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SHORT COMMUNICATION

An FPTAS for minimizing the product of two non-negative linear cost functions

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Abstract We consider a quadratic programming (QP) problem (Π) of the form $\min x^T Cx$ subject to $Ax \geq b, x \geq 0$ where $C \in \mathbb{R}_+^{n\times n}$, $\text{rank}(C) = 1$ and $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. We present an fully polynomial time approximation scheme (FPTAS) for this problem by reformulating the QP (Π) as a parameterized LP and "rounding" the optimal solution. Furthermore, our algorithm returns an extreme point solution of the polytope. Therefore, our results apply directly to 0–1 problems for which the convex hull of feasible integer solutions is known such as spanning tree, matchings and sub-modular flows. They also apply to problems for which the convex hull of the dominant of the feasible integer solutions is known such as s, t-shortest paths and s, t-min-cuts. For the above discrete problems, the quadratic program Π models the problem of obtaining an integer solution that minimizes the *product* of two linear non-negative cost functions.

Keywords Quadratic programming · Approximation schemes · Combinatorial optimization

Mathematics Subject Classification (2000) 90C20 · 90C26 · 90C27 · 90C29

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1 Introduction

In this paper, we consider the following special case of the non-convex quadratic programming (QP) problem (Π).

$$\begin{aligned} \min\ & x^T C x \\ & Ax \geq b \\ & x \geq 0, \end{aligned}$$

where $C \in \mathbb{R}_+^{n \times n}$, $\text{rank}(C) = 1$ and $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and let $P = \{x \in \mathbb{R}_+^n | Ax \geq b\}$. Since a rank-1 matrix is not necessarily positive semi-definite, $x^T C x$ is a non-convex function and thus, Π is a non-convex QP problem. We can rewrite $C = c_1 c_2^T$ for some $c_1, c_2 \in \mathbb{R}_+^n$ as $C \in \mathbb{R}_+^{n \times n}$ and $\text{rank}(C) = 1$. Therefore, the objective function $x^T C x$ can be reformulated as a product of two non-negative linear functions,

$$x^T (c_1 c_2^T) x = (c_1^T x) \cdot (c_2^T x),$$

and the problem Π models the problem of minimizing the product of two non-negative linear cost functions over a polyhedral set. This problem is known to be NP-hard [6].

Definition 1 A function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-concave if and only if the upper level sets

$$S_u = \{x \in \mathbb{R}^n | f(x) \geq u\},$$

are convex for all $u \in \mathbb{R}$.

It is well known that the objective function $(c_1^T x) \cdot (c_2^T x)$ in problem Π is *quasi-concave* where $c_1, c_2 \in \mathbb{R}_+^n$. Therefore, the minimum for the problem Π is attained at an extreme point of the polytope P [1].

In this paper, we present a fully polynomial time approximation scheme (FPTAS) for the problem Π that obtains $(1 + \epsilon)$-approximate solution for any $\epsilon > 0$ in time polynomial in the input size and $\frac{1}{\epsilon}$.

Theorem 1 *Given a rank-1 matrix $C = c_1 c_2^T$, $c_1, c_2 \in \mathbb{R}_+^n$, a polytope $P \subset \mathbb{R}_+^n$ and $\epsilon > 0$, there is a $(1 + \epsilon)$-approximation algorithm $\mathcal{A}$ for the problem Π,*

$$\min_{x \in P} x^T C x = \min_{x \in P} (c_1^T x) \cdot (c_2^T x),$$

in time polynomial in the input size and $\frac{1}{\epsilon}$. Furthermore, $\mathcal{A}$ returns a solution that is an extreme point of P. If $u = \max_{x \in P} c_2^T x$ and $l = \min_{x \in P} c_2^T x$, then $\mathcal{A}$ solves $O\left(\frac{\log \frac{u}{l}}{\epsilon}\right)$ linear minimization problems over P.

We would like to note that an FPTAS for this problem is known due to Kern and Woeginger [5]. However, our work is independent of [5] and our algorithm differs significantly giving an interesting alternate approach to solve the problem with a reduced running time. To compare, the algorithm presented in [5] solves $\Omega\left(\frac{\log \frac{U}{L}}{\epsilon^2} \log\det(A)\right)$ linear minimization problems over P to obtain a $(1+\epsilon)$-approximation where U and L are upper and lower bounds on the objective function respectively.

Since our algorithm obtains an extreme point approximate solution for the problem Π, our result applies directly to the problem of minimizing a rank-1 quadratic objective over a set of 0–1 points when the description of the convex hull or the dominant of the 0–1 points is known.

Corollary 1 *Let $S \subset \{0,1\}^n$, $c_1 \in \mathbb{R}^n_+$, $c_2 \in \mathbb{R}^n_+$. If we can optimize over either the convex hull of S or the dominant of S (i.e. $dom(S) = \{x \in \{0,1\}^n | \exists x' \in S, x \geq x'\}$) in polynomial time, then there is an FPTAS for the problem $\Pi : \min_{x \in S}(c_1^T x) \cdot (c_2^T x)$.*

In particular, we obtain an FPTAS for minimizing the rank-1 quadratic objective when S is one of the following combinatorial sets over an undirected graph $G = (V, E)$ using Corollary 1.

1. Spanning trees of G.
2. Matchings in G.
3. s, t-paths in G where $s, t \in V$.
4. s, t-cuts in G where $s, t \in V$.

1.1 Related work

1.1.1 General QPs

The general QP problem is the following.

$$\min_{x \in \mathbb{R}^n} f(x) = (a^T x + x^T C x) \text{ subject to } Ax \geq b.$$

Here $a \in \mathbb{R}^n$, $C \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. It is known that the objective function f is convex if and only if the matrix C is positive semi-definite. The problem is referred to as a convex QP if the objective is convex and can be solved in polynomial time. On the other hand, if f is not convex, the problem is referred to as non-convex QP and is in general NP-hard to solve [7,9]. The non-convex QP problem has been studied widely in literature and finds important applications in numerous fields such as portfolio analysis, VLSI design, optimal power flow and economic dispatch. The bibliography of Gould and Toint [3] is an extensive list of references in non-convex QP and its applications.

