ADVANCE TOPICS IN ANALYSIS - REAL

NOTES COMPILED BY KATO LA

8 September 2011

Introductions

15 September 2011

Nested Interval Theorem: If $A_1 = [a_1, b_1], A_2 = [a_2, b_2], \cdots, A_n = [a_n, b_n], \cdots$ and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$

$$\Rightarrow \bigcap_{i=1}^{\infty} A_i \neq \emptyset$$

Proof: The set of left-hand endpoints $A = [a_1, a_2, \cdots]$ has an upper bound, say b_1.

Because A has an upper bound, it has a least upper bound, say L.

It turns out that $L \in \bigcap_{i=1}^{\infty} A_i$. i.e., Larger than every a_i and smaller than every b_i.

L is in every $[a_n, b_n]$ because it is a least upper bound for A, so $L \geqslant a_i \, \forall \, i$. The b_i's are all upper bounds for A and since L is the least upper bound, $L \leqslant b_i \, \forall \, i$. $\Rightarrow L$ is in every interval. ♣

Proof of Uncountability of $\mathbb{R}$ via the Nested Interval Theorem:
Suppose $\mathbb{R}$ is countable such that $x_1, x_2, x_3, \cdots$ where $x_i \in \mathbb{R}$. Let $[a_1, b_1]$ be some interval containing x_1. Divide $[a_1, b_1]$ into two disjoint closed intervals. One of those [or neither] will not contain x_2, call the interval $[a_2, b_2]$, continue $\cdots \bigcap_{i=1}^{\infty} [a_i, b_i]$ ***must*** be empty. $\Rightarrow\Leftarrow$
This is impossible since we already know the Nested Interval Theorem holds. Thus, our assumption of $\mathbb{R}$ being countable is false. $\therefore \mathbb{R}$ is uncountable. ♣

Example: The $\mathbb{Q}$ numbers between zero and one are each holding an umbrella.

Date: Fall 2011.

22 September 2011

Definitions: Given a function $f : A \longmapsto B$. f is *onto* if for every point $b \in B$ there is an $a \in A$ such that $f(a) = b$. f is said to *maps into* B if every $f(a)$ is in B. f is *one-to-one* if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

Example 1.5.1: $f(x) = \dfrac{1}{\pi}\arctan(x) + \dfrac{1}{2}$

Suppose Liz claims to have a one-to-one correspondence between $\mathbb{N}$ and (0, 1).

$$\begin{aligned} 1 &\longleftrightarrow 0.515151\cdots \\ 2 &\longleftrightarrow 0.333333\cdots \\ 3 &\longleftrightarrow 0.146810\cdots \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

However, the matching above is flawed, as we can find a decimal expansion guaranteed ***not*** to be on our list. We can create this unique decimal expansion by going down the diagonal by the following rule: Put a “5” as the n^{th} digit unless the n^{th} digit of the n^{th} number in the list is a “5”, then put a “6”. We will get the decimal expansion:

$$0.\_\_\_\_\cdots \longleftrightarrow 0.655\cdots$$

$\Rightarrow$ The unique decimal expansion differs in the n^{th} digit corresponding to the n^{th} decimal expansion in our supposed one-to-one correspondence list. Thus, we have constructed a decimal expansion between (0, 1) that is indeed not in our list because it differs in every digit compared to every number! $\Rightarrow\Leftarrow$ Therefore, (0, 1) is uncountable. ♣

Game Theory Approach to Show (0, 1) is Uncountable:
Let $A = \{a_0, a_1, a_2, \cdots\}$. Let $\sup A = L$ by the axiom of completeness. A subset $S \subseteq (0, 1)$ is chosen at the start of the game. Player A wins if $L \in S$. Player B wins if $L \notin S$. This game is a sure-win for Player B if S is countable. $S = \{x_1, x_2, x_3, \cdots\}$. Let $b_n = x_n$.

29 September 2011

A special case of an argument that mirrors the proof of Cantor's Theorem on page 32:

Claim: There are more subsets of $\mathbb{N}$ than elements of $\mathbb{N}$.

Proof:

$$\begin{array}{ll} \mathbb{N} & \text{Subsets of } \mathbb{N} \\ 1 \longleftrightarrow & \emptyset \\ 2 \longleftrightarrow & \{2,4,6,8,\cdots\} \\ 3 \longleftrightarrow & \{1,2,3,\cdots\} \\ 4 \longleftrightarrow & \{10,100,1000,\cdots\} \\ \vdots & \quad\vdots \end{array}$$

We display a subset of $\mathbb{N}$ that cannot be in the one-to-one mapping:

Let $W = \{$all the elements of $\mathbb{N}$ that are ***not*** in the set they are paired$\}$

In our one-to-one mapping, $W = \{1, 4, \cdots\}$

$\Rightarrow W$ is ***not*** in our one-to-one mapping; however, ***can*** W be in our mapping?
i.e., Eventually, In our mapping we have $n \longleftrightarrow \{1, 4, \cdots\}$

If $n \notin \{1, 4, \cdots\}$, then n is placed in W, so $n \in \{1, 4, \cdots\}$, but this is impossible.
If $n \in \{1, 4, \cdots\}$, then $n \notin W$, so $n \notin \{1, 4, \cdots\}$, but this is impossible as well.

