Learning in Rare Risks and Asset Price Implications

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November 3, 2014

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Key Words: learning, heterogenous beliefs, risk transfer, asset pricing.

Abstract

The impact of rare risks such as natural disasters, terrorism and epidemics are an increasing force in society. Unlike the case of common risks, our inexperience in rare risks creates differing views on the level of risk and the optimal form of risk management. Nonetheless, in both cases, the transfer of the financial risk can play a role in risk management. This paper illustrates how risk class impacts the price of risk transfer. In each risk class, two kinds of risk transfer assets are considered: an aggregate transfer (a share in an insurance company) and a disaggregated transfer specific to a particular risk (a catastrophe bond). Learning is harnessed to underline the differences between the risk classes and examine the change in prices over time. Results demonstrate persistent pricing benefits through disaggregation in rare risks. In contrast, these benefits are not maintained in common risks.

1 Introduction

Rare events excite the imagination because of their inherent uncertainty and irregularity. The magnitude of impact and the infrequency of event tends to misguide our sensibilities and lead us to a wide dispersion of beliefs. For instance, consider a national lottery. Although there is an objective probability of winning, it is reasonable that an average ticket purchaser would provide a probability of winning that differs from the objective probability. Unlike a common event such as the toss of a coin, individuals have a vast range of beliefs over rare events.

One particularly devastating rare event is a natural disaster. Natural disasters occur infrequently with large impacts on economies and communities (Cavallo and Noy 2011). To combat these impacts, communities, individuals and businesses seek to transfer their natural disaster risk through insurance and capital markets. Investors act as counterparties, trading risk for returns. However, in rare risks the returns are difficult to determine and quantify.[1] Further, it is realistic that investors have differing views on the probability of a natural disaster. 'In practice, the very concept of uncertainty implies that reasonable men may differ in their forecasts' (Miller 1977).

This paper aims to connect the attributes of rare risk classes through learning and dispersion of beliefs to asset markets. In current asset markets, natural disaster risk is a rare risk class that is traded in two ways. Firstly, an investor can purchase a catastrophe bond by providing principal to a bond issuer in return for coupons. In the event of a natural disaster the investor forfeits the principal and it is provided to the issuer.[2] If no event occurs during the bond duration the catastrophe bond lapses and the principal is returned to the investor. Catastrophe bonds are often multi-year

[1] For example, the ratings of catastrophe bonds face a different methodology to other bonds due to their inherent uncertainty.

[2] Remaining coupons may also be forfeited.

bonds of up to five years in length. In evaluating the investment value, an investor is calculating the likelihood of a natural disaster during that period. In this paper, catastrophe bonds are referred to as disaggregated since the likelihood refers to a single event.

Secondly, an investor seeking to capitalise on natural disaster risk transfer may invest in a reinsurance company that deals in natural disaster risk transfer. Since the catastrophe bond market is still very small compared to the reinsurance market, there are many risks that are not traded individually. Instead these risks are transferred via reinsurance. Indirectly, the investor is absorbing a suite of natural disaster risks through purchasing a share in a reinsurance company. The investor's return depends upon a group of natural disaster risks rather than a single event. In this paper, this risk transfer asset is referred to as the aggregated asset. The aggregated asset reflects the transfer of a group of risks rather than a single risk.[3]

This paper compares the pricing dynamics of aggregated and disaggregated risks across risk classes. By utilising the rare risk as a base, the paper explores the impact of learning through the arrival time of information. The paper illustrates that distinguishing between risk classes is important in market price determination. Further, in the time scale of asset markets the disaggregation benefits in rare risk classes are maintained whereas the disaggregation benefits in common risk classes are negligible. This provides important implications to the design of risk transfer assets in rare risk classes. As the transfer of rare risks becomes more prevalent, greater attention is needed to understand the different attributes of rare risk classes. This paper takes a preliminary step in elucidating the price impacts due to differences in learning for rare risk classes.

Section 2 of the paper provides a review of literature in the context of this paper. Sec-

[3]An alternative framing is that the aggregated asset constrains each investor to purchase the same amount of each risk.

tion 3 describes and explores the theoretical framework used in the analysis. For ease of explanation the learning and the pricing model are initially considered separately in Section 3. Following the explanation of each model independently, the models are integrated to provide the results of Section 4. Section 5 provides some numerical examples. Section 6 provides further applications of the results and Section 7 concludes.

2 Context

This paper lies at the intersection of several diverse fields of literature. The most similar literature relates to asset pricing with belief dispersion. This literature uses the competitive equilibrium of individuals to determine asset pricing. Varian (1985) considers the impact of subjective probabilities on asset prices in an Arrow-Debreu equilibrium and determines that if 'risk aversion declines less rapidly than ... in the case of logarithmic utility, an increase in the diversity of opinion will be associated with decreased asset prices'. Varian considers a static model in which individuals' subjective probabilities are unchanging. Morris (1996) illustrates that heterogenous priors can result in a persistent speculative premium in asset prices when there is learning. Unlike Varian (1985), Morris concentrates on risk neutral traders where assets are traded indefinitely. Whilst Lintner (1969) illustrates the analytical pricing solution for exponential utility maximising individuals with heterogenous normally distributed beliefs.[4] This paper follows in this vein of the literature by modelling risk neutral investors with heterogenous priors. Unlike Morris, the assets in this paper are modelled as fixed term contingent bonds without resale. Thus, within this model there is no speculative premium.

A second cluster of relevant papers in the financial literature focus on the pricing of

[4]Exponential utility implies constant absolute risk aversion.

assets under rare disasters and learning. Rare disasters are comparable to rare risks in this paper. However, rare disasters in the financial literature most commonly refers to economic and political disasters.[5] The flow on effects of these disasters are both analysed in the volume and pricing data and in learning behaviour (Cogley and Sargent (2008)). These papers tend to incorporate rare disasters through parameter uncertainty in the dividend pricing process and have the aim of explaining asset pricing inconsistencies (Barro 2006, Gabaix 2012).

Within the financial literature, a common method of modelling rare disasters is via a Poisson process. Koulovatianosi and Wieland (2011) introduce rare downward jumps through a Poisson process representing rare disasters and find that investors do not reach rational expectations about the average likelihood of rare disasters even after infinite time. Liu et al. (2003) also use a Poisson process to model rare risks to determine an optimal portfolio strategy. This paper follows in this tradition and models rare risks by a Poisson process.