1.1.2 Bi-criteria approximations

Bi-criteria variants of spanning tree and s, t-path problems have been considered in literature [8,4] where we are given two non-negative cost functions c_1 and c_2 over edges and the goal is to find a spanning tree (or a s,t-path) that minimizes cost c_1 subject to a budget constraint on cost c_2. Ravi and Goemans [8] give a bi-criteria $(1, 1+\epsilon)$-approximation for any fixed $\epsilon > 0$ for the spanning tree problem [the algorithm outputs a tree with optimal c_1 cost while violating the budget constraint by a factor $(1+\epsilon)$]. Hassin [4] gives a similar bi-criteria $(1, 1+\epsilon)$-approximation for the shortest path problem. While these bi-criteria approximation algorithms can be adapted to give a PTAS (though not an FPTAS) for the product objective version for the spanning tree and shortest path problems, they are specific to the spanning tree and the shortest path problems and can not be generalized.

2 $(1+\epsilon)$-approximation algorithm

We solve the problem Π via a parametric approach. Consider the following parametric problem $\Pi(B)$ where B is a given parameter.

$$\min c_1^T x \tag{1}$$

$$c_2^T x \leq B \tag{2}$$

$$x \in P \tag{3}$$

The following lemma follows directly from the properties of a basic feasible solution of a polytope (see Brønsted [2]).

Lemma 1 *Let $\tilde{x}(B)$ be a basic optimal solution of $\Pi(B)$ for any $B > 0$. Then $\tilde{x}(B)$ can be written as a convex combination of at most two extreme points of polytope P.*

Consider a basic optimal solution $\tilde{x}(B)$ of $\Pi(B)$ for any $B > 0$. From the above lemma, we know that $\tilde{x}(B)$ can be written as a convex combination of at most two extreme points of P (say $x^1, x^2 \in extr(P)$). Let $f(x) = (c_1^T x)\cdot(c_2^T x)$ for any $x \in \mathbb{R}^n_+$. It is known that the function f is quasi-concave and is minimized at an extreme point of the feasible set [1]. Therefore, the following problem:

$$\min\{f(x) |\ x = \alpha x^1 + (1-\alpha)x^2, 0 \leq \alpha \leq 1\},$$

is minimized at either x^1 or x^2 and we have the following lemma.

Lemma 2 *Let $\tilde{x}(B)$ be a basic optimal solution for $\Pi(B)$ for some $B > 0$. We can efficiently find an extreme point $x \in extr(P)$ such that*

$$(c_1^T x) \cdot (c_2^T x) \leq (c_1^T \tilde{x}(B)) \cdot (c_2^T \tilde{x}(B)) \leq (c_1^T \tilde{x}(B)) \cdot B. \tag{4}$$

Since we do not know the value of parameter B, we try different powers of $(1+\epsilon)$ and solve the parametric problem $\Pi(B)$ for each value of B.

Proof of Theorem 1 Let x^* be an optimal solution for the problem Π. There exists $j \in \mathbb{N}$ such that

$$(1+\epsilon)^{j-1} \leq c_2^T x^* < (1+\epsilon)^j.$$

Consider the problem $\Pi(B)$ for $B = (1+\epsilon)^j$ and let $\tilde{x}(B)$ be a basic optimal solution for $\Pi(B)$. Clearly, $c_1^T \tilde{x}(B) \le c_1^T x^*$ as x^* is a feasible solution for $\Pi(B)$. From Lemma 2, we can find $x \in extr(P)$ such that

$$c_1^T x \cdot c_2^T x \le c_1^T \tilde{x}(B) \cdot B \le c_1^T x^* \cdot B \le c_1^T x^* \cdot c_2^T x^* (1+\epsilon).$$

We consider different values of the parameter B between l and u in powers of $(1+\epsilon)$. For each value of the parameter, we solve a linear minimization problem over P with an additional budget constraint over cost c_2. Therefore, we solve $O\left(\log_{(1+\epsilon)} \frac{u}{l}\right) \simeq O\left(\frac{\log \frac{u}{l}}{\epsilon}\right)$ different minimization problems (we can assume $l \neq 0$ since if $l = 0$, the optimum is 0 and this case can be easily checked). $\square$

2.1 Comparison with Kern and Woeginger [5]

The algorithm in [5] searches for the optimal objective value in powers of $(1+\epsilon)$. For each possible objective function value (say λ), the authors solve $O(\log_{(1+\epsilon)} \det A)$ linear minimization problems over P and output the best solution over all objective values. Hence, the total number of LPs solved by their method is $O\left(\log_{(1+\epsilon)} \frac{U}{L} \cdot \log_{(1+\epsilon)} \det A\right) \simeq O\left(\frac{\log \frac{U}{L} \cdot \log \det A}{\epsilon^2}\right)$ where U and L are the upper and lower bounds on the value of the objective function respectively. On the other hand, our algorithm solves $O\left(\log_{(1+\epsilon)} \frac{u}{l}\right) \simeq O\left(\frac{\log \frac{u}{l}}{\epsilon}\right)$ different LPs where u and l are the upper and lower bounds of the cost function c_2 and not the product of c_1 and c_2 cost functions. Furthermore, the running time is linear in $\frac{1}{\epsilon}$ compared to the quadratic in $\frac{1}{\epsilon}$ running time in [5].

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