In both cases, we have a contradiction. Regardless of our one-to-one mapping we select, we can construct the set W such that it does not appear on our list.
$\therefore$ There are indeed more subsets of $\mathbb{N}$ than elements of $\mathbb{N}$. ♣

Definition: A *sequence* is a function whose domain is $\mathbb{N}$. i.e., It is a fancy way of saying something is countable.

Definition: A sequence (a_n) *converges* to a real number L if $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that whenever $n \geqslant N$ it follows that $|a_n - L| < \varepsilon$.

Examples:
(i) $\lim\limits_{n\to\infty} a_n = t$

By definition, the expression means: $\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$ such that $\forall\ n \geqslant N \Rightarrow |a_n - t| < \varepsilon$.

(ii) $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

By definition, expression means: $\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$ such that $\forall\ n \geqslant N \Rightarrow \left|\dfrac{1}{n} - 0\right| < \varepsilon$.

6 October 2011

Definition: If $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant a_{n+1} \leqslant \cdots$, then (a_n) is an *increasing* sequence.

Definition: If $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant a_{n+1} \geqslant \cdots$, then (a_n) is a *decreasing* sequence.

There are no sequences that are properly described as "increasing and decreasing [except constant sequences].

Definition: A *monotone* sequence is a sequence that is either increasing or decreasing.

Monotone sequences ***do not*** imply convergence.

Examples:
(i) 1, 2, 3, 4,$\cdots$ does not converge

(ii) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ does converge (to zero).

Monotone Convergence Theorem: A monotone, bounded sequence converges.

Proof: Is L a limit?
Given any $\varepsilon > 0$, is there a point in the sequence after which all terms are in $(L-\varepsilon, L+\varepsilon)$?

Because L is the least upper bound there must be elements in the sequence between $L-\varepsilon$ and L or maybe L is the sequence [otherwise $L-\varepsilon$ would be an upper bound of the sequence less than the least upper bound].

Since the sequence is monotone, all terms after the n^{th} term are between a_n and L. ♣

Definition: Given a sequence a_n, an *infinite series* is a formal expression:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots .$$

Every infinite series has a corresponding *sequence of partial sums*

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n \end{aligned}$$

The series $\sum_{n=1}^{\infty} a_n$ is said to converge to A if and only if the sequence of partial sums converges to A denoted as:

$$\sum_{n=1}^{\infty} a_n = A$$

What about 1 - 1 + 1 - 1 + 1 - 1 + $\cdots$?

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 0 \\ s_3 &= 1 \\ s_4 &= 0 \\ &\vdots \end{aligned}$$

This series does not converge. Why? If we pair the one's $(1-1)+\cdots$, we get zero as our sum. If we pair the one's $1-(1+1)-\cdots$, we get one as our sum. This series is ***not well-defined***!

Two classic examples:

(i) Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2\cdot 2} \\ s_3 &= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} \\ s_4 &= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} \\ &\vdots \\ s_n &= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{n\cdot n} \\ &< 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n} \\ &= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 - \frac{1}{n} \end{aligned}$$

Thus, $s_n < 2 - \frac{1}{n}$

The s_n's are bounded by two and increasing so they have a limit and thus converge to an unknown limit, but is bounded by two.

(ii) Consider $\sum_{n=1}^{\infty} \frac{1}{n} \longrightarrow$ this actually diverges!

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2} \\ s_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \end{aligned}$$

Look at powers of two!

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2} \\ s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 2 \\ s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 2\frac{1}{2} \\ S_{2^n} &> \underbrace{1 + n\left(\frac{1}{2}\right)}_{\text{divergers!}} \end{aligned}$$

Here, s_n is increasing and unbounded. Thus, it does not converge.

13 October 2011

Examples of good subsequence use:

(i) $0 < b < 1$

$b > b^2 > b^3 > \cdots > 0$

$$\left.\begin{array}{ll} b^n & \to L \\ \underbrace{b^n \cdot b^n} & \to L^2 \\ & \\ b^{2n} & \to L^2 \end{array}\right\} \Rightarrow L = 0$$

(ii) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \cdots$

The above sequence does *not* converge.