In contrast to the rare disaster literature in finance, this paper focuses directly on the likelihood of rare disasters as the determinant of risk transfer asset prices. The uncertainty associated with the rare risk directly impacts the value of the risk transfer asset rather than having a shifting effect on the mean of the process. Additionally, this paper differs from the rare disaster literature since the number of disasters in each period is uncertain but losses are fixed. Many of the papers in the rare disaster literature focus on binary states such as a high or low state and the uncertainty lies in extent of loss in the disaster state. Although, methodologically distinct the outcome of the two frameworks is an equivalent compound random variable of disaster losses.

The third area of literature provides support for the subjectivity and heterogenity of beliefs in rare risks. Much of this literature is survey and experimental analysis from

[5] For example World Wars, the Depression, stock market crashes.

psychological literature. This literature shows the existence of heterogenous beliefs (Tversky and Kahneman 1974, Kahneman and Tversky 1972). In particular rare risks have heterogeneity due to personal experiences and can lead to overestimation or underestimation of risks (Hertwig et al. 2004). Kunreuther and Pauly (2004) suggest that individuals downward bias the probability of rare events because they occur so infrequently as not to warrant any deep thought. Whilst Viscusi and Zeckhauser (2006) provide survey evidence that individuals underestimate the risks of natural disasters even after they report having been impacted by a disaster. Cameron (2005) illustrates that individuals have varying beliefs on climate change risk and use information to Bayesian update their priors.

The intersection of these areas of literature leaves a gap that is filled by this paper. This paper uses a similar structure to the risk disaster literature to reflect rare risks through Poisson processes and allow updating through Bayesian learning. However, it seeks to determine asset prices via the competitive equilibrium methods of earlier papers. Finally, this paper acknowledges the differences between rare risks and common risks as elucidated in the psychology literature and cultivates these differences to provide comparative analysis between risk classes.

3 Model

3.1 Preliminaries

The theoretical framework of this paper concerns two risk transfer asset markets. In each market there are k independent risks in each risk class. The full set of risks is referred to as K. The first market is the disaggregated market where risk transfer assets are based on single risks. The disaggregated market has k assets. The second market is the aggregated market where all risks are transferred through one asset. The total expected loss of each market is the same. That is when faced with risk k, the potential loss associated with k is equal under the disaggregated and aggregated case. The exposure per risk is identical in each market. The only difference is in the design of the risk transfer instrument.

Investors $i = \{1, 2...n\}$ set the prices in these markets separately. Investors are able to buy from the aggregated and the disaggregated market. Investors set the asset prices by maximising their own utility. The market has a fixed supply of each asset, normalised to 1.

The focus of the model is the comparison of the pricing benefits from disaggregation between risk classes. This is defined as the difference in prices between the aggregated market price and the disaggregated market price in each risk class. Differences in risk classes are modelled though the learning process due to differences in the frequency of events.

3.2 The Learning Process: Poisson-Gamma Model

In line with existing literature, nature's underlying likelihood of an event is modelled using a Poisson process. To enable tractability the learning process is represented by a Poisson-Gamma model. For a risk $X \in K$, an investor i has beliefs over the Poisson rate parameter λ_X. Their beliefs are uncertain and follow a Gamma distribution $\lambda_{iX} \sim Gamma(\alpha_{iX}, \beta_{iX})$. The expected mean for an investor given their uncertainty is the mean of their Gamma distribution $\frac{\alpha_{iX}}{\beta_{iX}}$ and this will be used as their likelihood estimate of event X occurring. Investor i has parameters α_{iv} and β_{iv} for each risk $v \in \{1, 2...k\}$.

The β_{iX} parameter in the Gamma distribution represents the rate parameter and the spread of the distribution; a smaller β_{iX} represents a larger variance. The α_{iX} parameter allows a variety of shapes from logarithmic (smaller values of α_{iX}), left skewed (medium range values of α_{iX}), right skewed (large values of α_{iX}) and symmetric distributions. The Gamma distribution provides a convenient analytic form whilst allowing a wide dispersion of beliefs.

Individuals gather information on the rate parameter λ_X as they observe events over time. Using Baye's rule we can update each individual's prior given the new information in the current time period to provide the prior for the following period. The underlying likelihood is modelled as $X \sim Pois(\lambda_X)$ and the investor observes some number of events z during a time period. z is the information observed by the investor each period.

Bayes rules provides the relationship between the prior, posterior and observations:

$$p(\lambda = x \mid z) = \frac{p(\lambda = x)p(z \mid \lambda = x)}{p(z)}$$

Using the Gamma and Poisson distributions (dropping subscripts temporarily for neatness) results in:

$$\begin{aligned} p(z \mid \lambda = x) &= \frac{x^z e^{-x}}{z!} \\ p(\lambda = x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \end{aligned}$$

Combining these equations determines the posterior $p(\lambda \mid z)$

$$\begin{aligned} p(\lambda = x \mid z) &= \frac{\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \frac{x^z e^{-x}}{z!}}{p(z)} \\ &\propto x^{z+\alpha-1} e^{-(1+\beta)x} \end{aligned}$$

Since this is the kernel of the Gamma distribution, $\lambda|z \sim Gamma(\alpha + z, \beta + 1)$. The Gamma distribution is a conjugate prior of the Poisson distribution; providing an opportune computation of updating. The new parameters of the Gamma distribution can be defined as $\tilde{\alpha} = \alpha + z$ and $\tilde{\beta} = \beta + 1$. Therefore, the new mean is $\frac{\alpha+z}{\beta+1}$ and the new variance is $\frac{\alpha+z}{(\beta+1)^2}$.

Generalising this first period result, at time t, $\alpha_{iXt} = \alpha_{iX} + \sum_{s=0}^{t} z_s$ and $\beta_{iXt} = \beta_{iX} + t$.

Iterated over time the learning process results in a Gamma distribution for beliefs over λ_{iX} and a Negative Binomial predictive distribution over the occurrence of an event with mean $\frac{\alpha_{it}}{\beta_{it}}$ and variance $\frac{\alpha_{it}(1+\beta_{it})}{\beta_{it}^2}$. The Negative Binomial predictive distribution reflects the likelihood of an event from the perspective of an investor utilising all information until the present. Since investors are interested in the likelihood of an event to determine their expected returns the mean of the predictive distribution represents their updated belief for risk return assessment.

