

A Tight Characterization of the Performance of Static Solutions in Two-stage Adjustable Robust Linear Optimization

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Abstract In this paper, we study the performance of static solutions for two-stage adjustable robust linear optimization problems with uncertain constraint and objective coefficients and give a tight characterization of the adaptivity gap. Computing an optimal adjustable robust optimization problem is often intractable since it requires to compute a solution for all possible realizations of uncertain parameters [16]. On the other hand, a static solution is a single (here and now) solution that is feasible for all possible realizations of the uncertain parameters and can be computed efficiently for most dynamic optimization problems. We show that for a fairly general class of uncertainty sets, a static solution is optimal for the two-stage adjustable robust linear optimization problem. This is highly surprising in view of the usual perception about the conservativeness of static solutions. Furthermore, when a static solution is not optimal for the adjustable robust problem, we give a tight approximation bound on the performance of the static solution that is related to a measure of non-convexity of a transformation of the uncertainty set. We also show that our bound is at least as good (and in many case significantly better) as the bound given by the symmetry of the uncertainty set [9,8].

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1 Introduction

In most real world problems, problem parameters are uncertain at the optimization or decision-making phase and stochastic and robust optimization are two different paradigms to address this parameter uncertainty. In a stochastic optimization approach, we model uncertainty using probability distributions and optimize the expected value of the objective. Stochastic optimization was introduced by Dantzig [13] and Beale [1], and since then has been extensively studied in literature. We refer the reader to several textbooks including Infanger [19], Kall and Wallace [20], Prekopa [22], Shapiro [23], Shapiro et al. [24] and the references therein for a comprehensive review of stochastic optimization. While the stochastic optimization approach has its merits, it is by and large computationally intractable even when the constraint and objective functions are linear. Shapiro and Nemirovski [25] give hardness results for two-stage and multi-stage stochastic optimization problems where they show that the multi-stage stochastic optimization problem is computationally intractable even if approximate solutions are desired. Dyer and Stougie [14] show that a multi-stage stochastic optimization problem where the distribution of uncertain parameters in any stage also depends on the decisions in past stages is PSPACE-hard. Even for the stochastic problems that can be solved efficiently, it is difficult to estimate the probability distributions of the uncertain parameters from historical data.

Robust optimization is another paradigm for dynamic optimization that has been recently considered (see Ben-Tal and Nemirovski [3, 4, 5], El Ghaoui and Lembre [15], Bertsimas and Sim [10, 11], Goldfarb and Iyengar [17]). In a robust optimization approach, the uncertain parameters are assumed to belong to some uncertainty set. The goal is to construct a single (static) solution that is feasible for all possible realizations of the uncertain parameters from the set and optimizes the worst case objective value. We point the reader to the survey by Bertsimas et al. [6] and the book by Ben-Tal et al. [2] and the references therein for an extensive review of the literature in robust optimization. This approach is significantly more tractable and for a large class of problems, the robust problem is equivalent to the corresponding deterministic problem in computational complexity [6, 2]. However, the robust optimization approach is perceived to be highly conservative as it optimizes over the worst-case realization of uncertain parameters, and the solution is not adjustable to the uncertain parameter realization. However, computing an optimal adjustable robust solution is computationally intractable even for two-stage linear optimization problems. In fact, Feige et al. [16] show that it is hard to approximate a two-stage fractional set covering problem within a factor of $\Omega(\log n)$.

In this paper, we study the performance of static solutions for two-stage adjustable robust linear optimization problems under uncertain constraint and objective coefficients. Approximation bounds for the performance of static solutions in two-stage and multi-stage linear optimization problems have been studied in the literature. Bertsimas and Goyal [7] show that if the uncertainty set is perfectly *symmetric* (such as a hypercube or an ellipsoid), a static solution gives a 2-approximation for the adjustable robust linear covering problems under right hand side uncertainty. Bertsimas et al. [9] generalize the result for general convex compact uncertainty sets and show that the performance of the static solution depends on a measure of symmetry of the uncertainty set, introduced in

Minkowski [21]. They also extend the approximation bounds to multi-stage problems. However, the models in [7] and [9] consider linear optimization problems with only covering constraints of the form $\mathbf{a}^T \mathbf{x} \geq b$, $\mathbf{a} \in \mathbb{R}^n$ and $b \geq 0$. Moreover, the uncertainty appears only in the right hand side of the constraints. Therefore, they do not capture many important applications including revenue management and resource allocation problems where we require packing constraints with uncertain constraint coefficient. For instance, in a typical revenue management problem, the goal is to allocate scarce resources to a demand with uncertain resource requirements such that the total revenue is maximized. The constraints in the problem corresponding to resource capacities are packing constraints and the uncertainty related to resource requirements appears in the constraint coefficients.

1.1 Our Models

We consider the following two-stage adjustable robust linear optimization problem, Π_{AR} under uncertain constraint coefficients.

$$\begin{aligned} z_{\text{AR}} = \max \mathbf{c}^T \mathbf{x} + \min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y}(\mathbf{B})} \mathbf{d}^T \mathbf{y}(\mathbf{B}) \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{h} \\ \mathbf{x} \in \mathbb{R}_+^{n_1} \\ \mathbf{y}(\mathbf{B}) \in \mathbb{R}_+^{n_2}, \end{aligned} \quad (1.1)$$

where $\mathbf{A} \in \mathbb{R}^{m \times n_1}$, $\mathbf{c} \in \mathbb{R}_+^{n_1}$, $\mathbf{d} \in \mathbb{R}_+^{n_2}$, $\mathbf{h} \in \mathbb{R}_+^m$. The second-stage constraint matrix $\mathbf{B} \in \mathbb{R}_+^{m \times n_2}$ is uncertain and belongs to a full dimensional compact convex uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n_2}$. The decision variables \mathbf{x} represent the first-stage decisions before the constraint matrix \mathbf{B} is revealed. While $\mathbf{y}(\mathbf{B})$ represent the second-stage or recourse decision variables after observing the uncertain constraint matrix $\mathbf{B} \in \mathcal{U}$. Therefore, the (adjustable) second-stage decisions depend on the uncertainty realization. We can assume without loss of generality that \mathcal{U} is *down-monotone* (see Appendix A), since for any first-stage solution \mathbf{x} , the optimal solution $\mathbf{y}(\mathbf{B})$ for any $\mathbf{B} \in \mathcal{U}$ is also feasible for any $\hat{\mathbf{B}} \in \mathbb{R}_+^{m \times n_2}$ if $\hat{\mathbf{B}} \leq \mathbf{B}$.

We can model a two-stage resource allocation problem under uncertain resource requirements using (1.1) where $\mathbf{h} \in \mathbb{R}_+^m$ represent the resource capacities, \mathbf{A} denote the resource requirements of demands in first-stage, \mathbf{x} denote the first-stage allocation decisions, $\mathbf{B} \in \mathcal{U}$ denote the uncertain resource requirements in second-stage arising from an uncertainty set \mathcal{U} and $\mathbf{y}(\mathbf{B})$ denote the second-stage adjustable resource allocation decisions. However, computing an optimal adjustable robust solution is computationally intractable [16].

The corresponding static robust optimization problem, Π_{Rob} can be formulated as follows.

$$\begin{aligned} z_{\text{Rob}} = \max \mathbf{c}^T \mathbf{x} + \mathbf{d}^T \mathbf{y} \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \leq \mathbf{h}, \forall \mathbf{B} \in \mathcal{U} \\ \mathbf{x} \in \mathbb{R}_+^{n_1} \\ \mathbf{y} \in \mathbb{R}_+^{n_2}. \end{aligned} \quad (1.2)$$

Note that the second-stage solution \mathbf{y} is static and does not depend on the realization of uncertain \mathbf{B} . Both first-stage and second-stage decisions \mathbf{x} and \mathbf{y} are

selected before the second-stage uncertain constraint matrix is known and (\mathbf{x}, \mathbf{y}) is feasible for all $\mathbf{B} \in \mathcal{U}$. We can show that an optimal static robust solution to (1.2) can be computed efficiently and our goal is compare the performance of an optimal static robust solution with respect to the optimal adjustable robust solution of (1.1).

The above models have been considered in the literature. Ben-Tal and Nemirovski [4] consider the corresponding one-stage problem under uncertain constraint coefficients. They show that a static solution is optimal if the uncertainty set, \mathcal{U} is *constraint-wise* where each constraint $i = 1, \dots, m$ can be selected independently from a compact convex set \mathcal{U}_i . In other words, \mathcal{U} is a Cartesian product of $\mathcal{U}_i, i = 1, \dots, m$, i.e.,

$$\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \dots \times \mathcal{U}_m.$$

However, the authors do not discuss performance of static solutions if the constraint-wise condition on \mathcal{U} is not satisfied. Bertsimas and Goyal [8] consider a general multi-stage convex optimization problem under uncertain constraints and objective functions and show that the performance of the static solution is related to the symmetry of the uncertainty set, \mathcal{U} . However, the symmetry bound in [8] can be quite loose in many instances. For example, consider the case when \mathcal{U} is constraint-wise where each $\mathcal{U}_i, i = 1, \dots, m$ is a simplex, i.e.,

$$\mathcal{U}_i = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{e}^T \mathbf{x} \leq 1\}.$$

The symmetry of \mathcal{U} is $O(1/n)$ [9] and the results in [8] imply an approximation bound of $\Omega(n)$. While from Ben-Tal and Nemirovski [4], we know that a static solution is optimal.

1.2 Our contributions

In this paper, we analyze the performance of static solutions as compared to an optimal fully-adjustable solution for two-stage adjustable robust linear optimization problems under constraint and objective uncertainty. Our main contributions are summarized below.

Optimality of static solution. We give a tight characterization of the conditions under which a static solution is optimal for the two-stage adjustable robust problem. The optimality of static solutions depends on the geometric properties of a transformation of the uncertainty set. In particular, we show that the static solution is optimal if and only if the transformation of \mathcal{U} is convex. If \mathcal{U} is a constraint-wise set, we show that the transformation of \mathcal{U} is convex. Therefore, our result generalizes the result in Ben-Tal and Nemirovski [4] where the authors show that a static solution is optimal for one-stage adjustable robust problem if \mathcal{U} is a constraint-wise set. We also present other families of uncertainty sets for which the transformation is convex.

This result is quite surprising as the worst-case choice of $\mathbf{B} \in \mathcal{U}$ usually depends on the first-stage solution even if \mathcal{U} is constraint-wise unless when \mathcal{U} is a hypercube. For the case of hypercube, each uncertain element can be selected independently from an interval and in that case, the worst-case \mathbf{B} is independent of the first-stage decision. However, a constraint-wise set is a Cartesian product of general

convex sets. We show that if the transformation of \mathcal{U} is convex, there is an optimal recourse solution for the worst-case choice of $\mathbf{B} \in \mathcal{U}$ that is feasible for all $\mathbf{B} \in \mathcal{U}$. Furthermore, we show that our result can also be interpreted as the following min-max theorem.

$$\min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y} \geq \mathbf{0}} \{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h}\} = \max_{\mathbf{y} \geq \mathbf{0}} \min_{\mathbf{B} \in \mathcal{U}} \{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h}\}.$$

The inner minimization on the max-min problem implies that the solution \mathbf{y} must be feasible for all $\mathbf{B} \in \mathcal{U}$ and therefore, is a static robust solution. We would like to note that the above min-max result does not follow from the general saddle-point theorem [12].

Approximation bounds for the static solution. We give a tight approximation bound on the performance of the optimal static solution for the adjustable robust problem when the transformation of \mathcal{U} is not convex and the static solution is not optimal. We relate the performance of static solutions to a natural measure of non-convexity of the transformation of \mathcal{U} . We also present a family of uncertainty sets and instances where we show that the approximation bound is tight, i.e., the ratio of the optimal objective value of the adjustable robust problem (1.1) and the optimal objective value of the robust problem (1.2) is exactly equal to the bound given by the measure of non-convexity.

We also compare our approximation bounds with the bound in Bertsimas and Goyal [8] where the authors relate the performance of the static solutions with the symmetry of the uncertainty set. We show that our bound is stronger than the symmetry bound in [8]. In particular, for any instance, we can show that our bound is at least as good as the symmetry bound, and in fact in many cases, significantly better. For instance, consider the following uncertainty set

$$\mathcal{U} = \left\{ \mathbf{B} \in \mathbb{R}_+^{m \times n} \mid \sum_{i,j} B_{ij} \leq 1 \right\}.$$

In this case, $\text{sym}(\mathcal{U}) = 1/mn$ [9] and the symmetry bound is $\Omega(mn)$. However, our bound is 1, i.e., a static solution is optimal for the adjustable robust problem.

Models with both constraint and objective uncertainty. We extend our result to two-stage models where both constraint and objective coefficients are uncertain. In particular, we consider a two-stage model where the uncertainty in the second-stage constraint matrix, \mathbf{B} is independent of the uncertainty in the second-stage objective \mathbf{d} . Therefore, (\mathbf{B}, \mathbf{d}) belong to a convex compact uncertainty set \mathcal{U} that is a Cartesian product of the uncertainty set of constraint matrices, \mathcal{U}^B and uncertainty set of second-stage objective, \mathcal{U}^d .

We show that our results for the model with only constraint coefficient uncertainty can also be extended to this case of both constraint and objective uncertainty. In particular, we show that a static solution is optimal if and only if the transformation of \mathcal{U}^B is convex. Furthermore, if the transformation is not convex, then the approximation bound on the performance of the optimal static solution is related to the measure of non-convexity of the transformation of \mathcal{U}^B . Surprisingly, the approximation bound or the optimality of a static solution does not depend on the uncertainty set of objectives, \mathcal{U}^d . If the transformation of \mathcal{U}^B is convex, a

static solution is optimal for all convex compact uncertainty sets $\mathcal{U}^d \subseteq \mathbb{R}_+^{n_2}$. We also present a family of examples to show that our bound is tight for this case as well.

We also consider a two-stage adjustable robust model where in addition to the second-stage constraint matrix, \mathbf{B} and objective, \mathbf{d} , the right hand side \mathbf{h} of the constraints is also uncertain and

$$(\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U} = \mathcal{U}^{B,h} \times \mathcal{U}^d,$$

where \mathcal{U} is a convex compact set that is a Cartesian product of uncertainty set for (\mathbf{B}, \mathbf{h}) and uncertainty set for \mathbf{d} . For this case, we give a sufficient condition for the optimality of a static solution that is related to the convexity of the transformation of uncertainty set $\mathcal{U}^{B,h}$. Note again that the optimality of a static solution does not depend on the uncertainty set of objectives, \mathcal{U}^d . However, our approximation bounds do not extend for this case if the transformation of $\mathcal{U}^{B,h}$ is not convex.

Outline. The rest of the paper is organized as follows: In Section 2, we discuss the optimality of static solutions for the two-stage adjustable robust problem under constraint uncertainty and relate it to the convexity of an appropriate transformation of the uncertainty set. In Section 3, we introduce a measure of non-convexity for any compact set and in Section 4, we present a tight approximation bound for the performance of an optimal static solution for the adjustable robust problem, that is related to the measure of non-convexity of the transformation of the uncertainty set. In Section 5, we extend our result to two-stage models where both second-stage constraint and objective are uncertain.

