# Note on the Reformulation for Static Robust Problems for Convex Uncertainty Sets

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In this note, we provide revised proofs for Lemma 4 (page 291), Lemma 6 (page 304) and Lemma 8 (page 318) for the reformulation of $\Pi^I_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h})$ for general convex, compact and down-monotone uncertainty sets. The original proof in the paper assume that the uncertainty is a polytope.

**Proof of Lemma 4** For each $j \in [m]$, let

$$\mathcal{U}_j = \left\{ \frac{1}{h_j} \cdot \boldsymbol{B}^T \boldsymbol{e}_j \;\middle|\; \boldsymbol{B} \in \mathcal{U} \right\}.$$

Then,

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h}) &= \max_{\boldsymbol{y}} \left\{ \boldsymbol{d}^T \boldsymbol{y} \;\middle|\; \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h},\ \forall \boldsymbol{B} \in \mathcal{U}, \boldsymbol{y} \in \mathbb{R}^n_+ \right\} \\ &= \max_{\boldsymbol{y}} \left\{ \boldsymbol{d}^T \boldsymbol{y} \;\middle|\; \boldsymbol{b}_j^T \boldsymbol{y} \leq 1,\ \forall \boldsymbol{b}_j \in \mathcal{U}_j, j \in [m], \boldsymbol{y} \in \mathbb{R}^n_+ \right\} \end{aligned}$$

Consider a feasible solution $\boldsymbol{y}$, we have

$$\begin{aligned} & \boldsymbol{b}_j^T \boldsymbol{y} \leq 1,\ \forall\, \boldsymbol{b}_j \in \mathcal{U}_j,\ j \in [m] \\ \Leftrightarrow \quad & \boldsymbol{b}^T \boldsymbol{y} \leq 1,\ \forall\, \boldsymbol{b} \in \textstyle\bigcup_{j=1}^m \mathcal{U}_j \\ \Leftrightarrow \quad & \boldsymbol{b}^T \boldsymbol{y} \leq 1,\ \forall\, \boldsymbol{b} \in \mathsf{conv}\left( \textstyle\bigcup_{j=1}^m \mathcal{U}_j \right) \end{aligned}$$

where the last inference follows from the fact that if $\boldsymbol{b}_1^T \boldsymbol{y} \leq 1$ and $\boldsymbol{b}_2^T \boldsymbol{y} \leq 1$, then

$$(\alpha \boldsymbol{b}_1 + (1-\alpha)\boldsymbol{b}_2)^T \boldsymbol{y} = \alpha \boldsymbol{b}_1^T \boldsymbol{y} + (1-\alpha) \boldsymbol{b}_2^T \boldsymbol{y} \leq 1,$$

for all $0 \leq \alpha \leq 1$. In Theorem 3 on page 298, we show that

$$\mathsf{conv}(T(\mathcal{U}, \boldsymbol{h})) = \mathsf{conv}\left( \bigcup_{j=1}^m \mathcal{U}_j \right).$$

Therefore,

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h}) &= \max_{\boldsymbol{y}} \left\{ \boldsymbol{d}^T \boldsymbol{y} \;\middle|\; \boldsymbol{b}^T \boldsymbol{y} \leq 1,\ \forall \boldsymbol{b} \in \mathsf{conv}(T(\mathcal{U}, \boldsymbol{h}), \boldsymbol{y} \in \mathbb{R}^n_+ \right\} \\ &= \max_{\boldsymbol{y}} \left\{ \boldsymbol{d}^T \boldsymbol{y} \;\middle|\; \boldsymbol{y} \in (\mathsf{conv}(T(\mathcal{U}, \boldsymbol{h}))^\circ \bigcap \mathbb{R}^n_+ \right\} \end{aligned}$$

where $\mathcal{S}^\circ$ is the polar set of $\mathcal{S}$. Note that the last maximization problem can be viewed as the support function of the set

$$\mathcal{C} = (\mathsf{conv}(T(\mathcal{U}, \boldsymbol{h}))^\circ \bigcap \mathbb{R}^n_+.$$

Therefore, we can reformulate it as the Minkowski functional over the polar $\mathcal{C}^{\circ}$ as follows (see Proposition 3.2.5 in Chapter 5 of [1]).