In general, at time t: $z_t \backsim NegBin(\alpha_i + \sum_{s=0}^{t} z_s, \frac{1}{\beta_i+t+1})$. This implies that all investors observe the same information. Blackwell and Dubins (1962) show that in such a case investors' opinions will converge.[6] Even though opinions will eventually converge in all risk classes, the rate of convergence given arrival times of information will differ.

Proposition 1 illustrates that observed information affects investors' beliefs differently depending on their prior value of β. This implies that the arrival of information is important and the differences in the rate of information by risk classes will exaggerate this effect.

Proposition 1: Consider n investors where $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} > ... > \frac{\alpha_{n-1}}{\beta_{n-1}} > \frac{\alpha_n}{\beta_n}$. This does not imply, at any point in time $t > 0$ that $\frac{\alpha_1+\sum_{s=0}^{t} z_s}{\beta_1+t} > \frac{\alpha_2+\sum_{s=0}^{t} z_s}{\beta_2+t} > ... > \frac{\alpha_{n-1}+\sum_{s=0}^{t} z_s}{\beta_{n-1}+t} > \frac{\alpha_n+\sum_{s=0}^{t} z_s}{\beta_n+t}$. That is if investors are ranked based on their prior beliefs, this ranking is

[6] This result requires absolute continuity which is met in this case.

not consistent over time given the learning process.

Proof:

The proof follows from the basic concepts of Bayesian updating. If an individual has a prior with a larger variance then more weight will be placed on the observed data rather than the prior. Due to differences in prior variance, the rank order of investors's beliefs will shift over time. The Appendix provides a numerical example.

Depending on the value of λ convergence of beliefs can occur prior to convergence to the true underlying process. This is because the updating process produces a reinforcing cycle. Koulovatianosi and Wieland (2011) describe beliefs about rare disasters as having a degree of persistence due to the slow arrival time of information. Consider an individual updating their beliefs for a rare event. Having not seen a rare event their probability estimates are updated downwards, this continues until an event occurs. The event being rare by definition will occur very infrequently. Thus it is more often that an individual's updated beliefs is below the true underlying process. Since all individuals update downwards, convergence of beliefs can occur faster than convergence to the true underlying process. In fact once beliefs are close, they will remain close indefinitely.[7] Although this is an interesting result, the consequence of the convergence of beliefs to a point different from the truth has no impact on the pricing model. This is because the pricing model is a negotiation between individuals, whom equally ignorant will still arrive at a compatible price.

3.3 Pricing Process

The pricing process is governed by the beliefs of investors and their willingness to pay for the risk transfer. Risk transfer is available in two forms as an aggregated transfer

[7]This is because this learning model does not allow for uncertainty in the information and the information is observed by all individuals.

(e.g. share) or a disaggregated transfer (e.g. catastrophe bond). Since both these markets are incomplete, pricing is determined by exchange between parties and not by arbitrage.

For illustrative purposes consider the simplest case of two risks X and Y. Investors are given the option to be counterparties to these risks and can invest in the risks individually or in aggregate. There are three prices in the market of interest: price P_X for risk transfer of X, price P_Y for risk transfer of Y and the price P for the combined insurance product. The sum insured or principal for each risk is Xl_X and Yl_Y, where l_X and l_Y are size of loss for events X and Y respectively. l_X and l_Y are constants but not necessarily equal. X and Y are random variables representing the number of events that occur in a time period. Investors choose the holdings of each risk asset η_X and η_Y.

Proposition 2 establishes that in a market of homogenous investors there are no disaggregation pricing benefits. That is the price of risk transfer in the aggregated and disaggregated markets are identical.

Proposition 2: If investors have the same initial wealth, beliefs and preferences there is no price difference between the aggregated and disaggregated risks.

Proof:

Consider n investors who have identical beliefs and preferences. Investor j seeks to maximise their expected value $E_j[U_j] = E_j[U_j(w + \sum_{v \in K} \eta_{jv}(P_v - L_v))]$. Where w is initial wealth, P_v is the price of the risk v, L_v is the loss incurred in the event risk v occurs and η_{jv} is the share of the risk that the individual absorbs. Set K is the set of all assets available in the market. Since investors have identical beliefs and preferences the subscript j can be dropped.

Let $\eta_v = \frac{1}{n}$ for each asset $v \in K$. This satisfies the market clearing condition since $\sum_{j=1}^{n} \eta_{jv} = 1$.

The proof is shown for a market where $K = \{X, Y\}$, however this is easily extended to a market of many assets. The price of assets in the disaggregated market will be determined first.

Utilising the first order condition for the first investor with respect to η_X provides $E[U'(w + \frac{1}{n}(P_X - Xl_X) + \frac{1}{n}(P_Y - Yl_Y))(P_X - Xl_X)] = 0$. Similarly for η_Y, $E[U'(w + \frac{1}{n}(P_Y - Yl_Y) + \frac{1}{n}(P_X - Xl_X))(P_Y - Yl_Y)] = 0$ is satisfied for the first investor. Solving these equations simultaneously provides the solutions P_X and P_Y.

In order to show that this is a unique equilibrium for the disaggregated market these prices must satisfy the first order conditions of all investors.[8] Since investors have the same first order conditions, P_X and P_Y provide a market equilibrium.

The second part of the proof concerns the price of the asset in the aggregated market. The single risk asset is $L = Xl_X + Yl_Y$.

Since X and Y are independent, the first order condition can be written as: $E[U'(w + \frac{1}{n}(P - Xl_X - Yl_Y))(P - Xl_X - Yl_Y)] = 0$.

Substituting $P = P_X + P_Y$ (the prices from the disaggregated market) provides $E[U'(w + \frac{1}{n}(P_X + P_Y - Xl_X - Yl_Y))(P_X + P_Y - Xl_X - Yl_Y)]$.

Rearranging this expression results in:

$E[U'(w + \frac{1}{n}(P_X + P_Y - Xl_X - Yl_Y))(P_X - Xl_X)] + E[U'(w + \frac{1}{n}(P_X + P_Y - Xl_X - Yl_Y))(P_Y - Yl_Y)] = 0$

This illustrates that $P = P_X + P_Y$ satisfies the first order conditions of all investors and P provides a market equilibrium in the aggregated market.