2 Optimality of static solutions

In this section, we present a tight characterization of the conditions under which a static robust solution computed from (1.2) is optimal for the adjustable robust problem (1.1). We introduce a transformation of the uncertainty set \mathcal{U} and relate the optimality of a static solution to the convexity of the transformation.

An optimal static solution for (1.2) can be computed efficiently. Note that a static solution (\mathbf{x}, \mathbf{y}) is feasible for all $\mathbf{B} \in \mathcal{U}$. To observe that an optimal static robust solution can be computed in polynomial time, consider the separation problem: given a solution \mathbf{x}, \mathbf{y} , we need to decide whether or not there exists $\mathbf{B} \in \mathcal{U}$ and $j \in \{1, \dots, m\}$ such that

$$\mathbf{e}_j^T (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}) > h_j,$$

and find a separating hyperplane if (\mathbf{x}, \mathbf{y}) is not feasible. Therefore, by solving m linear optimization problems over \mathcal{U} we can decide whether the given solution is feasible or obtain a separating hyperplane. From the equivalence of the separation and optimization [18], we can compute an optimal static robust solution in polynomial time. In fact, in Appendix B, we give a compact linear formulation to compute the optimal static solution for Π_{Rob} .

We can easily see that the static solution is a lower bound of the optimal value of the adjustable robust problem. Suppose $(\mathbf{x}^*, \mathbf{y}^*)$ is an optimal solution for Π_{Rob} . Then, $\mathbf{x} = \mathbf{x}^*, \mathbf{y}(\mathbf{B}) = \mathbf{y}^*$ for all $\mathbf{B} \in \mathcal{U}$ is feasible for Π_{AR} . Therefore,

$$z_{\text{AR}} \geq z_{\text{Rob}}. \tag{2.1}$$

We would like to study the conditions under which $z_{\text{AR}} \leq z_{\text{Rob}}$. Suppose $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}))$ for all $\mathbf{B} \in \mathcal{U}$ is a fully-adjustable optimal solution for Π_{AR} . Then

$$\begin{aligned} z_{\text{AR}} &= \mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y}(\mathbf{B})} \left\{ \mathbf{d}^T \mathbf{y}(\mathbf{B}) \mid \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} \\ z_{\text{Rob}} &\geq \mathbf{c}^T \mathbf{x}^* + \max_{\mathbf{y} \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall \mathbf{B} \in \mathcal{U} \right\}. \end{aligned}$$

Note that $\mathbf{h} - \mathbf{A}\mathbf{x}^* \geq \mathbf{0}$, since otherwise the second-stage problem becomes infeasible for Π_{AR} . In fact, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$. Otherwise, it is easy to see that $z_{\text{AR}} = z_{\text{Rob}}$: suppose $(h - Ax^*)_i = 0$ for some i . Since \mathcal{U} is a full-dimensional convex set, we can find $\hat{\mathbf{B}} \in \mathcal{U}$ such that $B_{ij} > 0$ for all $j = 1, \dots, n_2$. Therefore,

$$\min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y}(\mathbf{B}) \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y}(\mathbf{B}) \mid \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} \leq \max_{\mathbf{y} \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y} \mid \hat{\mathbf{B}}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} = 0,$$

which implies $z_{\text{AR}} = z_{\text{Rob}}$. Therefore, we need to study conditions under which

$$\max_{\mathbf{y} \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall \mathbf{B} \in \mathcal{U} \right\} \geq \min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y} \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\}, \quad (2.2)$$

where $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$.

2.1 One-stage models

Motivated by (2.2), we consider the following one-stage adjustable robust problem $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$:

$$\begin{aligned} z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}) &= \min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y}} \mathbf{d}^T \mathbf{y} \\ &\quad \mathbf{B}\mathbf{y} \leq \mathbf{h} \\ &\quad \mathbf{y} \in \mathbb{R}_+^n, \end{aligned} \quad (2.3)$$

where $\mathbf{h} \in \mathbb{R}_+^m$, $\mathbf{d} \in \mathbb{R}_+^n$ and $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$ is the convex, compact and down-monotone uncertainty set. The corresponding one-stage static robust problem $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ can be formulated as follows:

$$\begin{aligned} z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) &= \max_{\mathbf{y}} \mathbf{d}^T \mathbf{y} \\ &\quad \mathbf{B}\mathbf{y} \leq \mathbf{h}, \forall \mathbf{B} \in \mathcal{U} \\ &\quad \mathbf{y} \in \mathbb{R}_+^n. \end{aligned} \quad (2.4)$$

Ben-Tal and Nemirovski [4] study these one-stage models and show that the solution of $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ is optimal if the uncertainty set \mathcal{U} is a Cartesian product of its projection onto its row vector spaces.

Consider $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ as defined in (2.3). We can write the dual problem of the inner maximization problem.

$$z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}) = \min \{ \mathbf{h}^T \boldsymbol{\alpha} \mid \mathbf{B}^T \boldsymbol{\alpha} \geq \mathbf{d}, \mathbf{B} \in \mathcal{U}, \boldsymbol{\alpha} \in \mathbb{R}_+^m \}.$$

Substituting $\lambda = \mathbf{h}^T \boldsymbol{\alpha}$ and $\boldsymbol{\alpha} = \lambda \boldsymbol{\mu}$, we can reformulate $z_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ as follows.

$$z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}) = \min \{ \lambda \mid \lambda \mathbf{B}^T \boldsymbol{\mu} \geq \mathbf{d}, \mathbf{h}^T \boldsymbol{\mu} = 1, \mathbf{B} \in \mathcal{U}, \boldsymbol{\mu} \in \mathbb{R}_+^m \}. \quad (2.5)$$

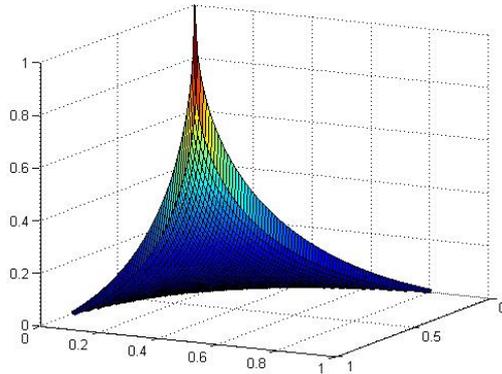


Fig. 1: The boundary of the set $T(\mathcal{U}, \mathbf{e})$ when $n = 3$.

2.2 Transformation of \mathcal{U}

Motivated from the formulation (2.5), we define the following transformation $T(\mathcal{U}, \mathbf{h})$ of the uncertainty set $\mathcal{U} \in \mathbb{R}_+^{m \times n}$ and $\mathbf{h} > \mathbf{0}$.

$$T(\mathcal{U}, \mathbf{h}) = \left\{ \mathbf{B}^T \boldsymbol{\mu} \mid \mathbf{h}^T \boldsymbol{\mu} = 1, \mathbf{B} \in \mathcal{U}, \boldsymbol{\mu} \geq \mathbf{0} \right\}. \quad (2.6)$$

For instance, if $\mathbf{h} = \mathbf{e}$, then $T(\mathcal{U}, \mathbf{e})$ is the set of all convex combinations of rows of $\mathbf{B} \in \mathcal{U}$ for all $\mathbf{B} \in \mathcal{U}$. Note that $T(\mathcal{U}, \mathbf{e})$ is not necessarily convex in general. We discuss several examples below to illustrate properties of $T(\mathcal{U}, \mathbf{h})$.

Example 1 (\mathcal{U} where $T(\mathcal{U}, \mathbf{h})$ is non-convex). Consider the following uncertainty set \mathcal{U} :

$$\mathcal{U} = \left\{ \mathbf{B} \in [0, 1]^{n \times n} \mid B_{ij} = 0, \forall i \neq j, \sum_{j=1}^n B_{jj} \leq 1 \right\}. \quad (2.7)$$

$T(\mathcal{U}, \mathbf{e})$ is non-convex. Figure 1 illustrates $T(\mathcal{U}, \mathbf{e})$ when $n = 3$. In fact, in Theorem 2, we prove that $T(\mathcal{U}, \mathbf{h})$ is non-convex for all $\mathbf{h} > \mathbf{0}$. On the other hand, in the following two lemmas, we show that $T(\mathcal{U}, \mathbf{h})$ can be convex for all $\mathbf{h} > \mathbf{0}$ for some interesting families of examples.

Example 2 (Constraint-wise uncertainty set). Suppose the uncertainty set \mathcal{U} is constraint-wise where each constraint $i = 1, \dots, m$ can be selected independently from a compact convex set \mathcal{U}_i . In other words, \mathcal{U} is a Cartesian product of $\mathcal{U}_i, i = 1, \dots, m$, i.e.,

$$\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \dots \times \mathcal{U}_m,$$

then $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. In particular, we have the following lemma.

Lemma 1 *Suppose the convex compact uncertainty set \mathcal{U} is a Cartesian product of its row vector spaces of the following form:*

$$\mathcal{U} = \{ \mathbf{B} \mid \mathbf{B}^T \mathbf{e}_j \in \mathcal{U}_j \},$$

where \mathcal{U}_j is a compact convex set in \mathbb{R}_+^n . Then $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$.

We provide a detailed proof of Lemma 1 in Appendix C. In Ben-Tal and Nemirovski [4], the authors show a static solution is optimal for the one-stage adjustable robust problem if \mathcal{U} is constraint-wise. In later discussion, we show that a static solution is optimal if and only if $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$; thereby generalizing the result in [4]. Note that constraint-wise uncertainty is quite analogous to independence in distributions for stochastic optimization problems.

Example 3 (Symmetric projections). Suppose the uncertainty set \mathcal{U} has symmetric projections, i.e., the projections of \mathcal{U} onto each of its row vector spaces are the same, then $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. In particular, we have the following lemma.

Lemma 2 Consider any convex compact uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$. For any $j = 1, \dots, m$, let

$$\mathcal{U}_j = \left\{ \mathbf{b} \mid \exists \mathbf{B} \in \mathcal{U}, \mathbf{b} = \mathbf{B}^T \mathbf{e}_j \right\}.$$

Suppose \mathcal{U} is such that $\mathcal{U}_i = \mathcal{U}_j$ for all $i, j \in \{1, \dots, m\}$. Then $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$.

We provide a proof of Lemma 2 in Appendix C.

We would like to note that a permutation invariant set is an important class of uncertainty set that is a special case of having symmetric projections. We define a set $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$ to be permutation invariant if for any $\mathbf{B} \in \mathcal{U}$ and any permutation σ of $[m]$, the matrix obtained by permuting the rows of \mathbf{B} , say \mathbf{B}^σ also belongs to \mathcal{U} . For any permutation invariant \mathcal{U} ,

$$\mathbf{b} \in \mathcal{U}_j \text{ for some } j = 1, \dots, m \Rightarrow \mathbf{b} \in \mathcal{U}_i, \forall i = 1, \dots, m.$$

This implies that \mathcal{U} has symmetric projections and $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$.

2.3 Main Theorem

Now, we introduce the main theorem which gives a tight characterization of the optimality of the static solution for the two-stage adjustable robust problem.

Theorem 1 (Optimality of Static Solutions) Let z_{AR} be the objective of the two-stage adjustable robust problem Π_{AR} defined in (1.1) and z_{Rob} be that of Π_{Rob} defined in (1.2). Then, $z_{\text{AR}} = z_{\text{Rob}}$ if and only if $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$.

Note that $z_{\text{AR}} = z_{\text{Rob}}$ implies that the optimal static robust solution for Π_{Rob} is also optimal for the adjustable robust problem, Π_{AR} . In order to prove Theorem 1, we first reformulate $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ and $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ as optimization problems over $T(\mathcal{U}, \mathbf{h})$. From (2.5) and the definition of $T(\mathcal{U}, \mathbf{h})$, we have the following lemma.

Lemma 3 Given $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$ and $\mathbf{h} > \mathbf{0}$, the one-stage adjustable robust problem $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ defined in (2.3) can be formulated as

$$z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}) = \min \{ \lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \mathbf{b} \in T(\mathcal{U}, \mathbf{h}) \}. \quad (2.8)$$

We can also reformulate $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ as an optimization problem over $\text{conv}(T(\mathcal{U}, \mathbf{h}))$ as follows.

Lemma 4 *Given $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$ and $\mathbf{h} > \mathbf{0}$, the one-stage static robust problem $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ defined in (2.4) can be formulated as*

$$z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) = \min \{ \lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \text{conv}(T(\mathcal{U}, \mathbf{h})) \}. \quad (2.9)$$

Proof Suppose

$$\mathcal{U} = \text{conv}(\mathbf{B}_1, \dots, \mathbf{B}_K)$$

where $\mathbf{B}_j \in \mathcal{U}, j = 1, \dots, K$ are the extreme points of \mathcal{U} . Let $\mathbf{b}_i^j = \mathbf{B}_j^T \mathbf{e}_i$, we can rewrite (2.4) as follows.

$$z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) = \max \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}_j \mathbf{y} \leq \mathbf{h}, \forall j = 1, \dots, K, \mathbf{y} \in \mathbb{R}_+^n \right\}.$$

Again, by writing the dual problem, we have

$$z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) = \min \left\{ \sum_{j=1}^K \mathbf{h}^T \boldsymbol{\alpha}^j \mid \sum_{j=1}^K \mathbf{B}_j^T \boldsymbol{\alpha}^j \geq \mathbf{d}, \boldsymbol{\alpha}^j \in \mathbb{R}_+^m \right\}$$

Denote $\theta_j = \mathbf{h}^T \boldsymbol{\alpha}^j$, $\lambda = \mathbf{e}^T \boldsymbol{\theta}$, we have

$$\begin{aligned} z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) &= \min \left\{ \lambda \mid \lambda \sum_{j=1}^K \frac{\theta_j}{\lambda} \mathbf{B}_j^T \frac{\boldsymbol{\alpha}^j}{\theta_j} \geq \mathbf{d}, \boldsymbol{\alpha}^j \geq \mathbf{0} \right\} \\ &= \min \left\{ \lambda \mid \lambda \sum_{j=1}^K \frac{\theta_j}{\lambda} \hat{\mathbf{b}}_j \geq \mathbf{d}, \hat{\mathbf{b}}_j \in T(\mathcal{U}, \mathbf{h}) \right\} \\ &= \min \{ \lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \text{conv}(T(\mathcal{U}, \mathbf{h})) \} \end{aligned}$$

where the second equality holds because $\mathbf{h}^T \frac{\boldsymbol{\alpha}^j}{\theta_j} = 1, j = 1, \dots, K$. \square

Note that the formulations (2.8) and (2.9) are bilinear optimization problems over $T(\mathcal{U}, \mathbf{h})$ and not necessarily convex even if $T(\mathcal{U}, \mathbf{h})$ is convex. However, the reformulations provide interesting geometric insights about the relation between the adjustable robust and static robust problems with respect to properties of \mathcal{U} . Figure 2 illustrates the geometric interpretation of $z_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ and $z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ in terms of the formulation in Lemma 3 and 4. Now, we are ready to prove Theorem 1.

