Quiz Sheet 4: Solutions

Question 1. *Let $A \subseteq \mathbb{R}^n$. Suppose $A_1 := B(0,1) = \{x \in \mathbb{R}^n : \|x\| < 1\}$ and $A_2 := \overline{B(0,1)} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Find int A_i, ext A_i, ∂A_i for $i = 1, 2$.*

$\underline{A_1}$: $\operatorname{int} A_1 = A_1 = B(0,1)$ since $B(0,1)$ is open and trivially contains all its open proper subsets.

Any $x \in \mathbb{R}^n$, s.t. $\|x\| = 1$, is a limit point of $B(0,1)$,[1] and so for any $B(x,\epsilon)$, we have $B(0,1) \cap B(x,\epsilon) \neq \emptyset$. Thus, $\operatorname{ext} A_1 \subseteq \overline{B(0,1)}^c$. Since $\overline{B(0,1)}^c$ is open, if $x_0 \in \overline{B(0,1)}^c$, then there exists $B(x_0,\delta) \subseteq \overline{B(0,1)}^c$, such that $B(x,\delta) \cap B(0,1) = \emptyset$. Thus, in fact, $\operatorname{ext} A_1 = \overline{B(0,1)}^c$.

By exclusion, $\partial A_1 = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

$\underline{A_2}$: Any $x \in \mathbb{R}^n$, s.t. $\|x\| = 1$, is a limit point of $\overline{B(0,1)}^c$,[2] which implies that $B(x,\epsilon) \cap \overline{B(0,1)}^c \neq \emptyset$. Thus B(0,1) is the maximal open set containing all open sets $U \subseteq \overline{B(0,1)}$, and we have $\operatorname{int} A_2 = B(1,0)$.

Now $x \in \operatorname{ext} A_2 \Leftrightarrow \exists B(x,\epsilon)$ s.t. $B(x,\epsilon) \cap A_2 = \emptyset \Leftrightarrow x \in A_2^c$. Thus, $\operatorname{ext} A_2 = A_2^c = \overline{B(0,1)}^c$.

Again, by exclusion, $\partial A_2 = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

Question 2. *Prove that $x \in \partial A \subseteq \mathbb{R}^n$ if and only if every open set containing x intersects both A and $\mathbb{R}^n \setminus A$.*

($\Rightarrow$) Let $x \in \partial A$ and $x \in U$ open. If $U \cap A = \emptyset \Rightarrow U \subseteq \operatorname{ext} A$, which is false. Thus $U \cap A \neq \emptyset$.

Similarly, if $U \cap \mathbb{R}^n \setminus A = \emptyset$, then $U \subseteq A$, which implies $U \in \operatorname{int} A$, which is a contradiction. Thus $U \cap \mathbb{R}^n \setminus A \neq \emptyset$.

($\Leftarrow$) If every U open, s.t. $x \in U$ intersects A, then $x \notin \operatorname{ext} A$. Similarly, if every such U intersects $\mathbb{R}^n \setminus A$, then $x \notin \operatorname{int} A$. Thus, $x \in \partial A$.

[1] To rigorously justify this consider $B(x,\epsilon)$ for $0 < \epsilon < 1$. Then $(1 - \epsilon/2)x$ will be contained in $B(x,\epsilon)$, and since $\|(1 - \epsilon/2)x\| = 1 - \epsilon/2$, x will be also contained in $B(0,1)$

[2] We can give a justification similar to the above. Consider $B(x,\epsilon)$ for $0 < \epsilon < 1$. Then $(1+\epsilon/2)x$ will be contained in $B(x,\epsilon)$, and since $\|(1 + \epsilon/2)x\| = 1 + \epsilon/2$, x will be also contained in $\overline{B(0,1)}$

Question 3. *In "Piece One" (invertibility of $\varphi'(a)$ implies φ is injective near a) of the proof of the Inverse Function Theorem we established the following. Let U be open in $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^n$ be of class C^1. If $\varphi'(a)$ is invertible then there exists $\alpha > 0$ s.t.*

$$\|\varphi(x_0) - \varphi(x_1)\| \geq \alpha\|x_0 - x_1\| \; (*)$$

for all $x_0, x_1 \in B(a, \epsilon)$ and some $\epsilon > 0$. Explain why local injectivity of φ follows from $(*)$.

If φ were not injective, then for some $x_0 \neq x_1$ in $\mathbb{R}^n$, we would have $\varphi(x_0) = \varphi(x_1)$, which would contradict $(*)$.

Question 4. *Give an example of $\varphi : U \to \mathbb{R}^n$ that is continuous on U and $B(a, \epsilon) \subseteq U$ open such that $\varphi(B(a, \epsilon)) \subseteq \mathbb{R}^n$ is NOT open, or explain why such result may occur.*

$\varphi : (-1, 1) \to \mathbb{R}$ defined by $\varphi(x) = x^2$ has the image $[0, 1)$, which is not open. Note that φ doesn't have a well-defined inverse on this interval.