# Quiz Sheet 7

**Instructions**: Turn in responses to TWO questions of your choice. This quiz will be timed for 15 minutes.

**Question 1.** Let $x, y, z \in \mathbb{R}^5$. Let

$$F(x, y, z) = 2x_2y_2z_1 + x_1y_5z_4$$
$$G(x, y) = x_1y_3 + x_3y_1$$
$$h(w) = w_1 - 2w_3$$

(a) Write AF and AG in terms of elementary alternating tensors, where $A : \mathcal{L}^k(V) \to \mathcal{A}^k(V)$ is defined by

$$Af := \sum_{\sigma \in \text{sym}(k)} (\text{sgn}\,\sigma) T_\sigma f \quad \text{for any } f \in \mathcal{L}^k(V)$$

*Hint: Write F and G in terms of elementary alternating tensors and use the identity that for any collection $f_1, ..., f_k$ of 1-tensors, $A(f_1 \otimes ... \otimes f_k) = f_1 \wedge ... \wedge f_k$ (e.g., Munkres, p. 241, Step 9).*

(b) Express $(AF) \wedge h$ in terms of elementary alternating tensors.

(c) Express $(AF)(x, y, z)$ as a function.

**Question 2.** Do all of the following:

(a) Given $x \in \mathbb{R}^n$, define what it means for a pair $(x, v)$ to be a *tangent vector* to $\mathbb{R}^n$ at $x$ and the set $\mathcal{T}_x(\mathbb{R}^n)$ to be the *tangent space* to $\mathbb{R}^n$ at $x$.

(b) Given an open set $A$ in $\mathbb{R}^n$, define what it means for a function $\omega$ to be a *$k$-tensor field* in $A$. Define what it means for a function $\omega$ to be a *differential form of order $k$* (or simply a *$k$-form*) on A.

(c) Let $e_1, ..., e_n$ be the canonical basis for $\mathbb{R}^n$, and $(x, e_1), ..., (x, e_n)$ be the canonical basis for $\mathcal{T}_x(\mathbb{R}^n)$. Define what it means for the 1-forms $\Phi_1, ...\Phi_n$ to be the *elementary 1-forms on $\mathbb{R}^n$*.

**Question 3.** Do all of the following:

(a) Given an ascending $k$-tuple $I = (i_1, ..., i_k)$ from the set $\{1, ..., n\}$, define what it means for a $k$-form $\Phi_I$ to be an *elementary $k$-form* on $\mathbb{R}^n$.

(b) Show that $\omega$ given by

$$\omega(x) = x_2 x_3 \Phi_{1,2} - x_3 \Phi_{1,3} + x_2 \cos x_1 \Phi_{2,3}$$

for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is a differential form of order 2.

(c) Evaluate $\omega(x)((x, v_1), (x, v_2))$ for $x = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}$.

*Hint: the determinants of* $2 \times 2$ *matrices.*