Hence there is no pricing difference in the equilibrium price between the aggregated and disaggregated markets when investors have the same beliefs and preferences.

[8] Uniqueness requires some technical conditions as shown in the appendix. Unique as long as there is concavity...check weak concavity.

Definition 1: An investor is considered optimistic if they have lower probability beliefs for every asset at one point in time. That is, investor i is optimistic if and only if at some t, $E_{it}(V) < E_{jt}(V)\ \forall V \in K$, $j \neq i$.

Since investors have heterogenous beliefs, for a large set of assets (K) and a large set of investors (n) the likelihood that one investor is optimistic reduces.

Lemma 1 establishes the conditions under which there are price differences between the aggregated and disaggregated risks for the risk neutral case. Short sale constraints are required due to risk neutrality to ensure that investors do not demand infinite amounts of each asset. If this were the case there would be no pricing solution.

Lemma 1 : Under short sale constraints with a finite number of risk neutral investors and finite number of risks,

(i) where one investor is optimistic there is no price difference between the aggregated and disaggregated risks, that is $P = \sum_{v=1}^{K} P_v$.

(ii) where no investors are optimistic there is a positive price difference between aggregated and disaggregated risks, that is $P > \sum_{v=1}^{K} P_v$.

Proof:

If investors are risk neutral their expected utility can be written as $E_j[U_j] = w + \eta_j P - \eta_j E_j[L]$ for the aggregated risk. The first order condition provides the result that $P = E_j[L]$. Similarly for the disaggregated risks $P_X = E_j[L_X]$ and $P_Y = E_j[L_Y]$.

Under short sale constraints each investor has a reservation price set by $P = E_j[L]$. The price of the market will be based on the investor with the lowest reservation price since for this investor $P - E_j[L] = 0$, but for all other investors $P - E_j[L] < 0$.

If (i) is satisfied then one investor is optimistic (has the lowest probability beliefs) and has the lowest reservation price for all assets. Suppose investor j is consistently optimistic then $E_j[.] < E_i[.]$ for all assets and investor j will purchase all assets.

The aggregated price will be $P = E_j[Xl_X + Yl_Y] = E_j[X]l_X + E_j[Y]l_Y$.

The disaggregated prices will be $P_X = E_j[X]l_x$ and $P_Y = E_j[Y]l_Y$. Thus $P - P_X - P_Y = 0$ and there is no difference in the prices between the aggregated and the disaggregated prices.

Alternatively if (ii) is satisfied, then consider an investor j with the lowest probability beliefs about asset X but not asset Y. Now $E_j[X] < E_i[X]\ \forall i \neq j$ but $\exists i\ s.t.\ E_i[Y] \leq E_j[Y]\ \forall i \neq j$. This ensures that no investor is optimistic.

So we have $P_X = E_j[X]l_X$ and $P_Y = E_i[Y]l_Y$.

In the aggregated market $P = E_B[Xl_X + Yl_Y]$ where B represents the investor with the lowest probability beliefs over the random variable $Xl_X + Yl_Y$. By definition $E_j[X] < E_B[X]$ and $E_i[Y] \leq E_B[Y]$.

$$\begin{aligned} P - P_X - P_Y &= E_B[X]l_X + E_B[Y]l_Y - E_j[X]l_X - E_i[Y]l_Y \\ &> 0 \end{aligned}$$

4 Results

This section combines the learning process and the pricing process. Lemma 2 and Proposition 3 provide general results a generic risk class. Propositions 4, 5, Lemma 3 and Theorem 1 in Section 4.1 provide results for the comparison of risk classes. Theorem 1 is the main result illustrating that the expected pricing benefits due to disaggregation are persistent in rare risk classes.

Lemma 2: Consider investors following a Poisson-Gamma learning process. If Lemma 1 (ii) is satisfied then the difference in price between the aggregated and disaggregated risks at time t is given by $D_t = \sum_{v=1}^{k} \left[\frac{\alpha_v^B}{\beta_v^B+t} - \frac{\alpha_{b_v}}{\beta_{b_v}+t} + \left(\frac{1}{\beta_v+t} - \frac{1}{\beta_{b_v}+t} \right) \sum_s v_S \right] l_v$; where the superscript B represents the individual with the lowest probability beliefs for the aggregated risk and b_v represents the individuals with the lowest probability beliefs for each disaggregated risk $v \in K$.

Proof: The proof follows directly by combining the learning process with the pricing process. For each asset, the investor with the lowest reservation price will set the market price for that risk.

The proof is shown for a market where $K = \{X, Y\}$, however this is easily extended to a market of many assets.

Using the Poisson-Gamma learning process and Lemma 1(ii):

$$\begin{aligned} P - P_X - P_Y &= E_B[X]l_X + E_B[Y]l_Y - E_{b_X}[X]l_X - E_{b_Y}[Y]l_Y \\ &= \frac{\alpha_y^B + \sum_s y_s}{\beta_y^B + t} l_Y + \frac{\alpha_x^B + \sum_s x_s}{\beta_x^B + t} l_X - \frac{\alpha_{b_Y} + \sum_s y_S}{\beta_{b_Y} + t} l_Y - \frac{\alpha_{b_X} + \sum_s x_S}{\beta_{b_X} + t} l_X \\ &= \left(\frac{\alpha_y^B + \sum_s y_S}{\beta_y^B + t} - \frac{\alpha_{b_Y} + \sum_s y_S}{\beta_{b_Y} + t} \right) l_Y + \left(\frac{\alpha_x^B + \sum_s x_S}{\beta_x^B + t} - \frac{\alpha_{b_X} + \sum_s x_S}{\beta_{b_X} + t} \right) l_X \\ &\equiv D_t \end{aligned}$$

Since l_X and l_Y are constants, D_t is a function of the information observed and the starting priors of investors.

This is easily generalisable to k assets: $D_t = \sum_{v=1}^{k} \left[\frac{\alpha_v^B}{\beta_v^B+t} - \frac{\alpha_{b_v}}{\beta_{b_v}+t} + \left(\frac{1}{\beta_v+t} - \frac{1}{\beta_{b_v}+t} \right) \sum_s v_S \right] l_v$.

