Appendix to *On the Minimal Edge Density of K_4-free 6-critical Graphs*

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A Omitted Proofs

A.1 Proofs for Section 3

A.1.1 Proof of Lemma 3.2

Proof. Suppose to the contrary that G contains an independent set A violating the desired inequality. Let G' be the subgraph induced by A and B, so $G' = G[A \cup B]$. Define f on $V(G')$ by $f(v) = d_{G'}(v)$ for $v \in A$, and $f(v) = \max\{d_{G'}(v) - i, 0\}$ for $v \in B_i$. Then we have

$$|E_{G'}(A, B)| > |A| + \sum_{i=1}^{\ell}(i+1)|B_i| \geq \sum_{v \in V(G')} d_{G'}(v) + 1 - f(v).$$

Hence the Lemma 3.2 applies to G'. Thus, we can find a non-empty induced subgraph H of G' (and hence of G) which is f_H-choosable for $f_H(v) = f(v) + d_H(v) - d_{G'}(v)$.

Since G is k-critical, there exists a $(k-1)$-coloring φ of $G \setminus H$. For each $v \in V(H)$, let $g(v)$ be the number of colors in $\{1, \ldots, k-1\}$ that are not used by the neighbors of v in φ. It suffices to show that $g(v) \geq f_H(v)$ for all $v \in V(H)$, since then φ could be extended to a $(k-1)$-coloring of G by using the f_H-choosability of H, thus contradicting that G is k-critical. Let $v \in V(H)$ be a vertex in B_i. Clearly v has at most $d_G(v) - d_H(v)$ neighbors in $G \setminus H$, so $g(v) \geq \max\{(k-1) - (d_G(v) - d_H(v)), 0\} = \max\{d_H(v) - i, 0\}$. Since $d_H(v) - d_{G'}(v) \leq 0$, a routine calculation shows that $\max\{d_H(v) - i, 0\} \geq \max\{d_{G'}(v) - i, 0\} + d_H(v) - d_{G'}(v) = f_H(v)$. Therefore $g(v) \geq f_H(v)$, completing the proof. □

A.2 Proofs for Section 4

A.2.1 Proof of Lemma 4.5

Proof. Let $H = G[N[x] \cap N[y]]$, the subgraph induced by x, y and their common neighbors $N(x) \cap N(y)$. If $|N(x) \cap N(y)| \geq k-2$, then $N[x] = N[y]$ as desired. So we may assume that $|N(x) \cap N(y)| = k-3$. Thus x has exactly one neighbor a not in H. Let $b_1, \dots, b_i$ be the neighbors of y not in H. Let S be the set of edges $\{ab_1, \dots, ab_i\}$ and suppose that $G \setminus \{x\} + S$ has a $(k-1)$-coloring φ.

Let $L = \{\varphi(v) : v \in N(x)\}$. If $|L| \leq k-2$, then φ extends to a $(k-1)$-coloring of G by letting $\varphi(x) \in [k-1] \setminus L$. But then S is an i-edge-addition, contrary to our assumption.

So we may suppose that $|L| = k-1$. Define a new $(k-1)$-coloring ψ by letting $\psi(x) = \varphi(y), \psi(y) = \varphi(a)$, and $\psi(z) = \varphi(z)$ for all $z \in V(G) \setminus \{x, y\}$. Since $\varphi(b_1), \dots, \varphi(b_i)$ are all distinct from $\varphi(a)$, ψ is a $(k-1)$-coloring of G, a contradiction. But then S is an i-edge-addition, contrary to our assumption. □

A.2.2 Proof of Lemma 4.10

Proof. We count the contribution of vertices, edges, and $T(\cdot)$ to each side. Since $R' = R \cup (V(W) - X)$, each vertex of R' is counted exactly once on the right side. The number of edges on the right is, by definition, $|E(G[R'])| - i$, so the net contribution from edges is $-2i(k-1)$ on the right side. Note that edges within $G[R]$ and $W - X$ are counted exactly once on each side, and each edge between X and $V(W) - X$ in $G_{R,\varphi}$ corresponds to at least one distinct edge between R and $V(W) - X$ in G, so $i \geq 0$. Finally, $T(G[R']) \geq T(G[R]) + T(W - X)$, and $T(W - X) \geq T(W) - |X|$, as W contains at most $|X|$ more disjoint cliques than $W - X$. Adding these inequalities proves the lemma. □