$$\begin{aligned} z^{I}_{\mathsf{Rob}}(\mathcal{U},\boldsymbol{h}) &= \min_{\boldsymbol{\lambda}} \left\{ \lambda \;\middle|\; \boldsymbol{d} \in \lambda \left( (\mathsf{conv}(T(\mathcal{U},\boldsymbol{h}))^{\circ} \bigcap \mathbb{R}^n_+ \right)^{\circ} \right\} \\ &= \min_{\boldsymbol{\lambda}} \left\{ \lambda \;\middle|\; \boldsymbol{d} \in \lambda \left( \mathsf{conv}(T(\mathcal{U},\boldsymbol{h}) \bigcup \mathbb{R}^n_- \right) \right\} \end{aligned}$$

where the second equation follows as

$$\left(\mathcal{S}_1 \bigcap \mathcal{S}_2\right)^{\circ} = \mathcal{S}_1^{\circ} \bigcup \mathcal{S}_2^{\circ}, \text{ and } (\mathcal{S}^{\circ})^{\circ} = \mathcal{S},$$

and $\left(\mathbb{R}^n_+\right)^{\circ} = \mathbb{R}^n_-$. Since $\boldsymbol{d} \in \mathbb{R}^n_+$, we have

$$\begin{aligned} z^{I}_{\mathsf{Rob}}(\mathcal{U},\boldsymbol{h}) &= \min_{\boldsymbol{\lambda}} \{ \lambda \mid \boldsymbol{d} \in \lambda \mathsf{conv}(T(\mathcal{U},\boldsymbol{h}) \} \\ &= \min_{\boldsymbol{\lambda}} \{ \lambda \mid \lambda \boldsymbol{b} \geq \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}(T(\mathcal{U},\boldsymbol{h}) \} \end{aligned}$$

which completes the proof. ■

Next, we provide a revised proof for Lemma 6 (page 304) for the reformulation of the following static robust problem, $\Pi^{I}_{\mathsf{Rob}}(\mathcal{U},\boldsymbol{h})$ with both uncertain constraint coefficients and objective coefficients for general convex, compact and down-monotone uncertainty sets.

$$\begin{aligned} z^{I}_{\mathsf{Rob}}(\mathcal{U},\boldsymbol{h}) = \max_{\boldsymbol{y}} \; \min_{\boldsymbol{d} \in \mathcal{U}^d} \; & \boldsymbol{d}^T \boldsymbol{y} \\ & \boldsymbol{B}\boldsymbol{y} \;\leq\; \boldsymbol{h}, \;\; \forall \boldsymbol{B} \in \mathcal{U}^{\boldsymbol{B}} \\ & \boldsymbol{y} \;\in\; \mathbb{R}^n_+. \end{aligned} \tag{1}$$

where $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$ and $\boldsymbol{h} > \boldsymbol{0}$.

**Proof of Lemma 6** We first introduce some notations. Let

$$\tilde{\mathcal{U}}^B = \left\{ [\boldsymbol{B} \;\; \boldsymbol{0}] \in \mathbb{R}^{m\times(n+1)}_+ \;\middle|\; \boldsymbol{B} \in \mathcal{U}^B \right\} \text{ and } \tilde{\mathcal{U}}^d = \left\{ \begin{pmatrix} -\boldsymbol{d} \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \boldsymbol{d} \in \mathcal{U}^d \right\}.$$

For each $j \in [m]$, let

$$\mathcal{U}_j = \left\{ \frac{1}{h_j} \cdot \boldsymbol{B}^T \boldsymbol{e}_j \;\middle|\; \boldsymbol{B} \in \mathcal{U} \right\} \text{ and } \tilde{\mathcal{U}_j} = \left\{ \begin{pmatrix} \boldsymbol{b} \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1}_+ \;\middle|\; \boldsymbol{b} \in \mathcal{U}_j \right\}.$$

Lastly, let

$$\tilde{\boldsymbol{h}} = \begin{pmatrix} \boldsymbol{h} \\ 0 \end{pmatrix}.$$

It is easy to see that

$$T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}}) = \left\{ \begin{pmatrix} \boldsymbol{b} \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1}_+ \;\middle|\; \boldsymbol{b} \in T(\mathcal{U},\boldsymbol{h}) \right\}.$$