Proposition 3: As $t \to \infty$, the price of the aggregated and the disaggregated risks converge.

Proof:

From Lemma 2 we have $D_t = \sum_{v=1}^{k} \left[\frac{\alpha_v^B}{\beta_v^B+t} - \frac{\alpha_{b_v}}{\beta_{b_v}+t} + \left(\frac{1}{\beta_v+t} - \frac{1}{\beta_{b_v}+t} \right) \sum_s v_S \right] l_v$. $D \to 0$ as $t \to \infty$.

4.1 Comparison of risk classes

In the following analysis, we consider the implications of the model for different risk classes. Risk classes are distinguished by the differences in the frequency of events. To incorporate this difference in the model, the rare risk class will have a much lower frequency than the common risk class. Further, since the comparison focuses on the price of risk assets it is required that the expected value in each risk class is set to be equal. If the expected value in each risk class were not equal the prices of assets would not be set to the same base and as a result would be incomparable.

Define a rare risk class as having an underlying probability of λ_R, each individual has $\lambda_{Rv}^j \sim Gamma(\alpha_{Rv}^j, \beta_{Rv}^j)$ as a prior over the assets within this risk class. A common risk class has an underlying probability of $\lambda_C = M\lambda_R$, and each individual has priors $\lambda_{Cv}^j = M\lambda_{Rv}^j \sim Gamma(\alpha_{Cv}^j, \beta_{Cv}^j)$, where $M > 1$. This asserts that an investor's beliefs are the same in the rare and the common case for a single event. However, in the common case there are more repetitions of the event represented by M resulting in the scaling of the underlying rate parameter.

The scaled beliefs ensure that the risk classes have different orders of magnitude. In addition, to ensure the same expected value of the assets the level of losses is scaled. In the case of common risks the loss associated with each asset v is l_v. Scaling this for the rare risk provides that the loss associated with each rare risk asset is Ml_v.

The following example motivates the construction of the common risk. Suppose the chance of a 1-year drought in a city is the rare risk in the model. This implies that the probability of this event in a time step t is λ_{Rt}. The cycle of drought is linked to the level of rainfall that follows a physical process that may not be fully understood. However, it is likely there is some overlap in the process of rainfall on a daily basis and drought. Suppose the common risk is that there is no rain on one day. This is a much more likely event than drought conditions over the year. If we consider that rainfall today is linked to drought conditions, then we could consider the common event to be M times more likely than the rare event. Alternatively, if the risk was rain on one day during the year there are 365 days in a year and M could be set to 365 to reflect that each day is a new draw from the distribution. This is an abstraction from the true meteorological process. Nonetheless the abstraction allows comparison of risk classes.

In keeping with the motivating example, the k assets in each risk class could represent various locations or different cities. For instance the set of rare risks can be considered as the risk of drought in a city where there are k cities. Whilst the set of common risks are no rain for a day in a city for k cities. The duration of all risk transfer assets is a single time step (for example a year), but events can occur multiple times (for instance per day) in a time step. Each time an event occurs the investor suffers a loss on the risk transfer asset. Thereby for each day of rain in city k, l_k is paid and if there is a drought over the year in city k, Ml_k is paid.

Proposition 4: If $\lambda^j_{Rv} \sim Gamma(\alpha^j_{Rv}, \beta^j_{Rv})$ and $\lambda^j_{Cv} \sim Gamma(\alpha^j_{Cv}, \beta^j_{Cv})$, then $\alpha^j_{Cv} = \alpha^j_{Rv}$ and $\beta^j_{Cv} = \frac{\beta^j_{Rv}}{M}$.

Proof:

For a Gamma distribution $\frac{mean}{variance} = \beta$.

The mean of λ^j_{Cv} is $\frac{\alpha^j_{Cv}}{\beta^j_{Cv}} = M\frac{\alpha^j_{Rv}}{\beta^j_{Rv}}$ and the variance is $\frac{\alpha^j_{Cv}}{(\beta^j_{Cv})^2} = M^2\frac{\alpha^j_{Rv}}{(\beta^j_{Rv})^2}$. Using the relationship of mean and variance, $\beta^j_{Cv} = \frac{\beta^j_{Rv}}{M}$ and $\alpha^j_{Cv} = \alpha^j_{Rv}$.

Proposition 4 implies that although the variance of an individual's beliefs are the same in the rare and the common for a single incident, the scaling up of the incident parameter in the common case leads to more variance. This is due to the cumulutive uncertainty. The variance of a single incident is now multiplied by M. However, when the loss is taken into account, the variance of the loss of the common case is smaller than that of the rare case.

Proposition 5 below establishes a baseline for comparison. Both risk classes begin without any disaggregation pricing benefits. That is the price differential between aggregated and disaggregated markets is zero within each risk class at $t = 0$.

Proposition 5: Given the two risk classes, $D_t^{comm} = D_t^{rare}$ at $t = 0$. That is the price differential is identical in both risk classes at $t = 0$.

Proof:

By constuction the individuals with the lowest probability beliefs in the set of common assets and the set of rare assets is the same. Thus these individuals will determine the prices in both risk classes.

Let B be the individual with the lowest probability beliefs in the common aggregated asset market.

Let b_v be the individual with the lowest probability beliefs in the common disaggregated asset market for asset v.

$$D_t^{comm} = \sum_{v=1}^{k} \left[\frac{M\alpha_{Rv}^B}{\beta_{Rv}^B} - \frac{M\alpha_{b_{Rv}}}{\beta_{b_{Rv}}} + \left(\frac{M}{\beta_{Rv}^B} - \frac{M}{\beta_{b_{Rv}}} \right) \sum_s v_S \right] l_v$$

$$D_t^{rare} = \sum_{v=1}^{k} \left[\frac{\alpha_{Rv}^B}{\beta_{Rv}^B} - \frac{\alpha_{b_{Rv}}}{\beta_{b_{Rv}}} + \left(\frac{1}{\beta_{Rv}^B} - \frac{1}{\beta_{b_{Rv}}} \right) \sum_s v_S \right] M l_v$$

$$\therefore D_t^{comm} = D_t^{rare}$$

Lemma 3: A $t \to \infty$, $E[D_t^{rare}] - E[D_t^{comm}] \to 0$. That is the expected price differential between risk classes asymptotically converges to zero.

