could be evenly diffused throughout the cell or lie on a sharp interface.
At this point we can now begin to ask some questions; How we treat these cells with mixed materials? What does one use for the conductivity of this cell? Is a simple averaging of the properties of two materials sufficient? Could we do something better? These are one of the main questions this report will begin to address. The next section begins to tackle the problem, starting with pure cell interfaces.

III. EFFECTIVE CONDUCTIVITY, $\tilde{k}$

As we saw above, the handling of a discontinuity in the differential form of the heat equation is non-trivial. How do we discretize this appropriately? By ignoring the fact that a change in the material conductivity affects the heat flow out of the discrete form (effectively pulling $k$ outside the derivative), we will be discarding important physics. We could perform a simple averaging of adjacent cell properties, but on what grounds can we justify this?

We can handle the discontinuity in material properties more physically by considering the flux between two adjacent cells of different materials. We know two things at this interface; the temperature infinitely close to the interface on both sides must approach the same value, and the heat flux into the interface must equal the flux out of the interface, which must equal the flux across the interface (assuming no surface or point source). As such, we have just created a statement about the heat flux, and therefore conductivity, between these two cells based on a physical law. The situation is depicted in Figure 3.

From this simple statement of conservation of flux we can write the following four statements:

$$F_L = -\frac{k_L}{\Delta x/2}(T_i - T_*)$$  \hspace{1cm}  $$F_R = -\frac{k_R}{\Delta x/2}(T_{i+1} - T_*)$$  \hspace{1cm}  (6)

$$F_* = -\frac{k_*}{\Delta x}(T_{i+1} - T_i)$$  \hspace{1cm}  $$F_L = F_* = F_R$$  \hspace{1cm}  (7)

From these statements, we can solve for the temperature $T_*$ at the interface, as well as the effective conductivity, $\tilde{k}$, between the two cells. These are given as:

$$T_* = \frac{k_R T_{i+1} + k_L T_i}{k_L + k_R} \hspace{1cm} \tilde{k} = \frac{2k_L k_R}{k_L + k_R}$$  \hspace{1cm}  (9)

Note the expression for $\tilde{k}$ is a sort of “reduced conductivity”, or in limit of $k_1 << k_2$, the conductivity behaves most like the smaller conductivity. Also notice that in the case of equal material properties the two cells conduct with the natural conductivity of their material. $T^*$, while we don’t use it here, also has its applications.

Comparing a solution computed using this effective conductivity to one that simply averages conductivity between neighboring cells, we see the effective conductivity is much more representative of the “true” solution, as is shown on
the right side of Figure 3\(^1\).

As such, we have implemented this effective conductivity in our code to best represent the conductivity between cells of different materials. Note that this method is limited to describing pure cells of one material type or another. The case of mixed material cells is more complicated and will be discussed next.

### IV. INTERFACE RECONSTRUCTION

#### A. Basic Principles and Method

Given a material volume fraction on a mesh, one can use gradients in this volume fraction to approximate the interfaces between different materials. For instance, let's take the situation where we have two materials with a transition between them. Each material is represented by a fractional value of the total material at that point, the material volume fraction. If we take the gradients of one material's volume fraction, we know that the surfaces of constant volume fraction must lie perpendicular to these gradients.

On this mesh, we will have pure cells of each material type, and at the transition there will be mixed cells. If this transition occurs quickly enough (spatially), we can approximate a material interface in these mixed material cells as a line perpendicular to the volume fraction gradient. This line will divide the mixed cells and will be placed based on the cell's volume fraction. Performing this for all the mixed cells will approximate and interface between the materials at a sub-cell scale.

For this project we weren't given a test case, so we created our own volume fractions. To represent a material on a mesh, we placed ones and zeros in a matrix to form various test material interfaces. The presence of a one meant that cell was pure of the material type in question, a zero meant there was none of that material in the cell. To create mixed cells we simply took block averages on this mesh. For example, if we started with an \( N \times N \) mesh of ones and zeros, by averaging cells in blocks of \( n \times n \), we returned a smaller matrix of dimensions \( (N/n) \times (N/n) \) where there are now ones, zeros and fractional values at the material interface. This was our artificial mixing process. For illustration, an example of this is shown in Figure 4.

