Quiz 7: Solutions

Question (1)**.** (Munkres, Ex. 1, p 243) *Let* $x, y, z \in \mathbb{R}^5$. *Let*

$$F(x, y, z) = 2x_2y_2z_1 + x_1y_5z_4$$
$$G(x, y) = x_1y_3 + x_3y_1$$
$$h(w) = w_1 - 2w_3$$

(a) *Write AF and AG in terms of elementary alternating tensors, where* $A : \mathcal{L}^k(V) \to \mathcal{A}^k(V)$ *is defined by*

$$Af := \sum_{\sigma \in \text{ sym } (k)} (\text{sgn } \sigma) T_\sigma f \quad \text{for any } f \in \mathcal{L}^k(V)$$

$$\begin{aligned} F =& 2\Phi_{2,2,1} + \Phi_{1,5,4} \\ =& 2\phi_2 \otimes \phi_2 \otimes \phi_1 + \phi_1 \otimes \phi_5 \otimes \phi_4 \quad \text{(by Munkres, Thm 26.4, p. 224)} \end{aligned}$$

$$\begin{aligned} AF =& 2A(\phi_2 \otimes \phi_2 \otimes \phi_1) + A(\phi_1 \otimes \phi_5 \otimes \phi_4) \\ =& 2\phi_2 \wedge \phi_2 \wedge \phi_1 + \phi_1 \wedge \phi_5 \wedge \phi_4 \quad \text{(by Munkres, p. 241, Step 9)} \\ =& -\phi_1 \wedge \phi_4 \wedge \phi_5 \quad \text{(by anticommutativity of the wedge product)} \\ =& -\Psi_{1,4,5} \quad \text{(by Munkres, Thm 28.1(5), p. 237)} \end{aligned}$$

where $\Psi_{1,4,5}$ is the corresponding elementary alternating tensor.

Similarly, $G = \Phi_{1,3} + \Phi_{3,1}$ and $AG = \Psi_{1,3} + \Psi_{3,1}$.

(b) *Express* $(AF) \wedge h$ *in terms of elementary alternating tensors.*

$$\begin{aligned} AF \wedge h =& (-\phi_1 \wedge \phi_4 \wedge \phi_5) \wedge (\phi_1 - 2\phi_3) \\ =& 2\phi_1 \wedge \phi_4 \wedge \phi_5 \wedge \phi_3 \\ =& 2\phi_1 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5 \\ =& 2\Phi_{1,3,4,5} \end{aligned}$$

(c) *Express* $(AF)(x, y, z)$ *as a function.*

By Theorem 27.7, Munkres,

$$AF = -\Psi_{1,4,5} = det \begin{vmatrix} x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \end{vmatrix}$$

Question (3)**.** *Show that ω given by $\omega(x) = x_2x_3\Phi_{1,2} - x_3\Phi_{1,3} + x_2 \cos x_1 \Phi_{2,3}$ for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is a differential form of order 2. Evaluate* $\omega(x)((x, v_1), (x, v_2))$ *for* $x = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}$.

$\omega \in C^\infty$ since each of its component functions is C^∞ (Lemma 29.2, Munkres, p 250). Once x is fixed $\omega(x) \in \mathcal{A}^k(\mathbb{R}^n) \in U \subseteq \mathbb{R}^n$ since it is a linear combination of the alternating tensors $\Psi_I(x)$.

Lastly, we have

$$\omega(x)((x, v_1), (x, v_2)) = \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 3 & -7 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 3 & -7 \end{vmatrix} = -7.$$