Proof of Theorem 1 Suppose $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. Let $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}), \mathbf{B} \in \mathcal{U})$ be an optimal fully-adjustable solution to Π_{AR} . Then

$$\begin{aligned} z_{\text{AR}} &= \mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y}(\mathbf{B}) \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y}(\mathbf{B}) \mid \mathbf{B} \mathbf{y}(\mathbf{B}) \leq \mathbf{h} - \mathbf{A} \mathbf{x}^* \right\} \\ &= \mathbf{c}^T \mathbf{x}^* + z_{\text{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A} \mathbf{x}^*), \end{aligned}$$

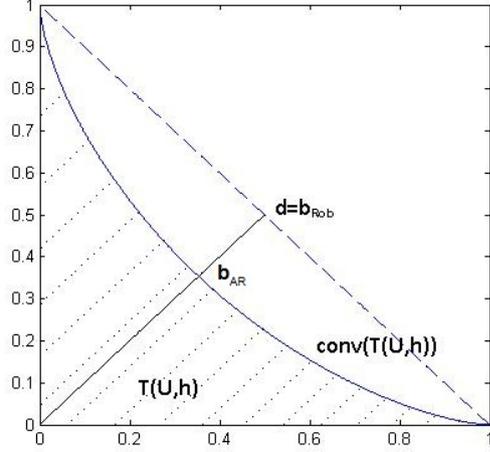


Fig. 2: A geometric illustration of $z_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ and $z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ when $\mathbf{d} = \frac{1}{2}\mathbf{e}$: For $z_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$, the optimal solution \mathbf{b} is the point where \mathbf{d} intersects with the boundary of $T(\mathcal{U}, \mathbf{h})$, while for $z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$, the optimal solution is $\mathbf{b} = \mathbf{d}$ since $\mathbf{d} \in \text{conv}(T(\mathcal{U}, \mathbf{h}))$.

where the second equation follows from (2.3). We can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$ as discussed earlier. Now,

$$\begin{aligned}
z_{\text{Rob}} &\geq \mathbf{c}^T \mathbf{x}^* + \max_{\mathbf{y} \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall \mathbf{B} \in \mathcal{U} \right\} \\
&= \mathbf{c}^T \mathbf{x}^* + z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\
&= \mathbf{c}^T \mathbf{x}^* + z_{\text{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\
&= z_{\text{AR}},
\end{aligned} \tag{2.10}$$

where the first inequality follows as \mathbf{x}^* is a feasible first-stage solution for the static robust problem. The second equation follows from (2.4). Equation (2.10) follows from Lemma 3 and Lemma 4 and the fact that $T(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*)$ is convex. Also, from (2.1) we know that $z_{\text{AR}} \geq z_{\text{Rob}}$ which implies $z_{\text{AR}} = z_{\text{Rob}}$.

Conversely, suppose $z_{\text{AR}} = z_{\text{Rob}}$. For the sake of contradiction, assume $T(\mathcal{U}, \mathbf{h})$ is non-convex for some $\mathbf{h} = \hat{\mathbf{h}}$. Then, there must exist $\hat{\mathbf{b}} \in \mathbb{R}_+^n$ such that $\hat{\mathbf{b}} \notin T(\mathcal{U}, \hat{\mathbf{h}})$ but $\hat{\mathbf{b}} \in \text{conv}(T(\mathcal{U}, \hat{\mathbf{h}}))$. Consider the following instance of Π_{AR} and Π_{Rob} :

$$\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}, \mathbf{h} = \hat{\mathbf{h}}, \mathbf{d} = \hat{\mathbf{b}}.$$

Note that in this case, we have $z_{\text{AR}} = z_{\text{AR}}^I(\mathcal{U}, \hat{\mathbf{h}})$ and $z_{\text{Rob}} = z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}})$. Therefore, by our assumption,

$$z_{\text{AR}}^I(\mathcal{U}, \hat{\mathbf{h}}) = z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}}).$$

Since $\hat{\mathbf{b}} \in \text{conv}(T(\mathcal{U}, \hat{\mathbf{h}}))$, $\alpha = 1, \mathbf{b} = \hat{\mathbf{b}}$ is a feasible solution for $z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}})$. Therefore, $z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}}) \leq 1$, which implies $z_{\text{AR}}^I(\mathcal{U}, \hat{\mathbf{h}}) \leq 1$. However, this would further imply that there exists some $\mathbf{b}_1 \in T(\mathcal{U}, \hat{\mathbf{h}})$ such that $\mathbf{b}_1 \geq \hat{\mathbf{b}}$. Since \mathcal{U} is down-monotone by

our assumption, so is $T(\mathcal{U}, \hat{\mathbf{h}})$ (see Appendix A). Therefore, $\hat{\mathbf{b}} \in T(\mathcal{U}, \hat{\mathbf{h}})$, which is a contradiction. \square

2.4 Min-Max Theorem interpretation

We can interpret a special case of Theorem 1 as a min-max theorem. Consider the case $\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}$, in which we have

$$z_{\text{AR}} = z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}), z_{\text{Rob}} = z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}).$$

Recall:

$$z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}) = \min_{B \in \mathcal{U}} \max_y \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} \right\}.$$

We define the following function:

$$f(\mathbf{y}, \mathbf{B}) = \begin{cases} \mathbf{d}^T \mathbf{y}, & \text{if } \mathbf{B}\mathbf{y} \leq \mathbf{h} \\ -\infty, & \text{otherwise.} \end{cases}$$

Now, we can express $z_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ and $z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ as follows:

$$z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}) = \min_{B \in \mathcal{U}} \max_y f(\mathbf{y}, \mathbf{B})$$

and

$$z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) = \max_y \min_{B \in \mathcal{U}} f(\mathbf{y}, \mathbf{B}).$$

Therefore, from Theorem 1, we have:

$$\min_{B \in \mathcal{U}} \max_y f(\mathbf{y}, \mathbf{B}) = \max_y \min_{B \in \mathcal{U}} f(\mathbf{y}, \mathbf{B}) \quad (2.11)$$

if and only if $T(\mathcal{U}, \mathbf{h})$ is convex. We would like to note that the min-max equality (2.11) does not follow from the general Saddle-Point Theorem [12] since $f(\mathbf{y}, \mathbf{B})$ is not a quasi-convex function of \mathbf{B} .

3 Measure of non-convexity

In this section, we introduce a measure of non-convexity for general compact sets and show that the performance of the optimal static solution is related to this measure of non-convexity of the transformation $T(\mathcal{U}, \mathbf{h})$ of the uncertainty set \mathcal{U} .

Definition 1 Given compact set S , we define the *measure of non-convexity*, $\kappa(S)$ as follows.

$$\kappa(S) = \min \{ \alpha \mid \text{conv}(S) \subseteq \alpha S \}. \quad (3.1)$$

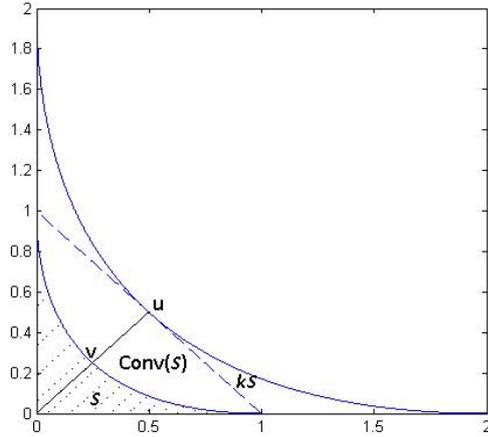


Fig. 3: A geometric illustration of $\kappa(\mathcal{S})$ when $n = 2$: \mathcal{S} is down-monotone and shaded with dot lines, $\text{conv}(\mathcal{S})$ is marked with dashed lines, and the outmost curve is the boundary of $\kappa \cdot \mathcal{S}$. Draw a line from the origin which intersects with the boundary of \mathcal{S} at v and the boundary of $\text{conv}(\mathcal{S})$ at u . $\kappa(\mathcal{S})$ is the largest ratio of such u and v 's.

For any compact set $\mathcal{S} \subseteq \mathbb{R}^n$, $\kappa(\mathcal{S})$ is the smallest factor by which \mathcal{S} must be scaled to contain the convex hull of \mathcal{S} . Note that it is not translation invariant, i.e., $\kappa(\mathcal{S} + \mathbf{v}) \neq \kappa(\mathcal{S})$. If \mathcal{S} is convex, then $\kappa(\mathcal{S}) = 1$. Therefore, if the uncertainty set \mathcal{U} is a Cartesian product of its row vector spaces, then $\kappa(T(\mathcal{U}, \mathbf{h})) = 1$ for all $\mathbf{h} > \mathbf{0}$ (Lemma 1). On the other hand, if \mathcal{S} is non-convex, then $\kappa(\mathcal{S}) > 1$. For instance, consider the following set:

$$\mathcal{S}^n = \left\{ \mathbf{x} \in \mathbb{R}_+^n \mid \sum_{j=1}^n x_j^{\frac{1}{2}} \leq 1 \right\}$$

Figure 3 illustrates set \mathcal{S}^n for $n = 2$ and its measure of non-convexity. In Theorem 2, we prove that \mathcal{S}^n is non-convex for $n > 1$ and $\kappa(\mathcal{S}^n) = n$. Moreover, consider the uncertainty set \mathcal{U} defined in (2.7):

$$\mathcal{U} = \left\{ \mathbf{B} \in [0, 1]^{n \times n} \mid B_{ij} = 0, \forall i \neq j, \sum_{j=1}^n B_{jj} \leq 1 \right\}.$$

We had mentioned earlier that $T(\mathcal{U}, \mathbf{e})$ is non-convex. In fact, we can show that $T(\mathcal{U}, \mathbf{e}) = \mathcal{S}^n$ in this case. We introduce a family of uncertainty sets \mathcal{U}^θ such that $T(\mathcal{U}, \mathbf{h})$ is non-convex for all $\mathbf{h} \geq \mathbf{0}$.

Theorem 2 Suppose the uncertainty set \mathcal{U} is:

$$\mathcal{U}^\theta = \left\{ \mathbf{B} \in [0, 1]^{n \times n} \mid B_{ij} = 0, \forall i \neq j, \sum_{j=1}^n B_{jj}^\theta \leq 1 \right\}.$$

with $\theta > 0$. Then,

1. $T(\mathcal{U}^\theta, \mathbf{h})$ can be written as:

$$T(\mathcal{U}^\theta, \mathbf{h}) = \left\{ \mathbf{b} \in \mathbb{R}_+^n \mid \sum_{j=1}^n \left(\frac{b_j}{h_j} \right)^{\frac{\theta}{\theta+1}} \leq 1 \right\} \quad (3.2)$$

2. The convex hull of $T(\mathcal{U}^\theta, \mathbf{h})$ can be written as:

$$\text{conv}(T(\mathcal{U}^\theta, \mathbf{h})) = \left\{ \mathbf{b} \in \mathbb{R}_+^n \mid \sum_{j=1}^n \frac{b_j}{h_j} \leq 1 \right\}. \quad (3.3)$$

3. $T(\mathcal{U}^\theta, \mathbf{h})$ is non-convex for all $\mathbf{h} > \mathbf{0}$.

4. $\kappa(T(\mathcal{U}^\theta, \mathbf{h})) = n^{\frac{1}{\theta}}$ for all $\mathbf{h} > \mathbf{0}$.

We present the proof of Theorem 2 in Appendix D.

4 Approximation bounds

In this section, we consider the general case where a static solution is not optimal for the adjustable robust problem, and with the measure of non-convexity defined in (3.1), we present a tight bound on the performance of the optimal two-stage static solution for the adjustable robust problem defined in (1.1). We also compare our bound with the symmetry bound introduced by Bertsimas and Goyal [8]. In particular, we show that our bound is at least as good as the symmetry bound, and is significantly better in many cases.

We prove the following theorem which illustrates the relationship between the performance of the static solution and the non-convexity of $T(\mathcal{U}, \mathbf{h})$.

Theorem 3 For any compact set $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$, let

$$\rho(\mathcal{U}) = \max\{\kappa(T(\mathcal{U}, \mathbf{h})) \mid \mathbf{h} > \mathbf{0}\}.$$

Let z_{AR} be the optimal value of Π_{AR} in (1.1) and z_{Rob} be the optimal value of Π_{Rob} in (1.2). Then,

$$z_{\text{AR}} \leq \rho(\mathcal{U}) \cdot z_{\text{Rob}}.$$

Furthermore, we can show that the bound is tight.

Proof Suppose $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}), \mathbf{B} \in \mathcal{U})$ is an optimal fully-adjustable solution for Π_{AR} . Based on the discussion in Theorem 1, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$. Then

$$\begin{aligned} z_{\text{AR}} &= \mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y}(\mathbf{B}) \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y}(\mathbf{B}) \mid \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} \\ &= \mathbf{c}^T \mathbf{x}^* + z_{\text{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*), \end{aligned}$$

and

$$\begin{aligned} z_{\text{Rob}} &\geq \mathbf{c}^T \mathbf{x}^* + \max_{\mathbf{y}} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall \mathbf{B} \in \mathcal{U} \right\} \\ &= \mathbf{c}^T \mathbf{x}^* + z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*). \end{aligned} \quad (4.1)$$

Let $\hat{\mathbf{h}} = \mathbf{h} - \mathbf{A}\mathbf{x}^*$ and $\kappa = \kappa(T(\mathcal{U}, \hat{\mathbf{h}}))$. From Lemmas 4, we can reformulate $z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}})$ as follows.

$$z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}}) = \min_{\mathbf{b} \in \text{conv}(T(\mathcal{U}, \hat{\mathbf{h}}))} \{\lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \lambda \geq 0\}. \quad (4.2)$$

Suppose $(\hat{\lambda}, \hat{\mathbf{b}})$ be the minimizer for $z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}})$ in (4.2). Therefore,

$$\hat{\mathbf{b}} \in \text{conv}(T(\mathcal{U}, \hat{\mathbf{h}})) \Rightarrow \frac{1}{\hat{\lambda}} \cdot \hat{\mathbf{b}} \in T(\mathcal{U}, \hat{\mathbf{h}}).$$

Now,

$$\begin{aligned} z_{\text{AR}}^I(\mathcal{U}, \hat{\mathbf{h}}) &= \min_{\mathbf{b} \in T(\mathcal{U}, \hat{\mathbf{h}})} \{\lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \lambda \geq 0\} \\ &\leq \kappa \cdot \hat{\lambda} \\ &= \kappa \cdot z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}}), \end{aligned} \quad (4.3)$$

where the first equation follows from the reformulation of $z_{\text{AR}}^I(\mathcal{U}, \hat{\mathbf{h}})$ in Lemma 3. The second inequality follows as $(1/\kappa)\hat{\mathbf{b}} \in T(\mathcal{U}, \hat{\mathbf{h}})$ and $\hat{\lambda}\hat{\mathbf{b}} \geq \mathbf{d}$ and the last equality follows as $z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}}) = \hat{\lambda}$. Therefore,

$$\begin{aligned} z_{\text{AR}} &= \mathbf{c}^T \mathbf{x}^* + z_{\text{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &\leq \mathbf{c}^T \mathbf{x}^* + \kappa \cdot z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \end{aligned} \quad (4.4)$$

$$\begin{aligned} &\leq \kappa \cdot (\mathbf{c}^T \mathbf{x}^* + z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*)) \\ &\leq \rho(\mathcal{U}) \cdot z_{\text{Rob}}, \end{aligned} \quad (4.5)$$

where (4.4) follows from (4.3) and the last inequality follows from (4.1) and the fact that $\kappa = \kappa(T(\mathcal{U}, \hat{\mathbf{h}})) \leq \rho(\mathcal{U})$.