**FIG. 4:** Initial clean material interface on the right (ones and zeros on matrix) and the artificially mixed volume fraction on the left.

#### B. Volume Fraction Gradient and Interface Line

To find gradients of volume fraction on the mesh, we used a linear least-squares method. For a given cell, by fitting a line in a least-squares sense to the values for volume fraction in the surrounding cells, in both the x and y directions, the slopes of these lines yield the components of the gradient in volume fraction. By performing this for the entire mesh we produce the leftmost plot in Figure 5.

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\(^1\) Figure courtesy of William Dai.
In these mixed cells the interface will be a line perpendicular to the cell’s volume fraction gradient. Clearly, the line perpendicular to the gradient vector goes as the negative reciprocal of the gradient vector components ($-x/y$). These initial interface lines are shown in the rightmost plot in Figure 5.

![Graph showing initial interface lines](image)

**FIG. 5**: The process to make the initial interface: find the gradients in the volume fraction, from these gradients find the perpendicular lines in each cell. In the cells with fractional volume fraction, use these perpendicular lines to create interface cells.

To begin to form our interface, what we call “interface cells” are created by dividing the cell by using the line perpendicular to the cell’s gradient vector. Initially, we divide the cell in half simply by defining the line to pass through the cell center. We perform this for cells that have a non-zero and non-unity volume fraction only. These initial cells are shown in the bottom of Figure 5.

These “interface cells” we create maintain the important information about the cells that are not pure. They are objects that store the cells material properties, corners, area, edge information and other details necessary for later calculation. The positions of the dividing line ends and the corners of the cells are stored as the “nodes” of the cell. Depending on the angle of the interface line, the number of nodes can be 3, 4 or 5.

### C. Interface Cell Area Resizing

The last step in recreating the interface requires moving each interface line based on the cell’s volume fraction. The angle of the line is fixed as it must remain perpendicular to the gradient, but its position in the cell ($y$-intercept) can change such that the fractional area “under” the interface line equals the cell’s volume fraction. To change the area of the cell, we simply need to move the positions of the two nodes which describe the line dividing the cell (the interface line). The manner in which we move these nodes depends on whether we are increasing or decreasing the cell’s current area and by how much.

Next, in each case we need to know how much to increase or reduce. For only small changes in cell area, the interface line may only need to be moved slightly thereby not significantly changing the cell structure. By moving the interface line too far, one of the nodes of this line will hit the corners of the cell. After this, this node will now need
to move along the other edge of the cell. After this happened, the cell will have either gained or lost a corner and an edge. For these cases the cell object must be redefined such that we properly manage the shape of the cell for later calculation. An example of this process is shown in Figure 6.

![Cells Diagram](image)

**FIG. 6:** Depending on the cell’s volume fraction the nodes describing the interface line must be moved in different ways. For a small enough change the nodes of the cell may simply be repositioned, but past a critical value the nodes must be redefined, adding a node and edge. An interface cell can be divided into its “triangular” and “rectangular” regions which allows for a simple means of determining how to resize the cell based on its volume fraction and gradient angle.

We can discuss the cell area in terms of its rectangular and triangular regions, as is shown in the first cell in Figure 6. Using this description, we find an algorithm for area resizing as follows:

- Volume fraction less than current fractional area (need to reduce area):
  - Volume fraction less than triangular area:
    - Completely remove rectangular cell area, reduce triangular region appropriately.
    - Note we now need to remove a node and edge in this case.
  - Volume fraction greater than triangular area:
    - Need to reposition the existing interface nodes accordingly, simply reduce the rectangular area.

- Volume fraction greater than current fractional area (need to increase area):
  - Volume fraction greater than triangular area plus twice the rectangular area:
    - Increase cell area such that only a triangular area is not occupied, reduce this triangle.
    - Note we now need to add a node and edge in this case
  - Volume fraction less than triangular area plus twice the rectangular area:
    - Need to reposition the existing interface nodes accordingly, simply increase rectangular area.

Note that for each above case, the dividing nodes are assigned based on gradient angle, so we need to handle the various ranges of angle for each case. This resizing code required a substantial amount of consideration and debugging.