A.2.3 Proof of Lemma 4.11

Proof. Since $G[R]$ is not a clique, $W \neq G$ and so W is smaller than G. As G is good, we find that W satisfies Conjecture 1.4.

By a routine calculation,

$$p(K_X) + \delta T(K_X) - \delta|X| = -|X|((k-1)|X| - (k^2 - 3 + \varepsilon - \delta)),$$

which is quadratic in $|X|$ with roots $\{0, \frac{k^2-3+\varepsilon-\delta}{k-1}\}$. Since $\varepsilon - \delta \geq 3 - k$, this quadratic is minimized over $|X| \in \{1, \dots, k-1\}$ at $|X| = 1$. Thus

$$p(K_X) + \delta T(K_X) - \delta|X| \geq p(K_1) - \delta = (k-1)\varepsilon - \delta.$$

Now, we consider three possibilities for W. First suppose that $W = K_k$. Observe that Lemma 4.10 holds without adding the term $\delta|X|$, as $T(W) = T(W - x)$ for every vertex $x \in X$. We then have

$$p_G(R') \leq p_G(R) - 2i(k-1) + p(K_k) - p(K_1) = p(R) - 2(i+1)(k-1) + (k-1)\varepsilon - 2\delta.$$

Since $\delta \geq (k-1)\varepsilon$ by Assumption 1.1, the desired inequality holds.

Next suppose that W is k-Ore and $W \neq K_k$. Using Lemma 2.5 and the fact that $\delta = 2(k-1)\varepsilon$ by Assumption 1.1, we have

$$\begin{aligned} p_G(R') &\leq p_G(R) - 2i(k-1) \\ &\quad + \left(k(k-3) + |V(G)|\varepsilon - \left(2 + \frac{|V(G)|-1}{k-1}\right)\delta \right) - (k^2 - k + \varepsilon - \delta) \\ &= p_G(R) - 2(i+1)(k-1) - \delta - (|V(G)|-1)\varepsilon \\ &\leq p_G(R) - 2(i+1)(k-1) - \delta. \end{aligned}$$

Finally suppose that W is not k-Ore. Now $p(W) \leq k(k-3) - P$ and

$$p_G(R') \leq p_G(R) - 2(i+1)(k-1) - (P + \varepsilon - \delta)$$

and we obtain the desired inequality since $P \geq 2\delta$ by Assumption 1.2. □

A.2.4 Proof of Proposition 4.13

Proof. Suppose that R is collapsible in G. Let R' be a critical extension of R with extender W and core X with respect to a $(k-1)$ coloring ϕ of $G[R]$. Without loss of generality we may assume that $\phi(\partial_G R) = 1$. By definition of collapsibility, x_1 is a cut-vertex of $G_{R,\varphi}$. Let $A = (V(G) \setminus R) \cup \{x_1\}$. A k-critical graph has no cut-vertex, and so W is a subgraph of $G_{R,\varphi}[A]$ and $X = \{x_1\}$.

Now suppose that $R' \neq V(G)$. Let $y \in V(G) \setminus R'$. Since G is k-critical, there exists a $(k-1)$-coloring ψ of $G - y$. ψ induces a $(k-1)$-coloring of R, and we may assume without loss of generality that $\psi(\partial_G R) = 1$. Now ψ induces a $(k-1)$-coloring of W, a contradiction. Hence we may assume that $R' = V(G)$ and thus the extension is spanning. It only remains to show that the extension is complete.