Then,

$$\begin{aligned} z^{I}_{\mathsf{Rob}}(\mathcal{U},\boldsymbol{h}) &= \max_{\boldsymbol{y},\mu} \left\{ \mu \;\middle|\; \mu \leq \boldsymbol{d}^T\boldsymbol{y}, \forall \boldsymbol{d} \in \mathcal{U}^d, \; \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h}, \forall \boldsymbol{B} \in \mathcal{U}^B, \; \boldsymbol{y} \in \mathbb{R}^n_+ \right\} \\ &= \max_{\boldsymbol{y},\mu} \left\{ \mu \;\middle|\; -\boldsymbol{d}^T\boldsymbol{y} + \mu + 1 \leq 1, \forall \boldsymbol{d} \in \mathcal{U}^d, \; \boldsymbol{b}_j^T \boldsymbol{y} \leq 1, \; \forall \boldsymbol{b}_j \in \mathcal{U}_j, j \in [m], \boldsymbol{y} \in \mathbb{R}^n_+ \right\}. \end{aligned}$$

Now, let

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{y} \\ \mu + 1 \end{pmatrix} \in \mathbb{R}^{n+1}_+,$$

we have

$$z^I_{\mathsf{Rob}}(\mathcal{U},\boldsymbol{h}) = \max_{\boldsymbol{v}} \left\{ \boldsymbol{e}_{n+1}^T \boldsymbol{v} - 1 \;\middle|\; \boldsymbol{d}^T\boldsymbol{v} \le 1, \forall \boldsymbol{d} \in \tilde{\mathcal{U}}^d,\ \boldsymbol{b}^T\boldsymbol{v} \le 1, \boldsymbol{b} \in T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}}),\ \boldsymbol{v} \in \mathbb{R}^{n+1}_+ \right\}$$

where $\boldsymbol{e}_{n+1} \in \mathbb{R}^{n+1}_+$ is the unit vector for the $(n+1)$-th coordinate. Following the revised proof of Lemma 4, we can write

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U},\boldsymbol{h}) &= \max_{\boldsymbol{v}} \left\{ \boldsymbol{e}_{n+1}^T \boldsymbol{v} \;\middle|\; \boldsymbol{v} \in \left(\mathsf{conv}\left(\mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}})\right) \bigcup \tilde{\mathcal{U}}^d\right)\right)^{\circ} \bigcap \mathbb{R}^{n+1}_+ \right\} - 1 \\ &= \min_{\gamma} \left\{ \gamma \;\middle|\; \boldsymbol{e}_{n+1} \in \gamma \left(\mathsf{conv}\left(\mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}})\right) \bigcup \tilde{\mathcal{U}}^d\right) \bigcup \mathbb{R}^{n+1}_-\right) \right\} - 1. \end{aligned}$$

Note that $\boldsymbol{e}_{n+1} \in \mathbb{R}^{n+1}_+$. Therefore,

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U},\boldsymbol{h}) &= \min_{\gamma} \left\{ \gamma \;\middle|\; \boldsymbol{e}_{n+1} \in \gamma \mathsf{conv}\left(\mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}})\right) \bigcup \tilde{\mathcal{U}}^d\right) \right\} - 1 \\ &= \min_{\gamma, \alpha \in [0,1]} \left\{ \gamma - 1 \;\middle|\; \gamma \boldsymbol{z} \ge \boldsymbol{e}_{n+1}, \boldsymbol{z} = (1-\alpha)\boldsymbol{b} + \alpha \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}})\right), \boldsymbol{d} \in \tilde{\mathcal{U}}^d \right\} \\ &= \min_{\lambda, \alpha \in [0,1]} \left\{ \lambda \;\middle|\; (1+\lambda) \boldsymbol{z} \ge \boldsymbol{e}_{n+1}, \boldsymbol{z} = (1-\alpha)\boldsymbol{b} + \alpha \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}})\right), \boldsymbol{d} \in \tilde{\mathcal{U}}^d \right\}. \end{aligned}$$