**D. Finding Sub-Cell Centers and Edges**

After rescaling these interface cells we need information about various geometric features. We will see later the position of the interface cell centroids will be important, so calculating these is necessary. To simplify this calculation, a Python package called Polygon was used. This package has several functions specific to the treatment of polygons, and from the nodes of each interface cell Polygon objects were created. Use of this package allowed for simple plotting of these cells as well as knowledge of the cell area (to confirm the resizing) and the cell center, which is simply stored as an \((x, y)\) coordinate in the cell.

The other important detail the interface reconstruction gives us are the edges of these interface cells. The edges of each cell are stored as simply two points (or nodes) of the cell’s edge. A convention was enforced to maintain a left, right, top, bottom order to the arrangement of nodes and edges. The left side is the starting side, with the lowest left point being the starting node. Left is given priority over bottom as is shown in Figure 7.
E. Resulting Interface

After performing an interface reconstruction using the above method for both volume fractions, an approximation to the material interface becomes clearly defined. In each mixed material cell an interface line divides the two material types. Generally in the case of a sharp interface (only a one or two mixed cells between pure cells), this reconstructed interface is relatively representative of the original interface, as shown in Figure 8.

FIG. 8: By performing the interface reconstruction on both material volume fractions we have an approximation for the positioning of the materials on a sub-cell scale. While the new interface represents the original, note that it is discontinuous between cells.

The interface is discontinuous as a line was used for the interface in each cell. This line was constrained first to be perpendicular to the gradient in volume fraction (slope), then placed by the cell’s volume fraction (y-intercept). As such, there is no more freedom in the fit to make the interface continuous. The use of a higher order polynomial or some other method could possibly be employed to smooth this interface, but that was beyond the scope of this work.

It is also important to mention that this method as described above fails when more than two material types are present at an interface. This method currently works as the gradients of different material volume fractions are anti-parallel. If three or more materials were present, this condition would most likely not be satisfied and there would be resulting overlap of interface cells in the reconstruction. This is clearly unphysical, and therefore a fundamental limitation of the aforementioned technique. Other codes handle this problem by putting certain constraints on the arrangement and alignment of materials on the mesh\(^1\). For example, there may be a requirement that the materials are always arranged in a certain order, or that the materials are always aligned based on the direction of some primary material species.
V. USING THE INTERFACE RECONSTRUCTION

From the interface reconstruction described in the last section, we can now approximate the location of materials in these mixed cells based on gradients in the volume fraction. Given this interface, we could easily divide the mixed cells each in two, yielding a mesh of only pure materials. In one way this greatly simplifies the issue; given only pure cells we already have a method to calculate the conductivity based on conservation of flux, which is shown to be an accurate representation. While these cells are no longer interacting across a purely vertical or horizontal interface the same underlying principle holds, so the calculation would simply be adjusted geometrically. The real issue is that if we divide these cells we are forced to move from the mesh we started with to a mesh that can handle the unstructured cells resulting from this interface. This poses a severe computational burden as a regular grid can be handled with much faster methods.

We wanted to somehow use the information from the interface reconstruction to improve the accuracy of the calculation on the original mesh, a way of representing the division of the cells on the interface while using only a single cell for the two materials. By using the information of the placement of the different material types in the cell, we hoped to find an alternative representation of the conductivity into these cells that would best represent the two divided cells. We do this by considering the directional effective conductivities.

A. Deriving Effective Conductivity with Cell Center of Mass

Given a cell divided as shown in Figure 9, we can discuss the flux from the upper sub-cell through the right boundary into the neighboring pure cell:

\[ \begin{align*}
F_L &= -\frac{k_L}{\Delta x_1} (T_\ast - T_i) \frac{a}{d} \\
F_R &= -\frac{k_R}{\Delta x_2} (T_{i+1} - T_\ast) \frac{a}{d} \\
F_\ast &= -\frac{k_\ast}{\Delta x_1 + \Delta x_2} (T_{i+1} - T_i) \frac{a}{d} \\
F_L &= F_\ast = F_R
\end{align*} \]

(10)

Notice in these we use the sub-cell centroid for determining the flux through the boundary, but as the flux is dotted with the normal to the surface as given by:

\[ \frac{\partial T}{\partial t} = -\nabla \cdot \vec{F} \rightarrow \int \frac{\partial T}{\partial t} dV = -\int \nabla \cdot \vec{F} dV = -\oint \vec{F} \cdot \hat{n} dA \]