Bolzano-Weierstrass Theorem: Every bounded sequence contains a convergent subsequence.

Note: The bounded condition is necessary. Consider $\{1, 2, 3, 4, \cdots\}$ The sequence is unbounded and $\nexists$ a convergent subsequence.

Proof: Let all terms of the sequence lie in $[-M, M]$. Divide $[-M, M]$ into two intervals: $[-M, 0], [0, M]$. One of these intervals contain infinitely many sequence terms, say $[0, M]$. Continue letting each new interval have half the length of the preceding interval and contain infinitely many sequence terms. Choose in each interval a point of the sequence [not equal to any point chosen earlier]. This defines a subsequence. ♣

Claim: The subsequence converges to L where L is in the intersection of the intervals $[-M, M] \cap [0, M] \cap \cdots$

Given any $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that ***all*** terms in the subsequence after a_N are in $(L - \varepsilon, L + \varepsilon)$. Why? The interval length is $\longrightarrow$ *zero*, so at some point, ***all*** intervals will be contained in $(L - \varepsilon, L + \varepsilon)$. ♣

20 October 2011

Midterm

27 October 2011

A little trick to show $x + \dfrac{1}{x} \geqslant 2$:

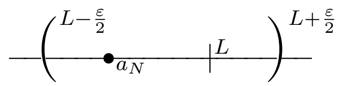
First, recall the arithmetic mean is greater than or equal to the geometric mean.

$$\frac{a+b}{2} \geqslant \sqrt{ab}$$
$$\frac{x + \frac{1}{x}}{2} \geqslant \sqrt{x \cdot \frac{1}{x}}$$
$$x + \frac{1}{x} \geqslant 2 \cdot 1$$

Definition: A *Cauchy sequence* (a_n) is one satisfying the following property:
$\forall\ \varepsilon > 0, \exists\ N \in \mathbb{N}$ such that whenever $m, n \geqslant N$, it follows $|a_m - a_n| < \varepsilon$.

Fact 1: If (a_n) converges, then it is Cauchy.

Picture:



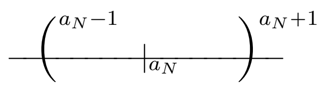
Any two of these can be at most, ε units apart.

$$a_1, a_2, \cdots, \underbrace{a_N}, \overbrace{a_{N+1}}, \cdots$$

After a_N, all points are in $\left(L - \dfrac{\varepsilon}{2}, L + \dfrac{\varepsilon}{2}\right)$

Fact 2: Every Cauchy sequence converges.

Picture:

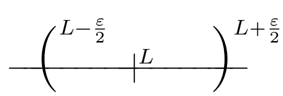


Note: The "tail" is in the above neighborhood. a_N is the point after which $|a_m - a_n| < \varepsilon$

Notice 1: Every Cauchy sequence is bounded.
Examine $|a_1|, |a_2|, \cdots, |a_{N-1}|, |a_N| + 1$. Note the largest value produced here is an upper bound. Likewise, the same could be constructed for a lower bound.
Let $\varepsilon = 1$. $|a_m - a_n| < 1 \; \forall \; m, n \geqslant N$. ♣

Notice 2: Every bounded sequence has a convergent subsequence. Suppose our Cauchy sequence (a_n) has a subsequence that converges to L.

Picture:



Note: After some term in the subsequence, all terms are in the above interval.

There is a term $a_N \in (a_n)$ ***after which*** any two terms differ by $\dfrac{\varepsilon}{2}$. Choose the larger of [the two] N and term K - after that point all subsequent terms will be in $\left(L - \dfrac{\varepsilon}{2}, L + \dfrac{\varepsilon}{2}\right)$ and all sequence terms will be written $\dfrac{\varepsilon}{2}$ of any subsequence term.

Conclusion: ***All*** terms will be in $\left(L - \dfrac{\varepsilon}{2}, L + \dfrac{\varepsilon}{2}\right)$ after that point.

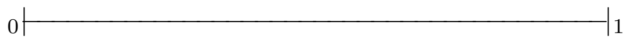
Theorem 2.7.3: If $\sum_{n=1}^{\infty} a_n$ converges, then $(a_n) \longrightarrow 0$.

Example: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$

Partial sums: $\underbrace{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \cdots}_{\text{Each of these partial sum values is approaching 1}}$ Thus, $\sum \frac{1}{2^n} = 1$

Chapter 3

Section 1: The Cantor Set



- Divide the above interval into thirds and delete the middle third [considered as an open interval, the deleted one].
- Now we have two closed intervals $\left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right]$. Repeat the process for each new set of intervals. We get $\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right]$.
- Repeat $\cdots \Rightarrow$ What remains is the ***Cantor Set*** denoted C.