Tightness of the bound. We can show that the bound is tight. In particular, given any scalar $\mu > 1$ and some $n \in \mathbb{Z}_+$, take $\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}, \mathbf{d} = \mathbf{e}, \mathbf{h} = \mathbf{e}$ and $\theta = \log_\mu n$. Consider the following uncertainty set:

$$\mathcal{U} = \left\{ \mathbf{B} \in [0, 1]^{n \times n} \mid B_{ij} = 0, \forall i \neq j, \sum_{j=1}^n B_{jj}^\theta \leq 1 \right\}.$$

For Π_{AR} , we have

$$\begin{aligned} z_{\text{AR}} &= \min_{\mathbf{B}} \max_{\mathbf{y}} \left\{ \mathbf{e}^T \mathbf{y} \mid B_{jj} y_j \leq 1, j = 1, \dots, n, \sum_{j=1}^n B_{jj}^\theta \leq 1 \right\} \\ &= \min_{\mathbf{B}} \left\{ \sum_{j=1}^n \frac{1}{B_{jj}} \mid \sum_{j=1}^n B_{jj}^\theta \leq 1 \right\}. \end{aligned}$$

This is a convex problem and by solving the KKT conditions, we have the optimal solution as $B_{jj} = n^{-\frac{1}{\theta}}$ for $j = 1, \dots, n$. Hence, the optimal value of $z_{\text{AR}} = n \cdot n^{\frac{1}{\theta}} = n^{1+\frac{1}{\theta}}$.

For Π_{Rob} , we have

$$z_{\text{Rob}} = \max_{\mathbf{y}} \left\{ \mathbf{e}^T \mathbf{y} \mid B_{jj} y_j \leq 1, \forall \mathbf{B} \in \mathcal{U}, j = 1, \dots, n. \right\}$$

The constraints essentially enforce $B_{jj} y_j \leq 1$ for all $B_{jj} \leq 1, j = 1, \dots, n$. We only need to consider the extreme case where $B_{jj} = 1$, which yields $y_j = 1$. Therefore, $z_{\text{Rob}} = n$ and

$$\frac{z_{\text{AR}}}{z_{\text{Rob}}} = \frac{n^{1+\frac{1}{\theta}}}{n} = n^{\frac{1}{\theta}} = \mu.$$

On the other hand, from Theorem 2, since $\kappa(T(\mathcal{U}, \mathbf{h})) = n^{\frac{1}{\theta}}$ for all $\mathbf{h} > \mathbf{0}$, we have $\rho(\mathcal{U}) = n^{\frac{1}{\theta}} = \mu$. Therefore, $z_{\text{AR}} = \rho(\mathcal{U}) \cdot z_{\text{Rob}}$, i.e., our bound is tight. \square

In Theorem 3, we prove a bound on the optimal objective value, z_{AR} of Π_{AR} with respect to the optimal objective value, z_{Rob} of Π_{Rob} . Note that this also implies a bound on the performance of the optimal static robust solution for Π_{Rob} for the adjustable robust problem, Π_{AR} . Furthermore, in using a static robust solution, $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ for the two-stage adjustable robust problem, we only implement the first-stage solution, $\hat{\mathbf{x}}$ and recompute the optimal second-stage solution, $\mathbf{y}(\mathbf{B})$ after the uncertain constraint matrix, \mathbf{B} is known. Therefore, the cost of such a solution would only be better than z_{Rob} which is at most $\rho(\mathcal{U}) \cdot z_{\text{AR}}$.

4.1 Comparison with symmetry bound [8]

Bertsimas and Goyal [8] consider a general two-stage adjustable robust convex optimization problem with uncertain convex constraints and under mild conditions, show that the performance of a static solution is related to the symmetry of the uncertainty set. In this section, we compare our bound $\kappa(\mathcal{U})$ defined in (3.1) with the symmetry bound of [8] for the case of two-stage adjustable robust linear optimization problem under uncertain constraints. The notion of symmetry is introduced by Minkowski [21].

Definition 2 Given a nonempty convex compact uncertainty set $\mathcal{S} \subseteq \mathbb{R}^m$ and a point $s \in \mathcal{S}$, the symmetry of s with respect to \mathcal{S} is defined as:

$$\text{sym}(s, \mathcal{S}) := \max\{\alpha \geq 0 \mid s + \alpha(s - \hat{s}) \in \mathcal{S}, \forall \hat{s} \in \mathcal{S}\}.$$

The symmetry of the set \mathcal{S} is defined as:

$$\text{sym}(\mathcal{S}) := \max\{\text{sym}(s, \mathcal{S}) \mid s \in \mathcal{S}\}. \quad (4.6)$$

The maximizer of (4.6) is called the point of symmetry.

In Bertsimas and Goyal [8], the authors prove the following bound on the performance of static solution for the two-stage adjustable robust convex optimization with uncertain constraints under some mild conditions.

$$z_{\text{AR}} \leq \left(1 + \frac{1}{\text{sym}(\mathcal{U})}\right) \cdot z_{\text{Rob}}$$

We show that for the case of two-stage adjustable robust linear optimization under uncertain constraints, our approximation bound in 3 is at least as good as the symmetry bound for all instances.

Theorem 4 Consider uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$. Then,

$$\max\{\kappa(T(\mathcal{U}, \mathbf{h})) \mid \mathbf{h} > \mathbf{0}\} \leq 1 + \frac{1}{\text{sym}(\mathcal{U})}.$$

Proof For a given $\mathbf{h} > \mathbf{0}$, from the definition of $\kappa(\cdot)$ in (3.1), we have

$$\text{conv}(T(\mathcal{U}, \mathbf{h})) \subseteq \kappa(T(\mathcal{U}, \mathbf{h})) \cdot T(\mathcal{U}, \mathbf{h}).$$

Therefore, it is sufficient to show

$$\text{conv}(T(\mathcal{U}, \mathbf{h})) \subseteq \left(1 + \frac{1}{\text{sym}(\mathcal{U})}\right) \cdot T(\mathcal{U}, \mathbf{h}) \quad (4.7)$$

for all $\mathbf{h} > \mathbf{0}$. Let

$$\mathbf{B}_0 = \arg \max\{\text{sym}(\mathbf{B}, \mathcal{U}) \mid \mathbf{B} \in \mathcal{U}\}$$

be the point of symmetry. Then, from the result in [9], we have

$$\left(1 + \frac{1}{\text{sym}(\mathcal{U})}\right) \cdot \mathbf{B}_0 \geq \mathbf{B}, \forall \mathbf{B} \in \mathcal{U}. \quad (4.8)$$

Now, given any $\mathbf{h} > \mathbf{0}$, consider an arbitrary $\mathbf{b} \in \text{conv}(T(\mathcal{U}, \mathbf{h}))$. There exists $\mathbf{B}_1, \dots, \mathbf{B}_K \in \mathcal{U}$ such that

$$\mathbf{b} = \sum_{j=1}^K \theta_j \mathbf{B}_j^T \boldsymbol{\lambda}^j, \quad \mathbf{h}^T \boldsymbol{\lambda}^j = 1, \quad \boldsymbol{\lambda}^j \in \mathbb{R}_+^m, \quad j = 1, \dots, K, \quad \mathbf{e}^T \boldsymbol{\theta} = 1, \quad \boldsymbol{\theta} \in \mathbb{R}_+^K.$$

From (4.8), since $\mathbf{B}_1, \dots, \mathbf{B}_K \in \mathcal{U}$, we have

$$\begin{aligned} \mathbf{b} &\leq \sum_{j=1}^K \theta_j \left(1 + \frac{1}{\text{sym}(\mathcal{U})}\right) \mathbf{B}_0^T \boldsymbol{\lambda}^j \\ &= \left(1 + \frac{1}{\text{sym}(\mathcal{U})}\right) \mathbf{B}_0^T \left(\sum_{j=1}^K \theta_j \boldsymbol{\lambda}^j\right) \in \left(1 + \frac{1}{\text{sym}(\mathcal{U})}\right) \cdot T(\mathcal{U}, \mathbf{h}). \end{aligned}$$

The last inequality holds because

$$\mathbf{h}^T \left(\sum_{j=1}^K \theta_j \boldsymbol{\lambda}^j\right) = \left(\sum_{j=1}^K \theta_j \mathbf{h}^T \boldsymbol{\lambda}^j\right) = \mathbf{e}^T \boldsymbol{\theta} = 1.$$

Since \mathcal{U} is down-monotone by assumption, so is $T(\mathcal{U}, \mathbf{h})$ (Appendix A), and we have

$$\mathbf{b} \in \left(1 + \frac{1}{\text{sym}(\mathcal{U})}\right) \cdot T(\mathcal{U}, \mathbf{h})$$

and therefore the theorem. \square

Uncertainty set	Our bound	Symmetric bound [8]
Constraint-wise set $\mathcal{U} = \mathcal{U}_1 \times \dots \times \mathcal{U}_m$	1	$1 + \frac{1}{\min_{1 \leq i \leq m} \text{sym}(\mathcal{U}_i)}$
Permutation invariant \mathcal{U}	1	$1 + \frac{1}{\text{sym}(\mathcal{U})}$
$\{\mathbf{B} : \ \mathbf{B}\ _{\theta_1} \leq 1, \ \mathbf{B}\ _{\theta_2} \leq r\} \subset \mathbb{R}_+^{m \times n}$	1	$1 + r(mn)^{\frac{1}{\theta_1}}$
$\{\mathbf{B} : \ \mathbf{B}\ _1 \leq 1\} \subset \mathbb{R}_+^{m \times n}$	1	$1 + mn$
$\{\mathbf{B} : \ \mathbf{B}\ _{\theta} \leq 1\} \subset \mathbb{R}_+^{m \times n}$	1	$1 + (mn)^{\frac{1}{\theta}}$
$\{\mathbf{B} : \sum_{j=1}^n B_{jj} \leq 1, B_{ij} = 0, \forall i \neq j\} \subset [0, 1]^{n \times n}$	n	$1 + n$
$\{\mathbf{B} : \sum_{j=1}^n B_{jj}^{\theta} \leq 1, B_{ij} = 0, \forall i \neq j\} \subset [0, 1]^{n \times n}$	$n^{\frac{1}{\theta}}$	$1 + n^{\frac{1}{\theta}}$

Table 1: A comparison between the non-convexity bound and the symmetry bound for various uncertainty sets. All the norms are entry-wise, i.e., $\|\mathbf{A}\|_p = (\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^p)^{1/p}$.

Theorem 4 states that our bound in Theorem 3 is at least as good as the symmetry bound. In fact, consider the following example:

$$\mathcal{U} = \left\{ \mathbf{B} \in [0, 1]^{n \times n} \mid \sum_{i,j} B_{ij} \leq 1 \right\}.$$

In this case, \mathcal{U} has symmetric projections. Therefore, from Lemma 2, $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$ and

$$\max\{\kappa(T(\mathcal{U}, \mathbf{h})) \mid \mathbf{h} > \mathbf{0}\} = 1.$$

On the other hand, \mathcal{U} is a simplex and as shown in Bertsimas et al. [9], $\text{sym}(\mathcal{U}) = \frac{1}{n^2}$. Therefore,

$$1 + \frac{1}{\text{sym}(\mathcal{U})} = n^2 + 1,$$

i.e., our bound is significantly better than the symmetry bound.

Furthermore, Table 1 compares our bound with the symmetry bound across several interesting uncertainty sets. Noticeably, our bound is better in all the examples.

5 Two-stage Model with Constraint and Objective uncertainty

In this section, we consider a two-stage adjustable robust linear optimization problem where both constraint and objective coefficients are uncertain. In particular, we consider the following two-stage adjustable robust problem, $\Pi_{\text{AR}}^{(B,d)}$.

$$\begin{aligned}
z_{\text{AR}}^{(B,d)} &= \max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} + \min_{(\mathbf{B}, \mathbf{d}) \in \mathcal{U}} \max_{\mathbf{y}(\mathbf{B}, \mathbf{d})} \mathbf{d}^T \mathbf{y}(\mathbf{B}, \mathbf{d}) \\
&\quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B}, \mathbf{d}) \leq \mathbf{h} \\
&\quad \mathbf{x} \in \mathbb{R}_+^{n_1} \\
&\quad \mathbf{y}(\mathbf{B}, \mathbf{d}) \in \mathbb{R}_+^{n_2},
\end{aligned} \tag{5.1}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n_1}$, $\mathbf{c} \in \mathbb{R}_+^{n_1}$, $\mathbf{h} \in \mathbb{R}_+^m$, and (\mathbf{B}, \mathbf{d}) are uncertain second-stage constraint matrix and objective that belong to a convex compact uncertainty set

$\mathcal{U} \subseteq \mathbb{R}_+^{m \times n_2} \times \mathbb{R}_+^{n_2}$. We consider the case where the uncertainty in constraint matrix, \mathbf{B} does not depend on the uncertainty in objective coefficients \mathbf{d} . Therefore,

$$\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d,$$

where $\mathcal{U}^B \subseteq \mathbb{R}_+^{m \times n_2}$ is a convex compact uncertainty set of constraint matrices and $\mathcal{U}^d \subseteq \mathbb{R}_+^{n_2}$ is a convex compact uncertainty set of the second-stage objective. As previous sections, we can assume without loss of generality that \mathcal{U}^B is down-monotone.