Proof:

By direct application of Blackwell and Dubins (1962) the beliefs within each risk class will converge eventually due to learning. If beliefs of individuals converge then the price differential between aggregated and disaggregated risks in a risk class also converge, implying $E[D_t^{rare}] \to 0$ and $E[D_t^{comm}] \to 0$. Thus $E[D_t^{rare}] - E[D_t^{comm}] \to 0$.

Theorem 1: When no investor has consistently lower probability beliefs, for sufficiently large$\infty > t$, $E\left[D_t^{rare}\right] - E\left[D_t^{comm}\right] > 0$. The expected difference in prices D_t is larger in the case of rare risks than common risks.

Proof:

Consider $E\left[D_t^{comm}\right]$ and $E\left[D_{Mt}^{rare}\right]$. Under Lemma 2, we know:

$$E\left[D_t^{comm}\right] = E\left\{\sum_{v=1}^{k}\left[\frac{\alpha_{Rv}^B}{\beta_{Rv}^B/M+t} - \frac{\alpha_{brv}}{\beta_{brv}/M+t} + \left(\frac{1}{\beta_{Rv}^B/M+t} - \frac{1}{\beta_{brv}/M+t}\right)\sum_{s=0}^{t} v_{cs}\right] l_v\right\}$$

$$E\left[D_{Mt}^{rare}\right] = E\left\{\sum_{v=1}^{k}\left[\frac{\alpha_{Rv}^B}{\beta_{Rv}^B+Mt} - \frac{\alpha_{brv}}{\beta_{brv}+Mt} + \left(\frac{1}{\beta_{Rv}^B+Mt} - \frac{1}{\beta_{brv}+Mt}\right)\sum_{s=0}^{Mt} v_{rs}\right] Ml_v\right\}$$

$$E\left[D_{Mt}^{rare}\right] = E\left\{\sum_{v=1}^{k}\frac{1}{M}\left[\frac{\alpha_{Rv}^B}{\beta_{Rv}^B/M+t} - \frac{\alpha_{brv}}{\beta_{brv}/M+t} + \left(\frac{1}{\beta_{Rv}^B/M+t} - \frac{1}{\beta_{brv}/M+t}\right)\sum_{s=0}^{Mt} v_{rs}\right] Ml_v\right\}$$

$$E\left[D_{Mt}^{rare}\right] = E\left\{\sum_{v=1}^{k}\left[\frac{\alpha_{Rv}^B}{\beta_{Rv}^B/M+t} - \frac{\alpha_{brv}}{\beta_{brv}/M+t} + \left(\frac{1}{\beta_{Rv}^B/M+t} - \frac{1}{\beta_{brv}/M+t}\right)\sum_{s=0}^{Mt} v_{rs}\right] l_v\right\}$$

Note that $\sum_{s=0}^{t} v_{cs} \sim Pois(Mt\lambda)$ and $\sum_{s=0}^{Mt} v_{rs} \sim Pois(Mt\lambda)$. Therefore, both random variables follow the same distribution.

Consider an arbitrary individual j the price setter for the aggregate risk in the rare class and a set of individuals $\{i_{r1}, i_{r2}, ...i_{rk}\}$ who are the price setters for the individual risks in the rare class.

Suppose $V_{r_{Mt}} \equiv \sum_{s=0}^{Mt} v_{rs} = 0$. $D_{Mt}^{rare} = \sum_{v=1}^{k}\left[\frac{\alpha_{Rv}^j}{\beta_{Rv}^j/M+t} - \frac{\alpha_{irv}}{\beta_{irv}/M+t}\right] l_v$.

Suppose $V_{c_t} \equiv \sum_{s=0}^{t} v_{cs} = 0$.

Now, $D_t^{comm} = \sum_{v=1}^{k} \left[\frac{\alpha_{Rv}^{j}}{\beta_{Rv}^{j}/M+t} - \frac{\alpha_{irv}}{\beta_{irv}/M+t} \right] l_v$. So, $D_{Mt}^{rare} = D_t^{comm}$.

Similarly suppose $V_{c_t} = V_{r_{Mt}} = 1$.

Now, $D_{Mt}^{rare} = \sum_{v=1}^{k} \left[\frac{\alpha_{Rv}^{j}}{\beta_{Rv}^{j}/M+t} - \frac{\alpha_{i_{Rv}}}{\beta_{i_{Rv}}/M+t} + \left(\frac{1}{\beta_{Rv}^{j}/M+t} - \frac{1}{\beta_{i_{Rv}}/M+t} \right) \right] l_v$ and

$D_t^{comm} = \sum_{v=1}^{k} \left[\frac{\alpha_{Rv}^{j}}{\beta_{Rv}^{j}/M+t} - \frac{\alpha_{i_{Rv}}}{\beta_{i_{Rv}}/M+t} + \left(\frac{1}{\beta_{Rv}^{j}/M+t} - \frac{1}{\beta_{i_{Rv}}/M+t} \right) \right] l_v$. Again, $D_{Mt}^{rare} = D_t^{comm}$.

It is easy to induce that when $V_{c_t} = V_{r_{Mt}}$ the price differentials of each risk class is identical. Since j and $\{i_{r1}, i_{r2}, ...i_{rk}\}$ were chosen arbitrarily this is true for any set of investors $(j, \{i_{r1}, i_{r2}, ...i_{rk}\})$.