Questions:
(i) Is anything in C? - Yes. Namely 0, 1, $\frac{1}{3}, \frac{2}{3}, \cdots$ all the endpoints.

(ii) Are all the points in C rational numbers? - No!
(iii) Does C contain any intervals? - No! [intuitively].

Amazing Fact: C is uncountable!

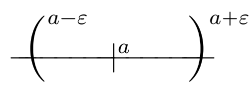
Every infinite string of zero's and one's determine a point in C.
$10011010111 \cdots \rightsquigarrow$ Can be considered directions to determine the progression of divisions. i.e., Point is in right of first division, point is in left half at the next one-third. We can also view the Cantor set process as leading us with all numbers between zero and one that have no one's in their base three representation. This is uncountable as well.

3 November 2011

Open and Closed Sets

Remember: $V_\varepsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\}$

Picture:



Definition: A subset S of $\mathbb{R}$ is called *open* if every point of S has a neighborhood that lies completely in S. *Or* If $\forall\ x \in S$, there is an $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq S$. [Set theory notation].

Are the following sets *open*?

(i) $\mathbb{R}$ - Yes, in fact any neighborhood of $\mathbb{R}$ will lie in $\mathbb{R}$.
(ii) (0, 1) - Yes, if not, then our notation for "open" would be incorrect.

(iii) $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ - No, because the elements of the neighborhood of any point in the sequence will *not* be in the sequence.
(iv) $\{0\}$ - No, any neighborhood around zero will contain points not in zero.
(v) $\{x \in \mathbb{R} \mid x \geqslant 10\}$ - No, because looking at the point 10, no matter how small of a neighborhood we create, it will contain points *not* in the set.
(vi) $\mathbb{Q}$ - No, the same idea as part (iii). No matter how big or small the neighborhood we create around any rational number, there will be elements in the neighborhood that are *not* rational.
(vii) $(0, 1) \cup (9, 10)$ - Yes, union of open sets is open.

So if S is ***not*** open, it means that $\exists$ an $x \in S$ such that ***every*** neighborhood of x contains points not in S.

$\Rightarrow$ No x's exist in $\emptyset$. So the empty set cannot be said to be not open. So $\emptyset$ is open.

Theorem: (a) The union of any number of open sets is open.
(b) The intersection of a finite number of open sets is open.

Note: Why is *finite* necessary in part (b)? - To circumvent if the following happens:

$$\text{Let } s_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$
$$\bigcap_{n=1}^{\infty} s_n = \{0\}$$

The above intersection is ***not*** open!

Proof: (a) Suppose S_α is a collection of open sets. [By placing an n as a subscript, the usual implication is countability, which is not necessary when talking about open sets.]

Let $x \in \bigcup S_\alpha$.

Then $x \in S_t$ where S_t is one of many sets in the union.

But S_t is open, so there is a $V_\varepsilon(x) \subseteq S_t \subseteq \bigcup S_\alpha$.

So x has a neighborhood that lies completely in the union. Hence, the union is open. ♣

(b) Suppose $S_1, \cdots, S_n$ is a collection of open sets.

Let $x \in \bigcap_{n=1}^{n} S_i$.

Thus, $x \in S_1, x \in S_2, \cdots, x \in S_n$. But the S's are open.

There are neighborhoods of x such that $V_{\varepsilon_1} \subseteq S_1, V_{\varepsilon_2} \subseteq S_2, \cdots, V_{\varepsilon_n} \subseteq S_n$.

Find the smallest ε, say ε^*. [We can find a "smallest" ε because there is a finite number of subsets]. Then $V_{\varepsilon^*}(x)$ will be contained in all the ***others*** and will

be a subset of $\bigcap_{n=1}^{n} S_i$. ♣

i.e., Of all the ε-neighborhoods, the ε^*-neighborhood will be contained in each of the ε-neighborhoods. $\Rightarrow V_{\varepsilon^*}(x)$ will be in each $S_1, \cdots, S_n$. $\Rightarrow V_{\varepsilon^*}(x)$ will be contained in $\bigcap_{n=1}^{n} S_i$.

Definition: x is a *limit point* [or *cluster* point or *accumulation* point] of a set A if every neighborhood of x contains points of A other than x. Note that x may or may not be in A.

Set	Set of Limit Points
$\mathbb{R}$	$\mathbb{R}$
$(0,1)$	$[0,1]$
$\{0\}$	$\emptyset$
$\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$	$\{0\}$
$\mathbb{Z}$	$\emptyset$
$(0,1]$	$[0,1]$
$\mathbb{Q}$	$\mathbb{R}$

One can see x is a limit point of a set A if there is a sequence of points in A that have x as a limit. i.e., A limit of a sequence (a_n) is a *limit point* of $\{a_1, a_2, \cdots\}$.