We formulate the corresponding static robust problem, $\Pi_{\text{Rob}}^{(B,d)}$, as follows.

$$\begin{aligned} z_{\text{Rob}}^{(B,d)} = \max_{\mathbf{x}, \mathbf{y}} \min_{\mathbf{d} \in \mathcal{U}^d} \mathbf{c}^T \mathbf{x} + \mathbf{d}^T \mathbf{y} \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \leq \mathbf{h}, \forall \mathbf{B} \in \mathcal{U}^B \\ \mathbf{x} \in \mathbb{R}_+^{n_1} \\ \mathbf{y} \in \mathbb{R}_+^{n_2}. \end{aligned} \quad (5.2)$$

We can compute an optimal static robust solution efficiently. It is easy to see that the separation problem for (5.2) can be solved in polynomial time. In fact, we can also give a compact LP formulation to compute an optimal static robust solution similar to (1.2). Now, suppose the optimal solution of $\Pi_{\text{Rob}}^{(B,d)}$ is $(\mathbf{x}^*, \mathbf{y}^*)$, then $\mathbf{x} = \mathbf{x}^*, \mathbf{y}(\mathbf{B}, \mathbf{d}) = \mathbf{y}^*$ for all $(\mathbf{B}, \mathbf{d}) \in \mathcal{U}$ is a feasible solution to $\Pi_{\text{AR}}^{(B,d)}$. Therefore,

$$z_{\text{AR}}^{(B,d)} \geq z_{\text{Rob}}^{(B,d)}. \quad (5.3)$$

We prove the following main theorem.

Theorem 5 Let $z_{\text{AR}}^{(B,d)}$ be the optimal objective value of $\Pi_{\text{AR}}^{(B,d)}$ in (5.1) defined over the uncertainty $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$. Let $z_{\text{Rob}}^{(B,d)}$ be the optimal objective value of $\Pi_{\text{Rob}}^{(B,d)}$ in (5.2). Also, let

$$\rho(\mathcal{U}^B) = \max_{\mathbf{h} > \mathbf{0}} \kappa(T(\mathcal{U}^B, \mathbf{h})).$$

Then,

$$z_{\text{AR}}^{(B,d)} \leq \rho(\mathcal{U}^B) \cdot z_{\text{Rob}}^{(B,d)}.$$

Furthermore, the bound is tight.

If $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$, then $\rho(\mathcal{U}^B) = 1$ and $z_{\text{AR}}^{(B,d)} \leq z_{\text{Rob}}^{(B,d)}$. In this case, Theorem 5 implies that a static solution is optimal for the adjustable robust problem, $\Pi_{\text{AR}}^{(B,d)}$. Therefore, if \mathcal{U}^B is constraint-wise or has symmetric projections then $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$ (Lemmas 1 and 2). In general, the performance of static solution depends on the worst-case measure of non-convexity of $T(\mathcal{U}^B, \mathbf{h})$ for all $\mathbf{h} > \mathbf{0}$. Surprisingly, the approximation bound for the static solution does not depend on the uncertain set of objectives, \mathcal{U}^d .

To prove the Theorem 5, we need to introduce the following one-stage models as in Section 2, $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ and $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$.

$$\begin{aligned} z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}) = \min_{(\mathbf{B}, \mathbf{d}) \in \mathcal{U}} \max_{\mathbf{y}} \mathbf{d}^T \mathbf{y} \\ \mathbf{B}\mathbf{y} \leq \mathbf{h} \\ \mathbf{y} \in \mathbb{R}_+^{n_2}, \end{aligned} \quad (5.4)$$

$$\begin{aligned}
z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) &= \max_{\mathbf{y}} \min_{\mathbf{d} \in \mathcal{U}^d} \mathbf{d}^T \mathbf{y} \\
&\quad \mathbf{B}\mathbf{y} \leq \mathbf{h}, \quad \forall \mathbf{B} \in \mathcal{U}^B \\
&\quad \mathbf{y} \in \mathbb{R}_+^n.
\end{aligned} \tag{5.5}$$

where $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$ and $\mathbf{h} > \mathbf{0}$. Similar to Lemma 3 and Lemma 4, we can reformulate the above problems as optimization problems over the transformation set $T(\mathcal{U}^B, \mathbf{h})$.

Lemma 5 *The one-stage adjustable robust problem $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ defined in (5.4) can be written as:*

$$z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}) = \min\{\lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \mathbf{b} \in T(\mathcal{U}^B, \mathbf{h}), \mathbf{d} \in \mathcal{U}^d\}.$$

Proof Consider $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$, by writing the dual of its inner maximization problem, we have

$$\begin{aligned}
z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}) &= \min\{\mathbf{h}^T \boldsymbol{\alpha} \mid \mathbf{B}^T \boldsymbol{\alpha} \geq \mathbf{d}, (\mathbf{B}, \mathbf{d}) \in \mathcal{U}, \boldsymbol{\alpha} \in \mathbb{R}_+^m\} \\
&= \min\left\{\lambda \mathbf{h}^T \left(\frac{\boldsymbol{\alpha}}{\lambda}\right) \mid \lambda \mathbf{B}^T \left(\frac{\boldsymbol{\alpha}}{\lambda}\right) \geq \mathbf{d}, \mathbf{h}^T \boldsymbol{\alpha} = \lambda, (\mathbf{B}, \mathbf{d}) \in \mathcal{U}, \boldsymbol{\alpha} \in \mathbb{R}_+^m\right\}. \\
&= \min\{\lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \mathbf{b} \in T(\mathcal{U}^B, \mathbf{h}), \mathbf{d} \in \mathcal{U}^d\},
\end{aligned}$$

where the last equality holds because $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$. \square

Lemma 6 *The one-stage static robust problem $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ defined in (5.5) can be written as:*

$$z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) = \min\{\lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \text{conv}(T(\mathcal{U}^B, \mathbf{h})), \mathbf{d} \in \mathcal{U}^d\}.$$

Proof Suppose

$$\mathcal{U} = \text{conv}((\mathbf{B}_1, \mathbf{d}_1), \dots, (\mathbf{B}_K, \mathbf{d}_K))$$

where $(\mathbf{B}_j, \mathbf{d}_j), j = 1, \dots, K$ are the extreme points of \mathcal{U} . We can rewrite (5.5) as follows.

$$z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) = \max\{z \mid \mathbf{B}_j \mathbf{y} \leq \mathbf{h}, z - \mathbf{d}_j^T \mathbf{y} \leq 0, \forall j = 1, \dots, K, \mathbf{y} \in \mathbb{R}_+^n\}.$$

Again, by writing the dual problem, we have

$$z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) = \min\left\{\sum_{j=1}^K \mathbf{h}^T \boldsymbol{\alpha}_j \mid \sum_{j=1}^K \mathbf{B}_j^T \boldsymbol{\alpha}_j \geq \sum_{j=1}^K \beta_j \mathbf{d}_j, \mathbf{e}^T \boldsymbol{\beta} = 1, \boldsymbol{\alpha}_j \in \mathbb{R}_+^m, \beta_j \in \mathbb{R}_+^K\right\}.$$

Note that $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$, \mathbf{d} can be chosen regardless of \mathbf{B} . Therefore, denote $\theta_j = \mathbf{h}^T \boldsymbol{\alpha}_j$, $\lambda = \mathbf{e}^T \boldsymbol{\theta}$ and we have

$$\begin{aligned}
z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}) &= \min\left\{\sum_{j=1}^K \theta_j \mathbf{h}^T \left(\frac{\boldsymbol{\alpha}_j}{\theta_j}\right) \mid \lambda \sum_{j=1}^K \frac{\theta_j}{\lambda} \mathbf{B}_j^T \left(\frac{\boldsymbol{\alpha}_j}{\theta_j}\right) \geq \mathbf{d}, \mathbf{d} \in \mathcal{U}^d, \boldsymbol{\alpha}_j \geq \mathbf{0}\right\} \\
&= \min\left\{\lambda \mid \lambda \sum_{j=1}^K \frac{\theta_j}{\lambda} \hat{\mathbf{b}}_j \geq \mathbf{d}, \hat{\mathbf{b}}_j \in T(\mathcal{U}^B, \mathbf{h}), \mathbf{d} \in \mathcal{U}^d\right\} \\
&= \min\{\lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \text{conv}(T(\mathcal{U}^B, \mathbf{h})), \mathbf{d} \in \mathcal{U}^d\},
\end{aligned}$$

where the second equality holds because $\mathbf{h}^T \left(\frac{\boldsymbol{\alpha}_j}{\theta_j}\right) = 1, j = 1, \dots, K$. \square

Now, we are ready to prove Theorem 5.

Proof of Theorem 5 Suppose $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}, \mathbf{d}), (\mathbf{B}, \mathbf{d}) \in \mathcal{U})$ is a fully-adjustable optimal solution for $\Pi_{\text{AR}}^{(B,d)}$. As discussed earlier, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$. Then,

$$\begin{aligned} z_{\text{AR}}^{(B,d)} &= \mathbf{c}^T \mathbf{x}^* + \min_{(\mathbf{B}, \mathbf{d}) \in \mathcal{U}} \max_{\mathbf{y}(\mathbf{B}, \mathbf{d}) \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y}(\mathbf{B}, \mathbf{d}) \mid \mathbf{B}\mathbf{y}(\mathbf{B}, \mathbf{d}) \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} \\ &= \mathbf{c}^T \mathbf{x}^* + z_{\text{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*), \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} z_{\text{Rob}}^{(B,d)} &\geq \mathbf{c}^T \mathbf{x}^* + \max_{\mathbf{y} \geq \mathbf{0}} \min_{\mathbf{d} \in \mathcal{U}^d} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall \mathbf{B} \in \mathcal{U}^B \right\} \\ &= \mathbf{c}^T \mathbf{x}^* + z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*). \end{aligned} \quad (5.7)$$

Let $\hat{\mathbf{h}} = \mathbf{h} - \mathbf{A}\mathbf{x}^*$ and $\kappa = \kappa(T(\mathcal{U}^B, \hat{\mathbf{h}}))$. Suppose $(\hat{\lambda}, \hat{\mathbf{b}}, \hat{\mathbf{d}})$ is an optimal solution for $\Pi_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}})$. Therefore

$$\hat{\mathbf{b}} \in \text{conv}(T(\mathcal{U}^B, \hat{\mathbf{h}})) \Rightarrow \frac{1}{\kappa} \cdot \hat{\mathbf{b}} \in T(\mathcal{U}^B, \hat{\mathbf{h}}).$$

Also,

$$\hat{\lambda} \cdot \hat{\mathbf{b}} \geq \hat{\mathbf{d}} \Rightarrow (\kappa \hat{\lambda}) \cdot \left(\frac{1}{\kappa} \hat{\mathbf{b}} \right) \geq \hat{\mathbf{d}},$$

which implies that $(\kappa \hat{\lambda}, \hat{\mathbf{b}}/\kappa, \hat{\mathbf{d}})$ is a feasible solution to $\Pi_{\text{AR}}^I(\mathcal{U}, \hat{\mathbf{h}})$ and

$$z_{\text{AR}}^I(\mathcal{U}, \hat{\mathbf{h}}) \leq \kappa \cdot z_{\text{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}}).$$

From (5.6), we have

$$\begin{aligned} z_{\text{AR}}^{(B,d)} &= \mathbf{c}^T \mathbf{x}^* + z_{\text{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &\leq \mathbf{c}^T \mathbf{x}^* + \kappa \cdot z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &\leq \kappa \cdot (\mathbf{c}^T \mathbf{x}^* + z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*)) \\ &\leq \kappa \cdot z_{\text{Rob}}^{(B,d)}, \end{aligned} \quad (5.8)$$

where (5.8) holds because $\kappa \geq 1$, the last inequality holds from (5.7).

We can show that the bound is tight using the same family of uncertainty sets of matrices \mathcal{U}_θ^B in the discussion of Theorem 3:

$$\mathcal{U}_\theta^B = \left\{ \mathbf{B} \in [0, 1]^{n \times n} \mid B_{ij} = 0, \forall i \neq j, \sum_{j=1}^n B_{jj}^\theta \leq 1 \right\}.$$

Consider the following instance of $\Pi_{\text{AR}}^{(B,d)}$ and $\Pi_{\text{Rob}}^{(B,d)}$:

$$\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}, \mathbf{h} = \mathbf{e}, \mathcal{U}^d = \{\mathbf{e}\}.$$

From the discussion in Theorem 3, the bound in Theorem 5 is tight. \square

Note that surprisingly, the bound only depends on the measure of non-convexity of \mathcal{U}^B and is not related to \mathcal{U}^d . Therefore, if $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$, then a static solution is optimal for the adjustable robust problem, $\Pi_{\text{AR}}^{(B,d)}$ irrespective

of \mathcal{U}^d . As a special case where there is no uncertainty in \mathbf{B} , i.e., $\mathcal{U}^B = \{\mathbf{B}^0\}$, and the only uncertainty is in \mathcal{U}^d , $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$ and a static solution is optimal. In fact, the optimality of static solution in this case follows from von Neumann's Min-max theorem [26]. Therefore, we can interpret the result as a generalization of von Neumann's theorem.

General Case when \mathcal{U} is not a Cartesian product. For the general case where the uncertainty set \mathcal{U} of constraint matrices, \mathbf{B} and objective coefficients, \mathbf{d} is not a Cartesian product of the respective uncertainty sets, our bound of Theorem 5 does not extend. Consider the following example:

$$\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}, \mathbf{h} = \mathbf{e},$$

$$\mathcal{U} = \left\{ (\mathbf{B}, \mathbf{d}) \in \mathbb{R}_+^{n \times n} \times \mathbb{R}_+^n \mid \sum_{i=1}^n B_{ii} \leq \sum_{j=1}^n d_j \leq 1, \mathbf{d} \geq \frac{\epsilon}{n} \mathbf{e}, B_{ij} = 0, \forall i \neq j \right\}.$$

Now,

$$\begin{aligned} z_{\text{AR}}^{(B, d)} &= \min_{(\mathbf{B}, \mathbf{d}) \in \mathcal{U}} \max_{\mathbf{y}} \left\{ \mathbf{d}^T \mathbf{y} \mid B_{jj} y_j \leq 1, \forall j = 1, \dots, n, \mathbf{y} \geq \mathbf{0} \right\} \\ &= \min_{(\mathbf{B}, \mathbf{d}) \in \mathcal{U}} \sum_{j=1}^n \frac{d_j}{B_{jj}} \\ &\geq 1, \end{aligned}$$

where the second equation follows from the fact that at optimum of the outer minimization problem, $B_{jj} > 0$ for all $j = 1, \dots, n$ and $y_j = 1/B_{jj}$ for the inner maximization problem. Otherwise, if $B_{jj} = 0$ for some j , then y_j and $d_j y_j$ are both unbounded as $d_j > \epsilon/n > 0$. The last equality follows as for any $(\mathbf{B}, \mathbf{d}) \in \mathcal{U}$,

$$\sum_{j=1}^n B_{jj} \leq \sum_{j=1}^n d_j,$$

which implies that $B_{jj} \leq d_j$ for some $j \in [n]$.