(13)
we see that only the $x$ separation is important in this case. Here we will use the approximation that these sub-cell temperatures are equivalent to the original cell-center temperature, though later we discuss that this approximation should be improved. Under this assumption, we find the effective conductivity between these neighboring cells to be given as:

$$\tilde{k} = \frac{k_L k_R}{\Delta x_1} \frac{\Delta x_1 + \Delta x_2}{\Delta x_1 \Delta x_2}$$  \hspace{1cm} (14)$$

Note that this expression returns to the original expression for effective conductivity between two pure cells in the case that $\Delta x_1 = \Delta x_2$.

To find the actual effective conductivity between two of these interface cells, we must consider the flux between all neighboring sub-cells. Above we found the new effective conductivity, but the flux for a specific $k$ is only through a portion of the boundary as is shown in Figure 9. As such, we compute the effective conductivity through each sub-cell boundary and scale this based on the fractional portion of the cell edge the sub-cell is in contact with. These scaled conductivities are then summed:

$$F_{\text{total}} = \frac{a}{d} F_{\text{above}} + \frac{d-a}{d} F_{\text{below}}$$  \hspace{1cm} (15)

$$= -\frac{a}{d} \tilde{k}_{\text{above}} \nabla T - \frac{d-a}{d} \tilde{k}_{\text{below}} \nabla T$$  \hspace{1cm} (16)

$$= -(\frac{a}{d} \tilde{k}_{\text{above}} + \frac{d-a}{d} \tilde{k}_{\text{below}}) \nabla T$$  \hspace{1cm} (17)

$$= -\tilde{k}_{\text{tot}} \nabla T$$  \hspace{1cm} (18)

B. Cell-Center Approximation

As mentioned earlier, we are attempting to represent the information from the interface reconstruction about two cells effectively in one. By considering the conductivities through sub-cell edges we have begun this process, but are still missing part of the representation. This approximation currently uses the same value for both sub-cell temperatures, the cell-center temperature. In many instances this will be a poor approximation. We can’t afford to keep track of two temperatures as we wish to maintain the original mesh style, but we can find a better approximation for the sub-cell temperature. Methods considering the sub-cell centroid as well as the internal conductivity of the cell based on our interface reconstruction have been discussed but not thoroughly developed as time in the workshop was limited.

VI. COMPARISONS TO CELL AVERAGING

Using the interface reconstruction to calculate the effective conductivity for the interface cells and their neighbors we have now have a new means of describing the heat transfer between these cells. Earlier we had discussed cell averaging based on volume fraction to give us a simple approximation to a cell’s conductivity. As such, we have two methods to compare. We use a test case of two materials separated by at a circular interface. The interior material is of lower conductivity than the external material. First, we can look at the actual differences in effective conductivity directly. If we break a cell into four quadrants as shown in Figure 10, we can represent the effective conductivities with neighboring cells more easily.

Using this visualization we can compare the representations of the interface conductivities. In Figure 11 we show the effective conductivity using the cell averaging method and using the interface reconstruction. Superimposed on these plots is a black circle representing the actual material interface.

Comparing these two plots, immediately one notices that the interface reconstruction sharpens our representation of the material conductivity. This comes from a better representation of the conductivity based on the materials in contact between cells. For example, notice the cells at the top of the “interface method” plot. The top of the mixed cells are made of outer material, and the conductivity between this outside boundary and the above pure cells is based only on the outer material’s conductivity. Compare this to the same cells in the “cell averaging” plot. We see here that the averaging of the materials in the mixed cells changes the effective conductivity between two regions that should conduct uniformly. Cell averaging appears to diffuse the actual material interface.

To more quantitatively compare these methods we need a “true” solution to compare to. In contrast to the single material case, we don’t have an analytic solution. Alternatively we could compare results found by dividing the mesh along the interface, but this requires coding an unstructured mesh method which we unfortunately didn’t have time to
FIG. 10: For visualization we divide the cell into four quadrants to represent the effective conductivities with the cell’s neighbors.