Definition: A point of A that is *not* a limit point of A is called an *isolation point* of A. i.e., $(0,1) \cup \{6\} = A$. In classic cases, isolated points are points "outside" the set.

Definition: A set is *closed* if it contains all its limit points. i.e., (0, 1) is *not* closed because the set of limit points is [0, 1] and $0, 1 \notin (0,1)$.

Are the following sets *closed*?

(i) $\mathbb{R}$ - Yes, it is the fact $\mathbb{R}$ will contain all of its limit points.
(ii) $\emptyset$ - Yes.
(iii) (0, 1) - No, because zero is a limit point, but $0 \notin (0,1)$.
(iv) [0, 1] - Yes.
(v) [0, 1) - No, because one is a limit point, but $1 \notin [0,1)$.
(vi) $[0, \infty)$ - Yes.
(vii) $\mathbb{Q}$ - No.
(viii) $\mathbb{Z}$ - Yes.
(ix) $\{0\}$ - Yes. In fact, any set with a single thing in it will be closed.

Generalization: Every finite set is closed.

(x) $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ - No, because zero is not in the set.

10 November 2011

Definition: The *closure* of a set A of real number is the union of A with the set of limit points of A. **Notation**: $\mathrm{cl}(A)$ or $\overline{A}$.

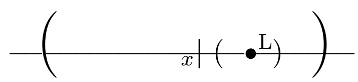
Theorem: If $A \subseteq \mathbb{R}$, then $\overline{A}$ is the smallest closed set containing A.

Proof: (a) Show $\overline{A}$ is closed. (b) Show $\overline{A}$ is the smallest closed set.

Let $\overline{A} = A \cup L$ where L is the set of all limit points of A.

Let x be a limit point of $\overline{A}$. Every neighborhood of x contains points of $\overline{A}$. Even more, every neighborhood of x contains either infinitely many points of A or infinitely many points of L. i.e., $x \in A$ or $x \in L$. In the first case, x would be a limit point of A, so $x \in \overline{A}$.

Picture:



Note: Given point L, the neighborhood of L ***must*** contain points of A. The outer neighborhood is the neighborhood of x.

In the second case, every neighborhood of x contains points of L, but each of those points of L have neighborhoods, within the neighborhood of x, that contains points of A. So every neighborhood of x contains points of A and x is a limit point of A.

Thus, $x \in \overline{A}$ since $\overline{A}$ is the set A together with all its limit points.

So $\overline{A}$ contains its limit points and $\overline{A}$ is closed.

We assume that B is a closed set containing A. Then we prove that $\overline{A} \subset B$.

$A \cup L = \overline{A} \subseteq B$
$x \in L$ means every neighborhood of x contains points of A.
$\Rightarrow$ Every neighborhood of x contains point of B.
$\Rightarrow$ x is a limit point of B.
$\Rightarrow$ $x \in B$ because B is closed. So it contains all its limit points.
$\therefore$ $\overline{A}$ is the smallest closed set. ♣

Corollary: $\overline{\overline{A}} = \overline{A}$

Definition: The *complement* of a set A of real numbers is the set of all real numbers *not* in A. **Notation**: $A^{\complement}$. **Notice**: $(A^{\complement})^{\complement} = A$.

Theorem: Let $A \subseteq \mathbb{R}$. A is open if and only if $A^{\complement}$ is closed. A is closed if and only if $A^{\complement}$ is open.

Theorem: The intersection of any number of closed sets is closed. The union of a finite number of closed sets is closed.

Wild Idea: Given the above definition and theorems, consider the following: Given sets such as all transcendentals, [0, 1), or {0} - How many sets can be created using closures and complements?

A Few Words on Point-Set Topology

Definition: A *topology* is a set [or universe] together with certain subsets are called *open sets*. The open sets must have the following properties:
(a) Ø and the universe set must be open.
(b) The union of any number of open sets must be another open set.
(c) The intersection of a finite number of open sets must be open.

Example: Let the universal set U be the following: $U = \{a, b, c, d\}$
The following are topologies:

Topology #1: $\emptyset, U, \{a, b\}, \{c, d\}$
Topology #2: $\emptyset, U, \{a, b\}, \{c, d\}, \{a, b, c\}$
Topology #3: $\emptyset, U$
Topology #4: $\emptyset, U$, every possible subset

What are the limit points of $\{a, b\}$?

Limit Points: $\{a, b\}, \{b, c, d\}, \{a, b, c, d\}, \emptyset$

Definition: A subset K of $\mathbb{R}$ is *compact* if and only if every sequence of points in K has a subsequence that converges to a point *in* K.