Note that

$$\begin{aligned} & (1+\lambda)\boldsymbol{z} \ge e_{n+1}, \boldsymbol{z} = (1-\alpha)\boldsymbol{b} + \alpha \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}})\right), \boldsymbol{d} \in \tilde{\mathcal{U}}^d \\ \Leftrightarrow\ & (1+\lambda) z_{n+1} \ge 1,\ z_i \ge 0, \forall i \in [n],\ \boldsymbol{z} = (1-\alpha)\boldsymbol{b} + \alpha \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}})\right), \boldsymbol{d} \in \tilde{\mathcal{U}}^d \\ \Leftrightarrow\ & (1+\lambda)\alpha \ge 1,\ (1-\alpha)\boldsymbol{b} - \alpha \boldsymbol{d} \ge \boldsymbol{0}, \boldsymbol{b} \in \mathsf{conv}\left(T(\mathcal{U}^B, \boldsymbol{h})\right), \boldsymbol{d} \in \mathcal{U}^d \end{aligned}$$

where the last step of induction holds because $b_{n+1} = 0$ for all $\boldsymbol{b} \in \mathsf{conv}(T(\tilde{\mathcal{U}}^B, \tilde{\boldsymbol{h}}))$ and $d_{n+1} = 1$ for all $\boldsymbol{d} \in \tilde{\mathcal{U}}^d$. Therefore,

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U},\boldsymbol{h}) &= \min_{\lambda,\alpha} \left\{ \lambda \;\middle|\; (1+\lambda)\alpha \ge 1,\ (1-\alpha)\boldsymbol{b} - \alpha\boldsymbol{d} \ge \boldsymbol{0}, \boldsymbol{b} \in \mathsf{conv}\left(T(\mathcal{U}^B, \boldsymbol{h})\right), \boldsymbol{d} \in \mathcal{U}^d \right\} \\ &= \min_{\lambda,\alpha} \left\{ \lambda \;\middle|\; \lambda \ge \frac{1}{\alpha} - 1,\ \left(\frac{1}{\alpha} - 1\right)\boldsymbol{b} \ge \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(T(\mathcal{U}^B, \boldsymbol{h})\right), \boldsymbol{d} \in \mathcal{U}^d \right\} \\ &= \min_{\lambda} \left\{ \lambda \;\middle|\; \lambda\boldsymbol{b} \ge \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(T(\mathcal{U}^B, \boldsymbol{h})\right), \boldsymbol{d} \in \mathcal{U}^d \right\}. \end{aligned}$$

which completes the proof. ■

We show that the similar approach works for Lemma 8 (page 318). Consider the following static robust problem

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U}^{B,h,d}) = \max_{\boldsymbol{y}} \ \min_{\boldsymbol{d} \in \mathcal{U}^d} \ & \boldsymbol{d}^T \boldsymbol{y} \\ & \boldsymbol{B}\boldsymbol{y} \ \le \ \boldsymbol{h},\ \ \forall (\boldsymbol{B}, \boldsymbol{h}) \in \mathcal{U}^{B,h} \\ & \boldsymbol{y} \ \in \ \mathbb{R}^n_+, \end{aligned} \tag{2}$$

where $\mathcal{U}^{B,h,d} = \mathcal{U}^{B,h} \times \mathcal{U}^d$.

**Proof of Lemma 8** We first introduce some notations. Let

$$\tilde{\mathcal{U}}^{B,h} = \left\{ [\mathsf{diag}^{-1}(\boldsymbol{h})\boldsymbol{B}\ \ \boldsymbol{0}] \in \mathbb{R}^{m\times(n+1)}_+ \;\middle|\; (\boldsymbol{B}, \boldsymbol{h}) \in \mathcal{U}^{B,h} \right\} \text{ and } \tilde{\mathcal{U}}^d = \left\{ \begin{pmatrix} -\boldsymbol{d} \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \boldsymbol{d} \in \mathcal{U}^d \right\}.$$

Note that if $h_j = 0$ for any $(\boldsymbol{B}, \boldsymbol{h}) \in \mathcal{U}^{B,h}$ and $j \in [m]$, then $\boldsymbol{y} = \boldsymbol{0}$ and Theorem 6 holds trivially. Therefore we can assume without loss of generality that $\boldsymbol{h} > \boldsymbol{0}$ and the above sets are well-defined. Moreover, for each $j \in [m]$, let

$$\mathcal{U}_j = \left\{ \begin{pmatrix} \boldsymbol{B}^T\boldsymbol{e}_j \\ \boldsymbol{h}^T\boldsymbol{e}_j \end{pmatrix} \;\middle|\; (\boldsymbol{B}, \boldsymbol{h}) \in \mathcal{U}^{B,h} \right\} \subseteq \mathbb{R}^{n+1}, \text{ and } \tilde{\mathcal{U}}_j = \left\{ \boldsymbol{B}^T\boldsymbol{e}_j \mid \boldsymbol{B} \in \tilde{\mathcal{U}}^{B,h} \right\} \subseteq \mathbb{R}^{n+1}.$$