Expected value is defined as: $E[X] = \sum_{x=0}^{\infty} xPr(x)$. We can rewrite the expected price differentials as:

$$E[D_t^{comm}] = \sum_{V_{c_t}=0}^{\infty} \sum_{v=1}^{k} \left[\frac{\alpha_{Rv}^{j}}{\beta_{Rv}^{j}/M+t} - \frac{\alpha_{i_{Rv}}}{\beta_{i_{Rv}}/M+t} + \left(\frac{1}{\beta_{Rv}^{j}/M+t} - \frac{1}{\beta_{i_{Rv}}/M+t} \right) V_{r_{ct}} \right] l_v Pr(V_{ct})$$

$$E[D_{Mt}^{rare}] = \sum_{V_{r_{Mt}}=0}^{\infty} \sum_{v=1}^{k} \left[\frac{\alpha_{Rv}^{j}}{\beta_{Rv}^{j}/M+t} - \frac{\alpha_{i_{Rv}}}{\beta_{i_{Rv}}/M+t} + \left(\frac{1}{\beta_{Rv}^{j}/M+t} - \frac{1}{\beta_{i_{Rv}}/M+t} \right) V_{r_{Mt}} \right] l_v Pr(V_{r_{Mt}})$$

From above we have shown that $D_{Mt}^{rare} = D_t^{comm}$ when $V_{c_t} = V_{r_{Mt}}$. It was also noted above that V_{c_t} and $V_{r_{Mt}}$ share the same distribution, thereby $Pr(V_{c_t} = x) = Pr(V_{r_{Mt}} = x)$. Thus combining these two notions provides that $E[D_{Mt}^{rare}] = E[D_t^{comm}]$.

The second part of the proof shows that $E[D_t^{rare}] > E[D_{Mt}^{rare}]$.

As mentioned in Lemma 3, it is clear that $E[D_t^{rare}] \to 0$ as $t \to \infty$. Thereby, it must be the case that at a sufficiently large t, $E[D_t^{rare}] > E[D_{Mt}^{rare}]$.

Sufficiently large t is required due to the risk neutrality of investors. In the case of risk neturality the pricing function $E[D_t^{rare}]$ is not a monotonic function. This is due to the extremity that one investor purchases all of each asset. If M is small enough, at small t there is a small possibility that $E[D_t^{rare}] < E[D_t^{comm}]$. Given the stochastic nature of Theorem 1, it is true for sufficiently large t. The size of the required t is inversely related to M. A numerical example demonstrates this in Section 5.

5 Numerical Examples

This section provides numerical examples to illustrate Theorem 1. The section also provides evidence that with large enough M, the constraint of $t > T$ in Theorem 1 is not restrictive.

Figure 1 displays a numerical example for 5 investors and 2 risks with fixed $\beta_{jR_V} = \beta_{iR_V}$.

Figure 1: D_t for common and rare

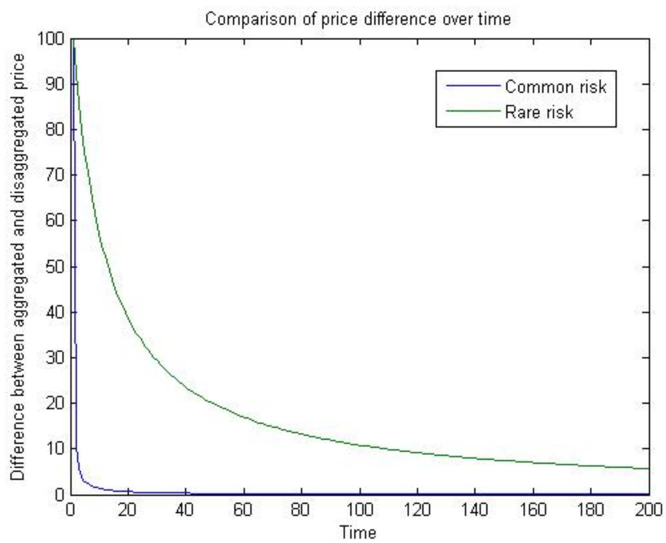


Figure 2 shows the non monotonicity of the pricing function for 5 investors and 2 risks.

Figure 2: $E\left[D_t\right]$for common and rare

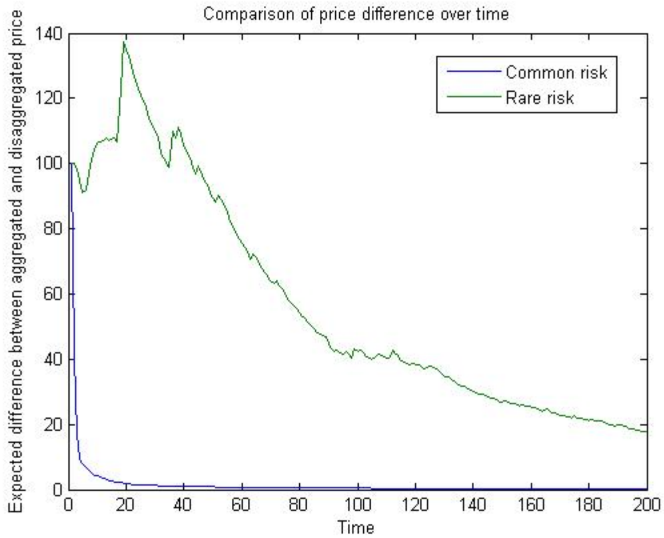
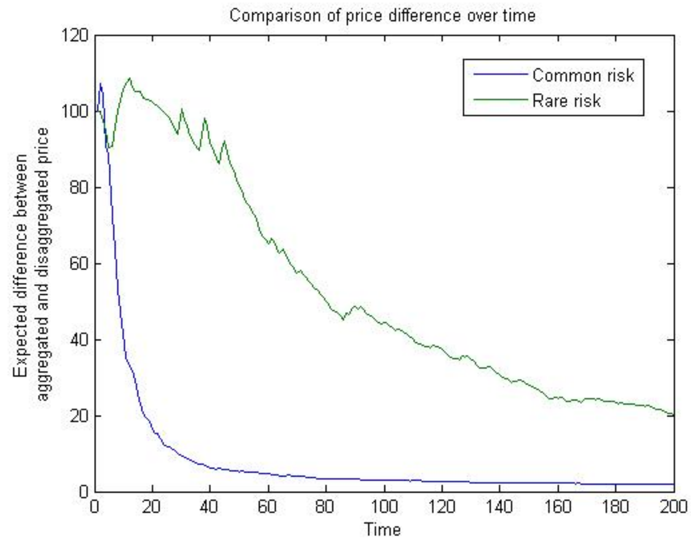