For the robust problem, $\Pi_{\text{Rob}}^{(B, d)}$, consider any static solution $\mathbf{y} \geq \mathbf{0}$. For all $j = 1, \dots, n$,

$$B_{jj} y_j \leq 1, \forall (\mathbf{B}, \mathbf{d}) \in \mathcal{U}.$$

Since there exist $(\mathbf{B}, \mathbf{d}) \in \mathcal{U}$ such that $B_{jj} = 1, y_j \leq 1$ for all $j = 1, \dots, n$. Moreover, $\mathbf{y} = \mathbf{e}$ is a feasible solution as $B_{jj} \leq 1$ for all $(\mathbf{B}, \mathbf{d}) \in \mathcal{U}$ for all $j \in [n]$. Therefore,

$$z_{\text{Rob}}^{(B, d)} = \min_{(\mathbf{B}, \mathbf{d}) \in \mathcal{U}} \mathbf{d}^T \mathbf{e} \leq \epsilon,$$

where the second inequality follows by setting $d_j = \epsilon/n$ for all $j = 1, \dots, n$. Therefore,

$$z_{\text{AR}}^{(B, d)} \geq \frac{1}{\epsilon} \cdot z_{\text{Rob}}^{(B, d)},$$

where $\epsilon > 0$ is arbitrary. Therefore, the performance of the optimal static robust solution as compared to the optimal fully adjustable solution can not be bounded by the measure of non-convexity as in Theorem 5.

5.1 Constraint, right-hand-side and objective uncertainty

In this section, we discuss the case where all of the right-hand-side vector, constraint and objective coefficients are uncertain. Consider the following adjustable robust problem, $\Pi_{\text{AR}}^{(B,h,d)}$.

$$\begin{aligned} z_{\text{AR}}^{(B,h,d)} &= \max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} + \min_{(\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U}^{B,h,d}} \max_{\mathbf{y}} \mathbf{d}^T \mathbf{y}(\mathbf{B}, \mathbf{h}, \mathbf{d}) \\ &\quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B}, \mathbf{h}, \mathbf{d}) \leq \mathbf{h} \\ &\quad \mathbf{x} \in \mathbb{R}_+^{n_1} \\ &\quad \mathbf{y}(\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathbb{R}_+^{n_2}, \end{aligned} \quad (5.9)$$

where $\mathbf{A} \in \mathbb{R}^{m \times n_1}$, $\mathbf{c} \in \mathbb{R}_+^{n_1}$. In this case, $(\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U}^{B,h,d}$ are uncertain and $\mathcal{U}^{B,h,d} \subseteq \mathbb{R}_+^{m \times n_2} \times \mathbb{R}_+^m \times \mathbb{R}_+^{n_2}$ is convex and compact. We consider the case that the uncertainties in constraint matrix, \mathbf{B} and right-hand-side vector, \mathbf{h} are independent of the uncertainty in the objective coefficients \mathbf{d} , i.e.,

$$\mathcal{U}^{B,h,d} = \mathcal{U}^{B,h} \times \mathcal{U}^d,$$

where $\mathcal{U}^{B,h} \subseteq \mathbb{R}^{m \times (n_2+1)}$ is the convex compact uncertainty set of constraint matrices and right-hand-side vectors, and $\mathcal{U}^d \subseteq \mathbb{R}^{n_2}$ is the convex compact set of the constraint coefficients.

The corresponding static robust version, $\Pi_{\text{Rob}}^{(B,h,d)}$, can be formulated as follows.

$$\begin{aligned} z_{\text{Rob}}^{(B,h,d)} &= \max_{\mathbf{x}, \mathbf{y}} \min_{\mathbf{d} \in \mathcal{U}^d} \mathbf{c}^T \mathbf{x} + \mathbf{d}^T \mathbf{y} \\ &\quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \leq \mathbf{h}, \forall (\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B,h} \\ &\quad \mathbf{x} \in \mathbb{R}_+^{n_1} \\ &\quad \mathbf{y} \in \mathbb{R}_+^{n_2}. \end{aligned} \quad (5.10)$$

We can compute an optimal solution for (5.10) efficiently by solving a compact LP formulation for its separation problem. Now, we study the performance of static solution and show that it is optimal if $\mathcal{U}^{B,h}$ is constraint-wise. In particular, we have the following theorem.

Theorem 6 *Let $z_{\text{AR}}^{(B,h,d)}$ be the optimal value of $\Pi_{\text{AR}}^{(B,h,d)}$ defined in (5.9) and $z_{\text{Rob}}^{(B,h,d)}$ be the optimal value of $\Pi_{\text{Rob}}^{(B,h,d)}$ defined in (5.10). Suppose $\mathcal{U}^{B,h}$ is constraint-wise, then the static solution is optimal for $\Pi_{\text{AR}}^{(B,h,d)}$, i.e.,*

$$z_{\text{AR}}^{(B,h,d)} = z_{\text{Rob}}^{(B,h,d)}. \quad (5.11)$$

To prove Theorem 6, we need to introduce the one-stage models. Consider the one-stage adjustable robust problem, $\Pi_{\text{AR}}^I(\mathcal{U}^{B,h,d})$

$$\begin{aligned} \Pi_{\text{AR}}^I(\mathcal{U}^{B,h,d}) &= \min_{(\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U}^{B,h,d}} \max_{\mathbf{y}} \mathbf{d}^T \mathbf{y} \\ &\quad \mathbf{B}\mathbf{y} \leq \mathbf{h} \\ &\quad \mathbf{y} \in \mathbb{R}_+^n, \end{aligned} \quad (5.12)$$

where $\mathcal{U}^{B,h,d} = \mathcal{U}^{B,h} \times \mathcal{U}^d$. The corresponding one-stage static robust problem $\Pi_{\text{Rob}}^I(\mathcal{U}^{B,h,d})$ can be formulated as follows

$$\begin{aligned} z_{\text{Rob}}^I(\mathcal{U}^{B,h,d}) &= \max_{\mathbf{y}} \min_{\mathbf{d} \in \mathcal{U}^d} \mathbf{d}^T \mathbf{y} \\ &\quad \mathbf{B}\mathbf{y} \leq \mathbf{h}, \quad \forall (\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B,h} \\ &\quad \mathbf{y} \in \mathbb{R}_+^n, \end{aligned} \quad (5.13)$$

We can reformulate these models as optimization problems over $T(\mathcal{U}^{B,h}, \mathbf{e})$.

Lemma 7 *The one-stage adjustable robust problem $\Pi_{\text{AR}}^I(\mathcal{U}^{B,h,d})$ defined in (5.12) can be written as*

$$z_{\text{AR}}^I(\mathcal{U}^{B,h,d}) = \min\{ \lambda t \mid \lambda \mathbf{b} \geq \mathbf{d}, (\mathbf{b}, t) \in T(\mathcal{U}^{B,h}, \mathbf{e}), \mathbf{d} \in \mathcal{U}^d \}.$$

We present the proof of Lemma 7 in Appendix E.

Lemma 8 *The one-stage static-robust problem $\Pi_{\text{Rob}}^I(\mathcal{U}^{B,h,d})$ defined in (5.13) can be written as*

$$z_{\text{Rob}}^I(\mathcal{U}^{B,h,d}) = \min\{ \lambda t \mid \lambda \mathbf{b} \geq \mathbf{d}, (\mathbf{b}, t) \in \text{conv}(T(\mathcal{U}^{B,h}, \mathbf{e})), \mathbf{d} \in \mathcal{U}^d \}.$$

We present the proof of Lemma 8 in Appendix E. Now, with the formulations in Lemma 7 and Lemma 8, we are ready to prove Theorem 6.

Proof of Theorem 6 Suppose the optimal solution of $\Pi_{\text{Rob}}^{(B,h,d)}$ is $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, then $\mathbf{x} = \tilde{\mathbf{x}}, \mathbf{y}(\mathbf{B}, \mathbf{h}, \mathbf{d}) = \tilde{\mathbf{y}}$ for all $(\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U}$ is a feasible solution to $\Pi_{\text{AR}}^{(B,h,d)}$. Therefore,

$$z_{\text{AR}}^{(B,h,d)} \geq z_{\text{Rob}}^{(B,h,d)}. \quad (5.14)$$

On the other hand, suppose $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}, \mathbf{h}, \mathbf{d}), (\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U}^{B,h,d})$ is a fully-adjustable optimal solution for $\Pi_{\text{AR}}^{(B,h,d)}$. As discussed earlier, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$ for all \mathbf{h} such that $(\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B,h}$ for some \mathbf{B} . Then,

$$\begin{aligned} z_{\text{AR}}^{(B,h,d)} &= \mathbf{c}^T \mathbf{x}^* + \min_{(\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U}^{B,h,d}} \max_{\mathbf{y} \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} \\ &= \mathbf{c}^T \mathbf{x}^* + z_{\text{AR}}^I(\mathcal{U}^{B,h-A\mathbf{x}^*,d}), \end{aligned} \quad (5.15)$$

and

$$\begin{aligned} z_{\text{Rob}}^{(B,h,d)} &\geq \mathbf{c}^T \mathbf{x}^* + \max_{\mathbf{y} \geq \mathbf{0}} \min_{\mathbf{d} \in \mathcal{U}^d} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall (\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B,h} \right\} \\ &= \mathbf{c}^T \mathbf{x}^* + z_{\text{Rob}}^I(\mathcal{U}^{B,h-A\mathbf{x}^*,d}). \end{aligned} \quad (5.16)$$

Since $\mathcal{U}^{B,h}$ is constraint-wise, so is $\mathcal{U}^{B,h-A\mathbf{x}^*}$. Therefore, $T(\mathcal{U}^{B,h-A\mathbf{x}^*}, \mathbf{e})$ is convex from Lemma 1 and $T(\mathcal{U}^{B,h-A\mathbf{x}^*}, \mathbf{e}) = \text{conv}(T(\mathcal{U}^{B,h-A\mathbf{x}^*}, \mathbf{e}))$. From Lemma 7 and Lemma 8, this implies that

$$z_{\text{AR}}^I(\mathcal{U}^{B,h-A\mathbf{x}^*,d}) = z_{\text{Rob}}^I(\mathcal{U}^{B,h-A\mathbf{x}^*,d}).$$

Therefore, from (5.15) and (5.16), we have

$$z_{\text{AR}}^{(B,h,d)} \leq z_{\text{Rob}}^{(B,h,d)}.$$

Together with (5.14), we have $z_{\text{AR}}^{(B,h,d)} = z_{\text{Rob}}^{(B,h,d)}$. \square

We would like to note, we can not extend the approximation bounds similar to Theorem 5 in this case. In fact, the measure of non-convexity of $T(\mathcal{U}^{B,h}, \mathbf{e})$ is not even well defined in this case since $\mathcal{U}^{B,h}$ is not down-monotone.

6 Conclusion

In this paper, we study performance of static robust solution as an approximation of two-stage adjustable robust linear optimization problem under uncertain constraints and objective. We give a tight characterization of the performance of static solution and relate it to the measure of non-convexity of the transformation $T(\mathcal{U}, \cdot)$ of the uncertainty set \mathcal{U} . In particular, we show that a static solution is optimal if and only if $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. If $T(\mathcal{U}, \cdot)$ is not convex, the measure of non-convexity of $T(\mathcal{U}, \cdot)$ gives a tight bound on the performance of static solutions. For several interesting families of uncertainty sets such as constraint-wise or symmetric projections, we show that $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$; thereby, generalizing the result of Ben-Tal and Nemirovski [4]. Also, our approximation bound is strictly better than the symmetry bound in Bertsimas and Goyal [8].

We also extend our result to models where both constraint and objective coefficients are uncertain. We show that if $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$, where \mathcal{U}^B is the set of uncertain second-stage constraint matrices, \mathbf{B} and \mathcal{U}^d is the set of uncertain second-stage objective, then the performance of static solution is related to the measure of non-convexity of $T(\mathcal{U}^B, \cdot)$. In particular, a static solution is optimal if $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. Surprisingly, the performance of static solution does not depend on the uncertainty set \mathcal{U}^d . We also present several examples to illustrate such optimality and the tightness of the bound.

Our results develop new geometric intuition about the performance of static robust solutions for adjustable robust problems. The reformulations of the adjustable robust and static robust problems based on the transformation $T(\mathcal{U}, \cdot)$ of the uncertainty set \mathcal{U} give us interesting insights about properties of \mathcal{U} where the static robust does not perform well. Therefore, our results provide useful guidance in selecting uncertainty sets such that the adjustable robust problem can be well approximated by a static solutions.

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A Down-monotone

In this section, we show that in $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ defined in (2.3) and $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ defined in (2.4), we can assume \mathcal{U} to be *down-monotone* without loss of generality, where down-monotone is defined as follows.

Definition 3 A set $S \subseteq \mathbb{R}^n$ is *down-monotone* if $\mathbf{s} \in S$ and $\mathbf{t} \leq \mathbf{s}$ implies $\mathbf{t} \in S$.

Given $S \subseteq \mathbb{R}^n$, we can construct the down-hull of S , denoted by S^\downarrow as follows.

$$S^\downarrow = \{\mathbf{t} \in \mathbb{R}^n \mid \exists \mathbf{s} \in S : \mathbf{t} \leq \mathbf{s}\}. \quad (\text{A.1})$$

Given uncertainty set $\mathcal{U} \in \mathbb{R}_+^{m \times n}$ and $\mathbf{h} > \mathbf{0}$, if \mathcal{U} is down-monotone, then $\mathcal{U}^\downarrow = \mathcal{U}$. Therefore, $\Pi_{\text{AR}}^I(\mathcal{U}^\downarrow, \mathbf{h})$ is essentially the same problem with $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ and we have $z_{\text{AR}}^I(\mathcal{U}^\downarrow, \mathbf{h}) = z_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$. Similar arguments applies for $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ and $z_{\text{Rob}}^I(\mathcal{U}^\downarrow, \mathbf{h}) = z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$. On the other hand, if \mathcal{U} is not down-monotone, then $\mathcal{U} \subsetneq \mathcal{U}^\downarrow$. Then, we prove the following lemma.