Note that for each $\tilde{\mathcal{U}}_j$, $\tilde{\mathcal{U}}_j$ normalizes any vector $\boldsymbol{b} \in \mathcal{U}_j$ so that the last component is one, then replace it with zero. This is very similar to the perspective function (See page 39 in [2]), which indicates that $\tilde{\mathcal{U}}_j$ is convex provided that $\mathcal{U}_j$ is convex. Then,

$$z^I_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h}) = \max_{\boldsymbol{y},z} \left\{ z \;\middle|\; z \le \boldsymbol{d}^T\boldsymbol{y}, \forall \boldsymbol{d} \in \mathcal{U}^d,\ \boldsymbol{B}\boldsymbol{y} \le \boldsymbol{h}, \forall (\boldsymbol{B}, \boldsymbol{h}) \in \mathcal{U}^{B,h},\ \boldsymbol{y} \in \mathbb{R}^n_+ \right\}.$$

Similar to the previous proof, by setting

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{y} \\ z+1 \end{pmatrix} \in \mathbb{R}^{n+1}_+,$$

we have

$$z^I_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h}) = \max_{\boldsymbol{v}} \left\{ \boldsymbol{e}^T_{n+1}\boldsymbol{v} - 1 \;\middle|\; \boldsymbol{d}^T\boldsymbol{v} \le 1, \forall \boldsymbol{d} \in \tilde{\mathcal{U}}^d,\ \boldsymbol{b}^T_j\boldsymbol{v} \le 1, \boldsymbol{b}_j \in \tilde{\mathcal{U}}_j, j \in [m],\ \boldsymbol{v} \in \mathbb{R}^{n+1}_+ \right\}.$$

where $\boldsymbol{e}_{n+1} \in \mathbb{R}^{n+1}_+$ is the unit vector for the $(n+1)$-th coordinate. Following the revised proof of Lemma 4, we can write

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h}) &= \max_{\boldsymbol{v}} \left\{ \boldsymbol{e}^T_{n+1}\boldsymbol{v} \;\middle|\; \boldsymbol{v} \in \left(\mathsf{conv}\left(\mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right) \bigcup \tilde{\mathcal{U}}^d\right)\right)^\circ \bigcap \mathbb{R}^{n+1}_+ \right\} - 1 \\ &= \min_{\gamma} \left\{ \gamma \;\middle|\; \boldsymbol{e}_{n+1} \in \gamma\left(\mathsf{conv}\left(\mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right) \bigcup \tilde{\mathcal{U}}^d\right) \bigcup \mathbb{R}^{n+1}_-\right) \right\} - 1. \end{aligned}$$

Note that $\boldsymbol{e}_{n+1} \in \mathbb{R}^{n+1}_+$. Therefore,

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h}) &= \min_{\gamma} \left\{ \gamma \;\middle|\; \boldsymbol{e}_{n+1} \in \gamma \mathsf{conv}\left(\mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right) \bigcup \tilde{\mathcal{U}}^d\right) \right\} - 1 \\ &= \min_{\gamma, \alpha \in [0,1]} \left\{ \gamma \;\middle|\; \gamma\boldsymbol{z} \ge \boldsymbol{e}_{n+1}, \boldsymbol{z} = (1-\alpha)\boldsymbol{b} + \alpha\boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right), \boldsymbol{d} \in \tilde{\mathcal{U}}^d \right\} - 1. \end{aligned}$$