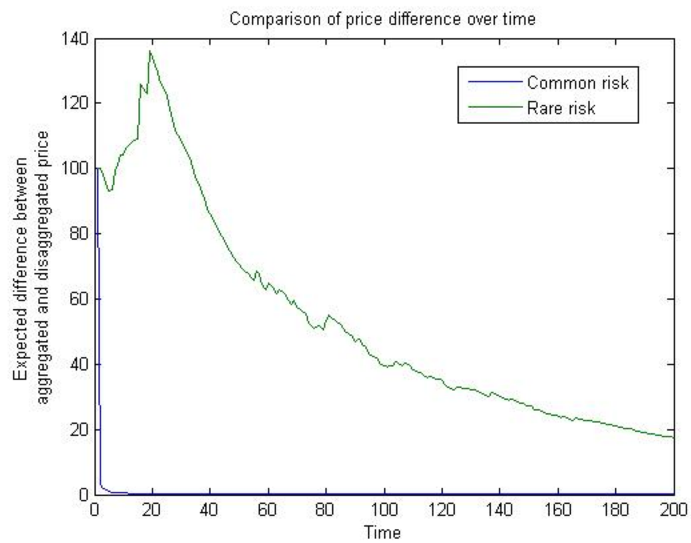
Figure 3 uses the same parameters as Figure 2 apart from the changing degree of rarity, M. The large the value of M, the more rare the event. Figure 3 indicates that as long as M is sufficiently large the expected pricing benefits from disaggregation are consistently larger for rare risks over time. In the event that a rare risk is not sufficiently sporadic, the pricing benefits are still maintained after sufficiently large t, as expressed in Theorem 1.

The reason for this clarification is that the pricing benefits from disaggregation vary with time and depend on the arrival of information. In the case of a rare event occurring early in the learning process, investors updated beliefs can initially be closer than in the common case. The degree of rarity determines whether Theorem 1 holds for all t or alternatively for sufficiently large t.

Figure 3: Theorem 1 and the degree of rarity M



$M = 10$



$M = 1000$

6 Discussion

The results of this paper are generalisable to different learning models and to risk aversion. The risk neutrality and no short sales assumptions are required for tractability and closed form solutions. Similarly the Poisson-Gamma learning model is not necessary to establish Theorem 1. The model was chosen in line with literature and to simplify calculations.

In the case of risk averse investors, short sales are allowed and investors purchase proportions of each asset rather than one individual holding the whole of an asset. To ensure a price is reached a limited supply of each asset is available. Under risk aversion, there is a level of monotonicity and the requirements of a sufficiently large t no longer have bearing. In fact, due to risk aversion even from the first period the disaggregation pricing benefits in the rare case are above that of the common.

As yet little attention has been given to nature's underlying probability of an event. The results of Section 4 hold for situations where the intensity of events is stochastic or deterministic. It is not necessary for the underlying true probability to be known or constant. It is only necessary to know that the event is possible and it has a non zero probability. In this model investors bargain amongst themselves for the appropriate price and whether that price is related to the true probability of an event is irrelevant.

Furthermore, this model as a description of the catastrophe bond market indicates that bond prices cannot be taken as reflecting the true probability of an event. The capital market provides little guidance to determine the true probability of a natural disaster. An attempt to backwards deduce the probability of a natural disaster from the price of a catastrophe bond could be misleading.

7 Conclusion

Rare risks are inherently different to common risks. Risk modellers have highlighted this by creating increasingly complex models of natural disasters, terrorism, stock market crashes and epidemic risks. Taking an alternate path this paper attempts to address the differences in a simpler and tractable model. In an effort to underpin the differences between risk classes this paper focuses attention on the rate of learning. As a key distinction between risk classes, learning reflects the uniqueness of rare risks and drives the differences between risk classes. Pricing differences are derived analytically for risk classes allowing for comparative analysis.

Recent issuance of catastrophe bonds suggests that significant demand is pushing prices below initial price guides.[9] This provides evidence that investors have differing views to bond issuers on the level of risk associated with the bond and the level of return required to match the risk. This paper suggests that these diverse opinions related to rare risks can be harnessed to improve pricing of risk transfer. In particular, issuers can take advantage of the diversity of beliefs by issuing risk transfer assets in disaggregated form.

In contrast, such a benefit is not afforded in the case of common risks. The high frequency of information leads to little difference between the price of the disaggregated and aggregated risks. Thereby, given non-negligible transactions costs, any pricing benefits due to disaggregation will be quickly expended within the time scale of asset markets.

In the growing market of risk transfer assets, rare risks are playing an increasingly important role. Rare risks are transferred by the issuance of catastrophe bonds and the purchase of reinsurance. The infrequent and erratic nature of rare risks results in a persistent diversity of beliefs. A market of disaggregated risks, such as catastrophe bonds, allows investors to invest directly based on their beliefs. Whereas a market of aggregated risks, such as insurance, limits the ability of investors to incorporate their beliefs into their investment strategy.

[9] For example CATMEx, Blue Danube II Ltd. (Series 2013-1) Allianz, Travellers, Residential Re 2013.

Although the range of investor beliefs decreases over time, the time scale of rare events is on a magnitude very different to that of asset markets. This ensures that within the framework of asset markets the variety of investor beliefs and the availability of pricing benefits through disaggregation are maintained.

8 Appendix

Proposition 1: Proof by example

Consider two investors 1 and 2 with prior parameters $\alpha_1 = 1$, $\beta_1 = 5$, $\alpha_2 = 5$, $\beta_2 = 12$.

At time $t = 0$, $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$.

However, suppose at time $t = 1$ the investors have seen 10 events. Bayesian updating provides that investor 1 has $\alpha_1 = 11$ and $\beta_1 = 6$ and investor 2 has $\alpha_2 = 15$ and $\beta_2 = 12$.

Thereby at $t = 1$ we have $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$ and the rank order has been reversed. $\square$

Numerical Examples:

Figure 1 parameters: 5 investors, 2 assets, $\lambda_1 = 0.003$, $\lambda_2 = 0.005$, $M = 100$ and expected loss=1000.

α_1	β_1	α_2	β_2
0.02	0.08	0.042	0.12
0.016	0.08	0.048	0.12
0.012	0.08	0.036	0.12
0.0136	0.08	0.0336	0.12
0.0176	0.08	0.03	0.12

Figure 2 parameters: 5 investors, 2 assets, $\lambda_1 = 0.003$, $\lambda_2 = 0.005$, $M = 100$ and expected loss=1000.

α_1	β_1	α_2	β_2
0.065	26	0.042	12
0.036	18	0.072	18
0.045	30	0.033	11
0.0255	15	0.056	20
0.0484	22	0.0225	9

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