Lemma 9 Given uncertainty set $\mathcal{U} \in \mathbb{R}_+^{m \times n}$ and $\mathbf{h} > \mathbf{0}$, let $z_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ be the optimal value of $\Pi_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$ defined in (2.3), $z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ be the optimal value of $\Pi_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$ defined in (2.4). Suppose \mathcal{U} is not down-monotone, let \mathcal{U}^\downarrow be defined as in (A.1). Then,

$$z_{\text{AR}}^I(\mathcal{U}^\downarrow, \mathbf{h}) = z_{\text{AR}}^I(\mathcal{U}, \mathbf{h}), \quad z_{\text{Rob}}^I(\mathcal{U}^\downarrow, \mathbf{h}) = z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h}).$$

Proof Consider an arbitrary $\mathbf{X} \in \mathcal{U}^\downarrow$ and $\mathbf{X} \notin \mathcal{U}$, i.e., $\mathbf{X} \in \mathcal{U}^\downarrow \setminus \mathcal{U}$. From (A.1), there exists $\mathbf{B} \in \mathcal{U}$ such that $\mathbf{X} \leq \mathbf{B}$. Since \mathbf{B} , \mathbf{X} and \mathbf{y} are all non-negative, any $\mathbf{y} \in \mathbb{R}_+^n$ satisfies $\mathbf{B}\mathbf{y} \leq \mathbf{h}$ also satisfies $\mathbf{X}\mathbf{y} \leq \mathbf{h}$. Therefore,

$$\max\{\mathbf{d}^T \mathbf{y} | \mathbf{B}\mathbf{y} \leq \mathbf{h}, \mathbf{y} \in \mathbb{R}_+^n\} \leq \max\{\mathbf{d}^T \mathbf{y} | \mathbf{X}\mathbf{y} \leq \mathbf{h}, \mathbf{y} \in \mathbb{R}_+^n\}.$$

Take minimum over all $\mathbf{B} \in \mathcal{U}$ on the left side, we have

$$\min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y}} \{\mathbf{d}^T \mathbf{y} | \mathbf{B}\mathbf{y} \leq \mathbf{h}, \mathbf{y} \in \mathbb{R}_+^n\} \leq \max_{\mathbf{y}} \{\mathbf{d}^T \mathbf{y} | \mathbf{X}\mathbf{y} \leq \mathbf{h}, \mathbf{y} \in \mathbb{R}_+^n\}.$$

Since \mathbf{X} is arbitrarily chosen in $\mathcal{U}^\downarrow \setminus \mathcal{U}$, we can take minimum of all $\mathbf{X} \in \mathcal{U}^\downarrow \setminus \mathcal{U}$ on the right side

$$\min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y}} \{\mathbf{d}^T \mathbf{y} | \mathbf{B}\mathbf{y} \leq \mathbf{h}, \mathbf{y} \in \mathbb{R}_+^n\} \leq \min_{\mathbf{X} \in \mathcal{U}^\downarrow \setminus \mathcal{U}} \max_{\mathbf{y}} \{\mathbf{d}^T \mathbf{y} | \mathbf{X}\mathbf{y} \leq \mathbf{h}, \mathbf{y} \in \mathbb{R}_+^n\}.$$

Therefore, the minimizer of the outer problem of $\Pi_{\text{AR}}^I(\mathcal{U}^\downarrow, \mathbf{h})$ is in \mathcal{U} , which implies

$$\min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y}} \{\mathbf{d}^T \mathbf{y} | \mathbf{B}\mathbf{y} \leq \mathbf{h}, \mathbf{y} \in \mathbb{R}_+^n\} = \min_{\mathbf{X} \in \mathcal{U}^\downarrow} \max_{\mathbf{y}} \{\mathbf{d}^T \mathbf{y} | \mathbf{X}\mathbf{y} \leq \mathbf{h}, \mathbf{y} \in \mathbb{R}_+^n\}.$$

As a result, we have $z_{\text{AR}}^I(\mathcal{U}^\downarrow, \mathbf{h}) = z_{\text{AR}}^I(\mathcal{U}, \mathbf{h})$.

Similarly, any $\mathbf{y} \in \mathbb{R}_+^n$ satisfies $\mathbf{B}\mathbf{y} \leq \mathbf{h}$ for all $\mathbf{B} \in \mathcal{U}$ is guaranteed to be feasible to $\mathbf{X}\mathbf{y} \leq \mathbf{h}$ for all $\mathbf{X} \in \mathcal{U}^\downarrow \setminus \mathcal{U}$. Therefore, we can remove the constraints generated by $\mathbf{B} \in \mathcal{U}^\downarrow \setminus \mathcal{U}$ in $\Pi_{\text{Rob}}^I(\mathcal{U}^\downarrow, \mathbf{h})$ and conclude that $z_{\text{Rob}}^I(\mathcal{U}^\downarrow, \mathbf{h}) = z_{\text{Rob}}^I(\mathcal{U}, \mathbf{h})$. \square

Therefore, we can assume without loss of generality that \mathcal{U} is down-monotone in (2.3) and (2.4). Now, we generalize the result for the two-stage problems Π_{AR} in (1.1) and Π_{Rob} in (1.2). Consider the following adjustable robust problem, $\Pi_{\text{AR}}^\downarrow$

$$\begin{aligned} z_{\text{AR}}^\downarrow &= \max \mathbf{c}^T \mathbf{x} + \min_{\mathbf{B} \in \mathcal{U}^\downarrow} \max_{\mathbf{y}(\mathbf{B})} \mathbf{d}^T \mathbf{y}(\mathbf{B}) \\ &\quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{h} \\ &\quad \mathbf{x} \in \mathbb{R}_+^{n_1} \\ &\quad \mathbf{y}(\mathbf{B}) \in \mathbb{R}_+^{n_2}, \end{aligned} \tag{A.2}$$

and the corresponding two-stage static robust problem, $\Pi_{\text{Rob}}^\downarrow$

$$\begin{aligned} z_{\text{Rob}}^\downarrow &= \max \mathbf{c}^T \mathbf{x} + \mathbf{d}^T \mathbf{y} \\ &\quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \leq \mathbf{h}, \forall \mathbf{B} \in \mathcal{U}^\downarrow \\ &\quad \mathbf{x} \in \mathbb{R}_+^{n_1} \\ &\quad \mathbf{y} \in \mathbb{R}_+^{n_2}. \end{aligned} \tag{A.3}$$

Again, given uncertainty set $\mathcal{U} \in \mathbb{R}_+^{m \times n_2}$ and $\mathbf{h} > \mathbf{0}$, if \mathcal{U} is down-monotone, then $\mathcal{U}^\downarrow = \mathcal{U}$. Therefore, $\Pi_{\text{AR}}^\downarrow$ is essentially the same problem with Π_{AR} and we have $z_{\text{AR}}^\downarrow = z_{\text{AR}}$. Similarly, $z_{\text{Rob}}^\downarrow = z_{\text{Rob}}$. For the case when \mathcal{U} is not down-monotone, we prove the following lemma:

Lemma 10 Given uncertainty set $\mathcal{U} \in \mathbb{R}_+^{m \times n_2}$ and $\mathbf{h} > \mathbf{0}$, let z_{AR} and z_{Rob} be the optimal values of Π_{AR} defined in (1.1) and Π_{Rob} defined in (1.2), respectively. Suppose \mathcal{U} is not down-monotone, let \mathcal{U}^\downarrow be defined as in (A.1). Let z_{AR}^\downarrow and $z_{\text{Rob}}^\downarrow$ be the optimal values of $\Pi_{\text{AR}}^\downarrow$ defined in (A.2) and $\Pi_{\text{Rob}}^\downarrow$ defined in (A.3), respectively. Then,

$$z_{\text{AR}}^\downarrow = z_{\text{AR}}, \quad z_{\text{Rob}}^\downarrow = z_{\text{Rob}}.$$

Proof Suppose $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}), \mathbf{B} \in \mathcal{U}^\downarrow)$ is an optimal solution of $\Pi_{\text{AR}}^\downarrow$. Based on the discussion in Theorem 1, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$. Then,

$$\begin{aligned} z_{\text{AR}}^\downarrow &= \mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{B} \in \mathcal{U}^\downarrow} \max_{\mathbf{y} \in \mathbb{R}_+^{n_2}} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} \\ &= \mathbf{c}^T \mathbf{x}^* + z_{\text{AR}}^I(\mathcal{U}^\downarrow, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &= \mathbf{c}^T \mathbf{x}^* + z_{\text{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &\leq z_{\text{AR}}. \end{aligned}$$

The second equation holds from Lemma 9, and the last inequality holds because $\mathbf{x} = \mathbf{x}^*$ is a feasible first-stage solution. Therefore, $z_{\text{AR}}^\downarrow \leq z_{\text{AR}}$.

Conversely, suppose $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}(\mathbf{B}), \mathbf{B} \in \mathcal{U})$ is the optimal solution for Π_{AR} . Again, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\tilde{\mathbf{x}} > \mathbf{0}$. Using similar arguments, we have

$$\begin{aligned} z_{\text{AR}} &= \mathbf{c}^T \tilde{\mathbf{x}} + \min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y} \in \mathbb{R}_+^{n_2}} \left\{ \mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\tilde{\mathbf{x}} \right\} \\ &= \mathbf{c}^T \tilde{\mathbf{x}} + z_{\text{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\tilde{\mathbf{x}}) \\ &= \mathbf{c}^T \tilde{\mathbf{x}} + z_{\text{AR}}^I(\mathcal{U}^\downarrow, \mathbf{h} - \mathbf{A}\tilde{\mathbf{x}}) \\ &\leq z_{\text{AR}}^\downarrow. \end{aligned}$$

The last inequality holds because $\mathbf{x} = \tilde{\mathbf{x}}$ is a feasible first-stage solution for z_{AR}^\downarrow . Therefore, in both cases, we have $z_{\text{AR}} \leq z_{\text{AR}}^\downarrow$. Together with previous result, we have $z_{\text{AR}}^\downarrow = z_{\text{AR}}$. In the same way, we can show that $z_{\text{Rob}}^\downarrow = z_{\text{Rob}}$, we omit it here. \square

Next, we prove the following lemma which is crucial for proving Theorem 1.

Lemma 11 *Given a down-monotone set $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$, let $T(\mathcal{U}, \mathbf{h})$ be defined as in (2.6), then $T(\mathcal{U}, \mathbf{h})$ is down-monotone for all $\mathbf{h} > \mathbf{0}$.*

Proof Consider an arbitrary $\mathbf{h} > \mathbf{0}$ and $\mathbf{y} \in T(\mathcal{U}, \mathbf{h}) \subseteq \mathbb{R}_+^n$ such that

$$\mathbf{y} = \mathbf{B}^T \boldsymbol{\lambda}, \mathbf{h}^T \boldsymbol{\lambda} = 1, \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{B} \in \mathcal{U}.$$

Then, for any $\mathbf{z} \in \mathbb{R}_+^n$ such that $\mathbf{z} \leq \mathbf{y}$, set

$$\hat{B}_{ij} = \frac{z_j}{y_j} B_{ij}, i = 1, \dots, m, j = 1, \dots, n.$$

Clearly, $\hat{\mathbf{B}} \leq \mathbf{B}$ since $\mathbf{z} \leq \mathbf{y}$. Therefore, $\hat{\mathbf{B}} \in \mathcal{U}$ from the assumption that \mathcal{U} is down-monotone. Then,

$$\mathbf{z} = \hat{\mathbf{B}}^T \boldsymbol{\lambda}, \mathbf{h}^T \boldsymbol{\lambda} = 1, \boldsymbol{\lambda} \geq \mathbf{0}, \hat{\mathbf{B}} \in \mathcal{U},$$

which implies $\mathbf{z} \in T(\mathcal{U}, \mathbf{h})$. \square

B A compact formulation to compute Π_{Rob}

Suppose the uncertainty set \mathcal{U} is:

$$\mathcal{U} = \left\{ \mathbf{B} = \begin{bmatrix} \mathbf{b}_1^T \\ \mathbf{b}_2^T \\ \vdots \\ \mathbf{b}_m^T \end{bmatrix} \mid \sum_{i=1}^m \mathbf{P}_i \mathbf{b}_i \leq \mathbf{q}, \mathbf{b}_j \in \mathbb{R}_+^{n_2} \right\},$$

The problem Π_{Rob} defined in (1.2) is equivalent to

$$\begin{aligned} z_{\text{Rob}} &= \max \mathbf{c}^T \mathbf{x} + \mathbf{d}^T \mathbf{y} \\ \text{s.t. } & \max_{\mathbf{B} \in \mathcal{U}} \mathbf{e}_j^T (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}) \leq h_j, \forall j = 1, \dots, m \\ & \mathbf{x} \in \mathbb{R}_+^{n_1} \\ & \mathbf{y} \in \mathbb{R}_+^{n_2}, \end{aligned} \quad (\text{B.1})$$

For each j , by strong duality, we have

$$\max_{\mathbf{b}_j} \left\{ \mathbf{b}_j^T \mathbf{y} \mid \sum_{j=1}^m \mathbf{P}_j \mathbf{b}_j \leq \mathbf{q}, \mathbf{b}_j \geq \mathbf{0} \right\} = \min_{\boldsymbol{\alpha}_j \geq \mathbf{0}} \left\{ \mathbf{q}^T \boldsymbol{\alpha}_j \mid \mathbf{P}_j^T \boldsymbol{\alpha}_j \geq \mathbf{y}, \mathbf{P}_k^T \boldsymbol{\alpha}_j \geq \mathbf{0}, \forall k \neq j \right\}.$$

Therefore, we can convert (B.1) into the following form

$$\begin{aligned} z_{\text{Rob}} &= \max \mathbf{c}^T \mathbf{x} + \mathbf{d}^T \mathbf{y} \\ & \mathbf{e}_j^T \mathbf{A}\mathbf{x} + \boldsymbol{\alpha}_j^T \mathbf{q} \leq h_j, \forall j = 1, \dots, m \\ & \mathbf{y} - \mathbf{P}_j^T \boldsymbol{\alpha}_j \leq \mathbf{0}, \forall j = 1, \dots, m \\ & -\mathbf{P}_k^T \boldsymbol{\alpha}_j \leq \mathbf{0}, \forall j = 1, \dots, m, k \neq j \\ & \boldsymbol{\alpha}_j \in \mathbb{R}_+^m, \forall j = 1, \dots, m \\ & \mathbf{x} \in \mathbb{R}_+^{n_1} \\ & \mathbf{y} \in \mathbb{R}_+^{n_2}, \end{aligned}$$

which is a linear optimization problem and can be solved in polynomial time.