Example: Are the following sets *compact*?
(i) [0, 1] - Yes.
(ii) $[0, \infty)$ - No, consider $\mathbb{N}$.
(iii) $\{1\}$ - Yes, construct a sequence out of the elements of the set.
(iv) (0, 1) - No, even though a harmonic series is present within the set, as well as a convergent subsequence, what the subsequence converges to is *not* in the set.
(v) $\mathbb{Z}$ - No, look at (ii).
(vi) $\{1, 2\}$ - Yes, drop the number needed to create a sequence.

Generalization: Any finite set will be compact.

17 November 2011

Theorem: (a) The intersection of an arbitrary number of closed sets is closed.
(b) The union of a finite number of closed sets is closed.

Proof: To prove (a), use the generalization of $(A \cup B)^{\complement} = A^{\complement} \cup B^{\complement}$.

$$\left(\bigcap_{\alpha \in \Delta} G_\alpha \right)^{\complement} = \bigcup_{\alpha \in \Delta} G_\alpha^{\complement}.$$

The complement of $\bigcap G_\alpha$ is open and thus, $\bigcap G_\alpha$ is closed.

To prove (b), use the definition of closed.

$\bigcap_{\alpha \in \Delta} G_\alpha$ is closed.

Question: Let x be a limit point of $\bigcap G_\alpha$. Is $x \in \bigcap G_\alpha$?

If so, then $\bigcap G_\alpha$ is closed. If x is a limit point of $\bigcap G_\alpha$, then every neighborhood of x contains points of $\bigcap G_\alpha$.

x is in ***every*** G_α! Why? Every neighborhood of x contains points from ***each one*** of the G_α's.

So x is a limit point of every G_α. The G_α's are closed and thus, contain all their limit points.

Thus, x is in every G_α, and thus in $\bigcap G_\alpha$. ♣

Recall: A set $K \subseteq \mathbb{R}$ is compact if and only if every sequence of points in K has a subsequence that converges to a point in K.

Heine-Borel Theorem: A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Definition: An *open cover* of a set $A \subseteq \mathbb{R}$ is a collection of open sets whose union contains A. If an open cover contains a finite collection of ope sets whose union contains A, then that is called a *finite subcover*.

Examples: Open covers of (0, 1): (i) $\{(0,1)\}$, (ii) $\left\{\left(0, \frac{1}{2}\right), \left(0, \frac{2}{3}\right), \left(0, \frac{3}{4}\right), \cdots\right\}$,

(iii) $\left\{\left(0, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, 1\right)\right\}$.

Open cover of [0, 1]: $\left\{\left(-\frac{1}{100}, \frac{1}{100}\right), \left(0, \frac{1}{2}\right), \left(0, \frac{2}{3}\right), \cdots, \left(\frac{99}{100}, \frac{101}{100}\right)\right\}$.

$\Rightarrow$ A finite subcover: $\left\{\left(-\frac{1}{100}, \frac{1}{100}\right), \left(0, \frac{1}{2}\right), \cdots, \left(0, \frac{100}{101}\right), \left(\frac{99}{100}, \frac{101}{100}\right)\right\}$.

Theorem: The following are equivalent:
(a) K is compact.
(b) K is closed and bounded.
(c) Every open covering of K contains a finite subcover.

Proof: (a) $\Rightarrow$ (b)
Suppose K is not bounded. Let $K_1 > 10, K_2 > 10^2, K_3 > 10^3, \cdots$.
$\Rightarrow$ (K_n) diverges to infinity. $\nexists$ a convergent subsequence.
Contra-positive: Not bounded $\Rightarrow$ Not compact.
$\therefore$ Compact $\Rightarrow$ Bounded.

Compact $\Rightarrow^?$ Closed

Let L be a limit point of K. Is $L \in K$?
$\Rightarrow$ $\exists$ a sequence of points in $K \to L$. Thus, every subsequence converges to L. By definition of compact, $L \in K$.
$\therefore$ K is closed.

1 December 2011

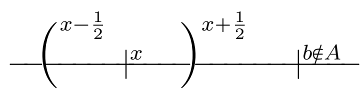
Theorem: The only subsets of $\mathbb{R}$ that are open and closed are $\mathbb{R}$ and $\emptyset$.

Proof: Suppose $A \subset \mathbb{R}, A \neq \emptyset$, and A is open and closed. [We want to show $A = \mathbb{R}$].
Since $A \neq \emptyset \Rightarrow x \in A$.

Idea: Show $\forall\ y > x$ such that $y \in A$. [Then show $\forall\ z < x$ such that $z \in A$].