FIG. 11: On the left is a visualization of the effective conductivity from using a cell averaging of conductivity based on volume fraction. On the right is the result if you use information from the interface reconstruction instead. The black ring represents the actual material interface. Notice the sharpening of our representation of the conductivity using the interface method compared to the cell averaging.

do. As a last resort we can perform a high resolution run where the original interface we artificially mixed to produce our volume fraction is used as the true interface. This is performed simply using the pure cell effective conductivity first discussed.

Using the true solution described above, the error for both methods was calculated. Various ratios between the two material conductivities were tested. This is plotted for both methods in Figure 12.

As Figure 12 shows, there is a clear trend with ratio between the material conductivities for the interface method, but no clear trend for the cell averaged method. For small ratios between $k_1$ and $k_2$ we see that the cell averaging method seems to be the more accurate of the two, but for conductivities over an order of magnitude different the interface reconstruction method becomes more accurate. This answers the earlier question; if the materials are significantly different conductors, when they are averaged it may be a poor representation. Note that the two methods agree completely for the unity ratio case.

If we look at contour plots of the error from both methods, we gain some insight into what is being misrepresented. Figure 13 plots the difference from our true solution for increasing $k$ ratio.

You’ll notice that the interface reconstruction method always underestimates the temperature of the interior material while the cell-averaging method initially underestimates, then overestimates for large enough conductivity. The structure of the cell-averaging error changes dramatically from $k_1/k_2 = 2$ to $k_1/k_2 = 10$, then is very consistent for larger proportionality. This suggests that the conductivity averaging process changes the effective “structure” of the conductivity in a less predictable way than the interface reconstruction method.
FIG. 12: L2 norm error for both the cell-averaging method and the interface reconstruction method with grid size. 8 ratios between the conductivities are plotted, showing that the cell averaging technique is more accurate for more similar materials, but the interface reconstruction is better for materials with significantly different conductivities.

The fact that interface method effectively cools the interior faster than it should while the cell averaging method cools the interior too slowly can be directly related to how the methods represent the material interface. As the cell averaging diffuses the interface, lower interior conductivities are spread out, effectively insulating the interior material. Opposite to this, the interface method sets the effective conductivity of an interface cell equal to the outside conductivity, even if the cell is mostly inside the interface. This cools the interior material faster than it should, an issue which could be accounted for by more properly considering the sub-cell temperatures.

VII. CONCLUSION

We have seen the method using the interface reconstruction to improve our estimations of conductivity between cells is in fact better than simple averaging of the material properties in a mixed cell for the case of significantly different conductivities at a material interface. For cases where the conductivities of the materials were similar, cell averaging did prove to be more accurate though the strange effect of the ratios between the conductivities showed that this method of may have issues. It was also clear that the interface method always underestimated the interior material temperature for the test geometry we used.

To improve on this current method, some means of describing the mixed cells internal mechanisms should be imposed. Currently both sub-cells use the cell-center temperature as the temperature at their centroid which is a poor approximation. Going through the derivation of effective conductivity considering a more proper treatment of the sub-cell center temperature yields a more complex result requiring some interpolation to find this sub-cell temperature based on neighboring cell-center values. The algorithm involved to account for this has been conceived, but the timeframe of the project prevented the implementation. It seems reasonable to believe that by considering these sub-cell temperatures properly, the results from the interface method should be as good if not better than the cell averaging method for all ratios between conductivity.

With regard to application of this method, one obvious example is at the finest resolutions of AMR style meshes separating two materials. To completely represent the material interface an unstructured mesh would be necessary, but maintaining the AMR type mesh is certainly computationally advantageous. These unstructured meshes will become particularly taxing when we move from 2D to 3D. As an alternative the interface reconstruction could mostly represent this interface while maintaining the original mesh structure. While the interface reconstruction coded for this project was 2D, a 3D version can certainly be envisioned though it may be substantially more tedious to code. It should be noted that while in 3D the interface reconstruction could prove to be relatively expensive, reconstruction is clearly a local operation and is therefore easily parallelizable.
FIG. 13: Plots of difference between “true” solution and solutions from both methods for increasing k ratio. Notice that the overall behavior for the interface method is more consistent, even for smaller k ratios. The interface method always underestimates interior temperature while the cell averaging method often overestimates.
VIII. ACKNOWLEDGEMENTS

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