C Proofs of Lemmas 1 and 2

Proof of Lemma 1 Consider any $\mathbf{v}_1, \mathbf{v}_2 \in T(\mathcal{U}, \mathbf{h})$. Therefore, for $j = 1, 2$,

$$\mathbf{v}_j = \mathbf{B}_j^T \boldsymbol{\lambda}^j, \mathbf{h}^T \boldsymbol{\lambda}^j = 1, \boldsymbol{\lambda}^j \geq \mathbf{0}, \mathbf{B}_j \in \mathcal{U}.$$

For any arbitrary $\alpha \in [0, 1]$, let $\mu_i = \alpha \lambda_i^1 + (1 - \alpha) \lambda_i^2$ and $\hat{\mathbf{b}}_i^j = \mathbf{B}_j^T \mathbf{e}_i$ for $i = 1, \dots, m$. Then,

$$\begin{aligned} \alpha \mathbf{v}_1 + (1 - \alpha) \mathbf{v}_2 &= \sum_{i=1}^m (\alpha \lambda_i^1 \mathbf{b}_i^1 + (1 - \alpha) \lambda_i^2 \mathbf{b}_i^2) \\ &= \sum_{i=1}^m \mu_i \left(\frac{\alpha \lambda_i^1}{\mu_i} \mathbf{b}_i^1 + \frac{(1 - \alpha) \lambda_i^2}{\mu_i} \mathbf{b}_i^2 \right) \\ &= \sum_{i=1}^m \mu_i \cdot \hat{\mathbf{b}}_i \\ &= \hat{\mathbf{B}}^T \boldsymbol{\mu}, \end{aligned}$$

where $\hat{\mathbf{b}}_i \in \mathcal{U}_i$ since $\hat{\mathbf{b}}_i$ is a convex combination of \mathbf{b}_i^1 and \mathbf{b}_i^2 for all $i = 1, \dots, m$ and \mathcal{U}_i is convex. Also, note that $\hat{\mathbf{B}} \in \mathcal{U}$ and $\mathbf{h}^T \boldsymbol{\mu} = \alpha \mathbf{h}^T \boldsymbol{\lambda}^1 + (1 - \alpha) \mathbf{h}^T \boldsymbol{\lambda}^2 = 1$, we have

$$\alpha \mathbf{v}_1 + (1 - \alpha) \mathbf{v}_2 \in T(\mathcal{U}, \mathbf{h}).$$

Therefore, $T(\mathcal{U}, \mathbf{h})$ is convex. \square

Proof of Lemma 2 Note that in (2.6), $\mathbf{h}^T \boldsymbol{\mu} = 1$, which implies $\mu_j \leq \frac{1}{h_j}$ for $j = 1, \dots, m$. We assume without loss of generality that $h_1 \leq h_j$ for $j = 2, \dots, m$. Note that \mathcal{U} is down-monotone, so is $\mathcal{U}_j, j = 1, \dots, m$. Therefore, for $j = 2, \dots, m$, we have

$$\frac{1}{h_j} \mathcal{U}_j \subseteq \frac{1}{h_1} \mathcal{U}_1 \subseteq T(\mathcal{U}, \mathbf{h})$$

where the second set inequality holds because we can take $\boldsymbol{\mu} = \frac{\mathbf{e}_1}{h_1}$ in (2.6). Note that \mathcal{U}_1 is convex, so is $\frac{1}{h_1} \mathcal{U}_1$. Now, consider an arbitrary $\mathbf{v} \in T(\mathcal{U}, \mathbf{h})$ such that

$$\mathbf{v} = \mathbf{B}^T \boldsymbol{\lambda}, \mathbf{h}^T \boldsymbol{\lambda} = 1, \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{B} \in \mathcal{U}.$$

Let $\mathbf{b}_j = \mathbf{B}^T \mathbf{e}_j$, we have

$$\begin{aligned} \mathbf{v} &= \sum_{j=1}^m \lambda_j \mathbf{b}_j \\ &= \sum_{j=1}^m \lambda_j h_j \cdot \frac{1}{h_j} \mathbf{b}_j \\ &= \frac{1}{h_1} \hat{\mathbf{b}}, \end{aligned}$$

where $\hat{\mathbf{b}} \in \mathcal{U}_1$. The last equation holds because $\mathbf{h}^T \boldsymbol{\lambda} = 1$ and $\frac{1}{h_j} \mathcal{U}_j \subseteq \frac{1}{h_1} \mathcal{U}_1$. Therefore,

$$T(\mathcal{U}, \mathbf{h}) = \frac{1}{h_1} \mathcal{U}_1,$$

which is convex. □

D Proof of Theorem 2

1. For given $\mathbf{h} > \mathbf{0}$, note that:

$$T(\mathcal{U}^\theta, \mathbf{h}) = \left\{ \left(\frac{b_1}{h_1}, \dots, \frac{b_n}{h_n} \right) \mid \mathbf{b} \in T(\mathcal{U}^\theta, \mathbf{e}) \right\}.$$

Therefore, it is sufficient to show:

$$T(\mathcal{U}^\theta, \mathbf{e}) = \mathcal{A} := \left\{ \mathbf{b} \in \mathbb{R}_+^n \mid \sum_{j=1}^n b_j^{\frac{\theta}{\theta+1}} \leq 1 \right\}.$$

Consider any $\mathbf{b} \in \partial \mathcal{A}$, i.e., $\mathbf{b} \in \mathbb{R}_+^n$ such that

$$\sum_{j=1}^n b_j^{\frac{\theta}{\theta+1}} = 1.$$

Set

$$\lambda_j = b_j^{\frac{\theta}{\theta+1}}, x_j = b_j^{\frac{1}{\theta+1}}.$$

Then,

$$\lambda_j x_j = b_j, \mathbf{e}^T \boldsymbol{\lambda} = 1, \sum_{j=1}^n x_j^\theta = 1,$$

which implies $\mathbf{b} \in T(\mathcal{U}^\theta, \mathbf{e})$. Since both \mathcal{A} and $T(\mathcal{U}^\theta, \mathbf{e})$ are down-monotone, this further implies $\mathcal{A} \subseteq T(\mathcal{U}^\theta, \mathbf{e})$.

Conversely, consider the following problem:

$$\max_{\lambda, \mathbf{x} \geq \mathbf{0}} \left\{ \sum_{i=1}^n (\lambda_j x_j)^{\frac{\theta}{\theta+1}} \mid \mathbf{e}^T \boldsymbol{\lambda} = 1, \sum_{j=1}^n x_j^\theta \leq 1 \right\}$$

From Holder's Inequality, we have

$$\sum_{i=1}^n (\lambda_j x_j)^{\frac{\theta}{\theta+1}} \leq (\mathbf{e}^T \boldsymbol{\lambda})^{\frac{\theta}{\theta+1}} \cdot \left(\sum_{j=1}^n x_j^\theta \right)^{\frac{1}{\theta+1}} \leq 1.$$

Therefore, for any $\mathbf{b} \in T(\mathcal{U}^\theta, \mathbf{e})$, we have

$$\sum_{j=1}^n b_j^{\frac{\theta}{\theta+1}} \leq 1,$$

which implies $\mathbf{b} \in \mathcal{A}$. Therefore, $T(\mathcal{U}^\theta, \mathbf{e}) \subseteq \mathcal{A}$.

2. Similarly, it is sufficient to show

$$\text{conv}(T(\mathcal{U}^\theta, \mathbf{e})) = \mathcal{B} := \left\{ \mathbf{b} \in \mathbb{R}_+^n \mid \sum_{j=1}^n b_j \leq 1 \right\}.$$

From (3.2), we see that $\mathbf{e}_j \in T(\mathcal{U}^\theta, \mathbf{e})$. For any $\mathbf{b} \in \partial \mathcal{B}$, by taking $\boldsymbol{\lambda} = \mathbf{b}$ as the convex multiplier, we have

$$\mathbf{b} = \sum_{j=1}^n b_j \mathbf{e}_j.$$

Therefore, $\partial \mathcal{B} \subseteq \text{conv}(T(\mathcal{U}^\theta, \mathbf{e}))$. Since both \mathcal{B} and $\text{conv}(T(\mathcal{U}^\theta, \mathbf{e}))$ are down-monotone, we have $\mathcal{B} \subseteq \text{conv}(T(\mathcal{U}^\theta, \mathbf{e}))$.

Conversely, consider the following problem:

$$\max_{\mathbf{b} \geq \mathbf{0}} \left\{ \mathbf{e}^T \mathbf{b} \mid \sum_{j=1}^n b_j^{\frac{\theta}{1+\theta}} \leq 1 \right\} = \max_{\mathbf{a} \geq \mathbf{0}} \left\{ \sum_{j=1}^n a_j^{\frac{1+\theta}{\theta}} \mid \mathbf{e}^T \mathbf{a} \leq 1 \right\}$$

Note that

$$f(\mathbf{x}) = \sum_{j=1}^n x_j^{\frac{1+\theta}{\theta}}$$

is a convex function, we have

$$\sum_{j=1}^n a_j^{\frac{1+\theta}{\theta}} \leq (\mathbf{e}^T \mathbf{a})^{\frac{1+\theta}{\theta}} \leq 1.$$

Therefore, for any $\mathbf{b} \in T(\mathcal{U}^\theta, \mathbf{e})$, we have $\mathbf{b} \in \mathcal{B}$. Also note that \mathcal{B} is convex, therefore, $\text{conv}(T(\mathcal{U}^\theta, \mathbf{e})) \subseteq \mathcal{B}$.

3. From (3.2) and (3.3), we see that

$$\frac{1}{n} \text{diag} \left(\frac{1}{h_j} \right) \cdot \mathbf{e} \in \text{conv}(T(\mathcal{U}^\theta, \mathbf{h})), \text{ but } \frac{1}{n} \text{diag} \left(\frac{1}{h_j} \right) \cdot \mathbf{e} \notin T(\mathcal{U}^\theta, \mathbf{h}).$$

Therefore, $T(\mathcal{U}^\theta, \mathbf{h})$ is non-convex for all $\mathbf{h} > \mathbf{0}$.

4. Now, we compute $\kappa(\mathcal{U}^\theta, \mathbf{h})$. Remember that

$$\kappa(\mathcal{U}^\theta, \mathbf{h}) = \min\{\alpha \mid \text{conv}(T(\mathcal{U}^\theta, \mathbf{h})) \subseteq \alpha T(\mathcal{U}^\theta, \mathbf{h})\} = \min\{\alpha \mid \frac{1}{\alpha} \text{conv}(T(\mathcal{U}^\theta, \mathbf{h})) \subseteq T(\mathcal{U}^\theta, \mathbf{h})\}.$$

From (3.3) and scaling, we can observe that it is equivalent to find the largest α such that the hyperplane

$$\left\{ \mathbf{b} \in \mathbb{R}_+^n \mid \sum_{j=1}^n \frac{b_j}{h_j} = \frac{1}{\alpha} \right\}$$

intersects with the boundary of $T(\mathcal{U}^\theta, \mathbf{h})$. Therefore, we formulate the following problem:

$$\begin{aligned} \kappa(\mathcal{U}^\theta, \mathbf{h})^{-1} &= \min_{\mathbf{b} \geq \mathbf{0}} \left\{ \sum_{j=1}^n \frac{b_j}{h_j} \mid \sum_{j=1}^n \left(\frac{b_j}{h_j}\right)^{\frac{\theta}{1+\theta}} = 1 \right\} \\ &= \min_{\mathbf{a} \geq \mathbf{0}} \left\{ \sum_{j=1}^n a_j^{\frac{1+\theta}{\theta}} \mid \sum_{j=1}^n a_j = 1 \right\} \end{aligned}$$

By solving KKT conditions for the convex problem above, the optimal solution is $\mathbf{a} = \frac{1}{n} \cdot \mathbf{e}$. Therefore, we have

$$\kappa(\mathcal{U}^\theta, \mathbf{h}) = (n \cdot n^{-\frac{1+\theta}{\theta}})^{-1} = n^{\frac{1}{\theta}},$$

which completes our proof. \square

E Proofs of Lemmas 7 and 8

Proof of Lemma 7 We can write the dual of the inner problem of (5.12):

$$\begin{aligned} z_{\text{AR}}^{(B, h, d)} &= \min\{\mathbf{h}^T \boldsymbol{\alpha} \mid \mathbf{B}^T \boldsymbol{\alpha} \geq \mathbf{d}, (\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U}^{B, h, d}, \boldsymbol{\alpha} \in \mathbb{R}_+^m\} \\ &= \min\left\{ \lambda \mathbf{h}^T \left(\frac{\boldsymbol{\alpha}}{\lambda}\right) \mid \lambda \mathbf{B}^T \left(\frac{\boldsymbol{\alpha}}{\lambda}\right) \geq \mathbf{d}, \mathbf{h}^T \boldsymbol{\alpha} = \lambda, (\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B, h}, \mathbf{d} \in \mathcal{U}^d, \boldsymbol{\alpha} \in \mathbb{R}_+^m \right\} \\ &= \min\{ \lambda t \mid \lambda \mathbf{b} \geq \mathbf{d}, (\mathbf{b}, t) \in T(\mathcal{U}^{B, h}, \mathbf{e}), \mathbf{d} \in \mathcal{U}^d \}, \end{aligned}$$

where the second equality holds because $\mathcal{U}^{B, h, d} = \mathcal{U}^{B, h} \times \mathcal{U}^d$. \square

Proof of Lemma 8 Suppose

$$\mathcal{U}^{B, h, d} = \text{conv}((\mathbf{B}_1, \mathbf{h}_1, \mathbf{d}_1), \dots, (\mathbf{B}_K, \mathbf{h}_K, \mathbf{d}_K))$$

where $(\mathbf{B}_j, \mathbf{h}_j, \mathbf{d}_j), j = 1, \dots, K$ are the extreme points of $\mathcal{U}^{B, h, d}$. We can rewrite (5.13) as follows.

$$z_{\text{Rob}}^{(B, h, d)} = \max\{z \mid \mathbf{B}_j \mathbf{y} \leq \mathbf{h}_j, z - \mathbf{d}_j^T \mathbf{y} \leq 0, \forall j = 1, \dots, K, \mathbf{y} \in \mathbb{R}_+^n\}.$$

By writing the dual problem, we have:

$$z_{\text{Rob}}^{(B, h, d)} = \min \left\{ \sum_{j=1}^K \mathbf{h}_j^T \boldsymbol{\alpha}_j \mid \sum_{j=1}^K \mathbf{B}_j^T \boldsymbol{\alpha}_j \geq \sum_{j=1}^K \beta_j \mathbf{d}_j, \mathbf{e}^T \boldsymbol{\beta} = 1, \boldsymbol{\alpha}_j \in \mathbb{R}_+^m, \boldsymbol{\beta} \in \mathbb{R}_+^K \right\}.$$

Note that $\mathcal{U}^{B, h, d} = \mathcal{U}^{B, h} \times \mathcal{U}^d$, \mathbf{d} can be chosen regardless of \mathbf{B} and \mathbf{h} . Therefore, denote $\theta_j = \mathbf{h}_j^T \boldsymbol{\alpha}_j$, $\lambda = \mathbf{e}^T \boldsymbol{\beta}$ and we have

$$\begin{aligned} z_{\text{Rob}}^{(B, h, d)} &= \min \left\{ \lambda \sum_{j=1}^K \frac{\theta_j}{\lambda} \mathbf{h}_j^T \left(\frac{\boldsymbol{\alpha}_j}{\theta_j}\right) \mid \lambda \sum_{j=1}^K \frac{\theta_j}{\lambda} \mathbf{B}_j^T \left(\frac{\boldsymbol{\alpha}_j}{\theta_j}\right) \geq \mathbf{d}, \mathbf{d} \in \mathcal{U}^d, \boldsymbol{\alpha}_j \geq \mathbf{0} \right\} \\ &= \min \left\{ \lambda \sum_{j=1}^K \frac{\theta_j}{\lambda} \hat{t}_j \mid \lambda \sum_{j=1}^K \frac{\theta_j}{\lambda} \hat{\mathbf{b}}_j \geq \mathbf{d}, (\hat{\mathbf{b}}_j, \hat{t}_j) \in T(\mathcal{U}^{B, h}, \mathbf{e}), \mathbf{d} \in \mathcal{U}^d \right\} \\ &= \min\{ \lambda t \mid \lambda \mathbf{b} \geq \mathbf{d}, (\mathbf{b}, t) \in \text{conv}(T(\mathcal{U}^{B, h}, \mathbf{e})), \mathbf{d} \in \mathcal{U}^d \}, \end{aligned}$$

where the second equality holds because $e^T \left(\frac{\alpha_j}{\theta_j} \right) = 1, j = 1, \dots, K$.

□