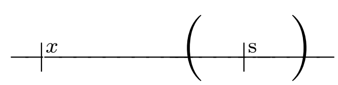
"Big" ***suppose*** $\exists$ real numbers greater than x such that the real numbers are in $A^{\complement}$.
$\Rightarrow \{x\}$ is bounded above by elements in $A^{\complement}$.

Picture:



$\Rightarrow \exists$ an element $s = \inf A^{\complement}$ such that $x < S$.

Picture:



Note: $s = \inf A^{\complement}$ such that $s > x$. All points left of s, but within the neighborhood of s are elements of A.

Where is s? $s \in A$? $s \in A^{\complement}$?

s is a limit point of A.
$\Rightarrow s \in A$ because A is closed.
s is a limit point of $A^{\complement}$ as well.
$\Rightarrow s \in A^{\complement}$.

How? There are two interpretations:

The left side of s, $\exists$ points of $A^{\complement}$. The points are ***not*** necessarily all of $A^{\complement}$. But, we can find telescoping intervals where we can find a point of $A^{\complement}$.
$\Rightarrow$ A sequence of points from $A^{\complement}$ that approach s.
$\Rightarrow s$ is a limit point of $A^{\complement}$.

OR $\rightarrow A$ is open.
$\Rightarrow \exists$ a neighborhood around s where it is completely contained in A.
If $s = \inf A^{\complement}$, then $\exists$ a lower bound greater than our supposed greatest lower bound.

Functions

Examples: (i) If $f(x) = x^2$, find $\lim_{x \to 3} f(x)$. [9]

(ii) If $f(x) = \begin{cases} 1 & \text{if} \quad x = 1 \\ 0 & \text{otherwise} \end{cases}$ find $\lim_{x \to 1} f(x)$. [0]

$\varepsilon - \delta$ **Definition**: Let $f : A \longrightarrow \mathbb{R}$ and c be a limit point of A. We write $\lim_{x \to c} f(x) = L$ if and only if $\forall\ \varepsilon > 0, \exists\ \delta > 0$ such that $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. [for $x \in A$].
Note: For every L-neighborhood, we can find a c-neighborhood.

Definition: A function $f : A \longrightarrow \mathbb{R}$ is *continuous at a point* $c \in A$ if and only if $\forall\ \varepsilon > 0, \exists\ \delta > 0$ such that $0 < |x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. [for $x \in A$].

Alternative Continuity Definition: A function is continuous if and only if the inverse image of an open set of B, is an open set of A.

Weird Function

$$f(x) = \begin{cases} 0 & \text{if} \quad x \text{ is irrational} \\ \dfrac{1}{n} & \text{if} \quad x = \dfrac{m}{n}, \text{in lowest terms} \\ 1 & \text{if} \quad x = 0 \end{cases}$$

What is $\lim_{x \to 5} f(x)$? - It does not exists.

Is $f(x)$ continuous at π? - Yes!
Is $f(x)$ continuous at 5? - No!

8 December 2011

Theorem 4.4.2: Let $f : A \longrightarrow \mathbb{R}$ be continuous. If $K \subseteq A$ is compact, then $f(K)$ is compact.

$$\begin{array}{ll} & y_1, y_2, \cdots \rightarrow \text{a sequence in } f(K)\text{, call it } (y_n) \\ f \text{ maps} & \uparrow \quad \uparrow \\ & x_1, x_2, \cdots \rightarrow \text{Points in } K \text{ that map to } f(K)\text{. This is a sequence in } K\text{, call it } (x_n). \end{array}$$

Since K is compact, this sequence has a subsequence that converges to a point in K. Let the following be our subsequence:

$$\begin{array}{ll} & u_1, u_2, u_3, \cdots \rightarrow u \in K \\ f \text{ maps} & \downarrow \quad \downarrow \quad \downarrow \\ & v_1, v_2, v_3, \cdots \rightarrow f(u) \in f(K)\text{, subsequence of } (y_n) \text{ above.} \end{array}$$

Does (y_n) have a subsequence that converges to a point in $f(K)$? - Yes, because f is continuous. ♣

Extreme Value Theorem: If $f : A \longrightarrow \mathbb{R}$ is continuous, then f has a maximum and minimum value on any compact subset of A. i.e., This means if K is compact subset of A, then $\exists\, x_1 \in K$ such that $f(x_1) \geqslant f(t)\ \forall\, t \in K$ and $\exists\, x_2 \in K$ such that $f(x_2) \leqslant f(t)\ \forall\, t \in K$.

Think of continuous functions on non-compact sets. Why does this theorem not hold?

Two examples to introduce the notion of *uniform continuity*:
The examples are in the book:
(i) $f(x) = x + 3 \rightarrow$ uniformly continuous.
(ii) $f(x) = x^2 \rightarrow$ *not* uniformly continuous.