Note that

$$\begin{aligned} & \gamma\boldsymbol{z} \ge e_{n+1}, \boldsymbol{z} = (1-\alpha)\boldsymbol{b} + \alpha\boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right), \boldsymbol{d} \in \tilde{\mathcal{U}}^d \\ \Leftrightarrow\quad & \gamma z_{n+1} \ge 1,\ z_i \ge 0, \forall i \in [n],\ \boldsymbol{z} = (1-\alpha)\boldsymbol{b} + \alpha\boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right), \boldsymbol{d} \in \tilde{\mathcal{U}}^d \\ \Leftrightarrow\quad & \gamma\alpha \ge 1,\ (1-\alpha)\boldsymbol{b} - \alpha\boldsymbol{d} \ge \boldsymbol{0}, \boldsymbol{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right), \boldsymbol{d} \in \mathcal{U}^d \end{aligned}$$

where the last statement holds because $b_{n+1} = 0$ for all $\boldsymbol{b} \in \mathsf{conv}(\cup_{j=1}^m \tilde{\mathcal{U}}_j)$ and $d_{n+1} = 1$ for all $\boldsymbol{d} \in \tilde{\mathcal{U}}^d$. Therefore,

$$z^I_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h}) = \min_{\gamma,\alpha} \left\{ \gamma - 1 \;\middle|\; \gamma \ge \frac{1}{\alpha},\ \frac{1-\alpha}{\alpha}\boldsymbol{b} \ge \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right), \boldsymbol{d} \in \mathcal{U}^d \right\}$$

Substitute by $\lambda = 1/\alpha - 1 \ge 0$, we have

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h}) &= \min_{\gamma,\lambda} \left\{ \gamma - 1 \;\middle|\; \gamma - 1 \ge \lambda,\ \lambda\boldsymbol{b} \ge \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right), \boldsymbol{d} \in \mathcal{U}^d \right\} \\ &= \min_{\lambda} \left\{ \lambda \;\middle|\; \lambda\boldsymbol{b} \ge \boldsymbol{d}, \boldsymbol{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right), \boldsymbol{d} \in \mathcal{U}^d \right\} \\ &= \min_{\lambda} \left\{ \lambda \;\middle|\; \lambda \sum_{j=1}^m \mu_j \frac{\boldsymbol{b}_j}{h_j} \ge \boldsymbol{d}, (\boldsymbol{b}_j, h_j) \in \mathcal{U}_j, \boldsymbol{e}^T\boldsymbol{\mu} = 1, \boldsymbol{\mu} \ge \boldsymbol{0}, \boldsymbol{d} \in \mathcal{U}^d \right\} \end{aligned}$$

For each $j \in [m]$, let

$$\theta_j = \frac{\mu_j / h_j}{\sum_{i=1}^{m} \mu_i / h_i}.$$

Note that

$$\mu_j = \frac{\theta_j h_j}{\sum_{j=1}^{m} \theta_j h_j}.$$

Then,

$$\begin{aligned} z^{I}_{\mathsf{Rob}}(\mathcal{U}, \boldsymbol{h}) &= \min_{\lambda} \left\{ \lambda \;\middle|\; \frac{\lambda}{\sum_{j=1}^{m} \theta_j h_j} \cdot \sum_{j=1}^{m} \theta_j \boldsymbol{b}_j \geq \boldsymbol{d}, (\boldsymbol{b}_j, h_j) \in \mathcal{U}_j, \boldsymbol{e}^T \boldsymbol{\theta} = 1, \boldsymbol{\theta} \geq \boldsymbol{0}, \boldsymbol{d} \in \mathcal{U}^d \right\} \\ &= \min_{\lambda} \left\{ \lambda \;\middle|\; \frac{\lambda}{t} \cdot \boldsymbol{b} \geq \boldsymbol{d}, (\boldsymbol{b}, t) \in \mathsf{conv}\left( T(\mathcal{U}^{B,h}, \boldsymbol{e}) \right), \boldsymbol{d} \in \mathcal{U}^d \right\} \\ &= \min_{\lambda} \left\{ \lambda t \;\middle|\; \lambda \boldsymbol{b} \geq \boldsymbol{d}, (\boldsymbol{b}, t) \in \mathsf{conv}\left( T(\mathcal{U}^{B,h}, \boldsymbol{e}) \right), \boldsymbol{d} \in \mathcal{U}^d \right\}. \end{aligned}$$

which completes the proof. ■

## References


[1] Lemarchal, C., and J. B. Hiriart-Urruty. “Convex analysis and minimization algorithms I.” Grundlehren der mathematischen Wissenschaften 305 (1996).

[2] Boyd, S., and Lieven Vandenberghe. *Convex optimization.* Cambridge university press (2004)