Minor Points

(i) We noted last time that $f : A \longrightarrow \mathbb{R}$ is continuous if and only if the inverse image of every open set in B is open in A. *But*, simply because a set S is open in A, it does not necessarily follow that its image, $f(S)$ is open in B.

Example: $f(x) = 1\ \forall\, x \in (0, 1) \rightarrow$ Finite sets are closed. $\Rightarrow$ Not open.

(ii) Why is the condition of *non-empty* necessary in the Axiom of Choice?
Does Ø have an upper bound? - Simple answer: Yes, six is an upper bound.

(iii) Why is the Cantor Set uncountable? A different argument:
Fact: The real numbers in the Cantor set are exactly those numbers in [0, 1] that can be expressed in base three without using any one's.

$\Rightarrow \nexists$ one's in base 3. $\Leftrightarrow \exists$ zero's or two's $\Leftrightarrow \exists$ zero's or one's [since we can re-assign all numbers in base 2]. $\Rightarrow$ Base is irrelevant. $\therefore$ All numbers in [0, 1] are uncountable, regardless of the base.

15 December 2011

Definition: Let $f : A \longrightarrow \mathbb{R}$. If $c \in A$, then f is called *continuous* at c if $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $|x - c| < \delta$ [assumed $x \in A$], then $|f(x) - f(c)| < \varepsilon$.

Exercise 4.4.2: $f(x) = \dfrac{1}{x^2}$ (†)
For example, suppose $\varepsilon = 0.01$ for our above function. Then at $\dfrac{1}{16}$, we have an ε-neighborhood of (0.0525, 0.0725) which corresponds to $c = 4$ [in other words, at four on the x-axis] with a neighborhood of (3.97, 4.03). **Note**: Because the function is inverted, $3.97 \mapsto 0.0725$ and $4.03 \mapsto 0.0525$.

We can show that if $c > 1$, then an $\delta = \dfrac{\varepsilon}{3}$ will always work. Since $c > 1$, the value of δ is not dependent on c, the function is uniformly continuous on $[1, \infty]$.

With the same function above, consider the same ε and look at two with an ε-neighborhood of (1.99, 2.01). The correspondence will be to $c = \dfrac{\sqrt{2}}{2}$ with a neighborhood of (0.705, 0.709). This is a case of ***not*** uniformly continuous. The value of δ depends on c.

Exercise 4.3.3: Let $y = ax + b$ be continuous on $\mathbb{R}$.
The "steps" to prove continuity are more or less the following:

(1) Given $\varepsilon > 0$, is there a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$?

(2) $|f(x) - f(c)| = |(ax + b) - (ac + b)| = |ax - ac| = |a||x - c|$

(3) **Proof**: Assume $|x - c| < \delta = \dfrac{\varepsilon}{|a|}$.
$|f(x) - f(c)| = \cdots = |a||x - c| < |a|\delta = |a|\dfrac{\varepsilon}{|a|} = \varepsilon$.

(†) Assume $c > 1$.

(1) Given $\varepsilon > 0$, does $\exists$ a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$?

(2)

$$\begin{aligned} |f(x) - f(c)| &= \left|\frac{1}{x^2} - \frac{1}{c^2}\right| \\ &= \left|\frac{c^2 - x^2}{c^2x^2}\right| \\ &\qquad\qquad \overset{\text{since } c^2x^2 > 1}{\downarrow} \\ &= \left|\frac{|x - c||x + c|}{c^2x^2}\right| < |x - c||x + c| < |x - c||c + c + 1| = |x - c||2c + 1| \end{aligned}$$

So let $\delta = \min\left(1, \dfrac{\varepsilon}{2c + 1}\right)$. **Notice**: We do not need absolute value since $c > 1$.

(3) $|f(x) - f(c)| = \cdots = |x - c||2c + 1| < 2c + 1\dfrac{\varepsilon}{2c + 1} = \varepsilon$

Notice: $c \geqslant 1$. So $\delta = \dfrac{\varepsilon}{2(1) + 1} = \dfrac{\varepsilon}{3}$ will always work.

Exercise 4.3.7: K contains the roots of $f(x)$.
Show $K^\complement$ is open? If $a \in K^\complement$, then $f(a) \neq 0$. Let $\varepsilon = \dfrac{1}{2}|f(a)|$. So the ε-neighborhood of $f(a)$ excludes zero. Since f is continuous, there is a δ-neighborhood of a such that ***all*** points map into the ε-neighborhood of $f(a)$. Thus, all those points are in $K^\complement$ [since they do not map to zero]. Thus, a has a neighborhood contained in $K^\complement$. This means that $K^\complement$ is open. $\Rightarrow K$ is closed.