Next suppose that $G[R' \setminus R]$ has an edge f not in $W[R' \setminus R]$. Since G is k-critical, there exists a $(k-1)$-coloring ψ of $G - f$. This induces a $(k-1)$-coloring of W, a contradiction. So we may assume that $E(G[R' \setminus R]) = E(W[R' \setminus R])$.

Since the extension is not complete, we may now assume that there exists an edge $ux_1 \in E(W)$ that corresponds to two edges $uv_1, uv_2 \in E(G)$, with $v_1, v_2 \in R$. Since G is k-critical, there exists a $(k-1)$-coloring ψ of $G - uv_1$. Since R is collapsible, $\psi(v_1) = \psi(v_2)$. Moreover, $\psi(u_1)$ is a distinct color from $\psi(u)$ and thus ψ extends to a $(k-1)$-coloring of G, a contradiction. Therefore R' is complete. Since R' was arbitrary, it follows that every critical extension of R has a core of size 1, is spanning, and is complete as desired.

We now prove the converse. Suppose to the contrary that every critical extension of R has a core of size 1, is spanning, and is complete, but that there exists a $(k-1)$-coloring φ of $G[R]$ with $\varphi(u) = 1, \varphi(v) = 2$ for some $u, v \in \partial_G R$. Let W be a k-critical subgraph of $G_{R,\varphi}$ and let R' be a critical extension of R with extender W and core X. Since $|X| = 1$, we may assume that $x_2 \notin X$. Since $v \in \partial_G R$ and R' is spanning, there is an edge $vz \in E(G)$ between R and $V(G) \setminus R$ with no corresponding edge in W. Thus R' is at least 1-incomplete, a contradiction. □

A.2.5 Proof of Lemma 4.14

Proof. Let φ be any $(k-1)$-coloring of $G[R]$. Let W be a k-critical subgraph of $G_{R,\varphi}$ and let R' be a critical extension of R with extender W and core X. By our hypothesis, $R' = V(G)$ and $|X| = 1$. We may assume without loss of generality that $X = \{x_1\}$. Now an edge between $E(\varphi^{-1}(C \setminus \{1\}) \cap R$ and $V(G) \setminus R$ is in $|E(G[R'])|$ but not in $|E(G[R])| + |E(W)| - |E(K_X)|$. As R' is at most i-incomplete, it follows that there are at most i such edges. Then for $c = 1$,

$$|E(\varphi^{-1}(C \setminus \{c\}) \cap R, V(G) \setminus R)| \leq i,$$

which proves that R is i-collapsible. □

A.2.6 Proof of Lemma 4.15

Proof. Since $|E(H) \setminus E(G[R])| \leq i$, it follows that $T(H) \leq T(G[R]) + i$. Hence $p(H) \geq p_G(R) - 2i(k-1) - i\delta$. Since G is tight and H is smaller than G, H satisfies Conjecture 1.4. Thus $p(H) \leq k(k-3) \leq p(G) + P + Q$. So $p_G(R) \leq p(G) + P + Q + 2i(k-1) + i\delta$ as desired. Furthermore, if H is not k-Ore, then $p(H) \leq k(k-3) - P = p(G) + Q$ as desired. □

A.2.7 Proof of Lemma 4.16

Proof. Suppose not. If $i = 0$, a collapsible subset clearly admits an edge-addition, as R contains at least two vertices which receive the same color in every $(k-1)$-coloring of $G[R]$. Hence we may assume that $i \geq 1$.

Let $R \subsetneq V(G)$ be an i-collapsible subset for some i with $1 \leq i \leq (k-3)/2$. Suppose to the contrary that G does not admit an $(i+1)$-edge-addition. For each $u \in \partial_G R$, let $w(u) = |\{uv \in E(G) | v \in V(G) \setminus R\}|$. Since G is $(k-1)$-edge connected, $\sum_{u \in \partial_G R} w(u) \geq k-1$. Let $\partial_G R = \{u_1, \ldots, u_s\}$ and assume without loss of generality that $w(u_1) \geq w(u_j)$ for all $j \geq 2$.

Claim A.1. *$w(u_2) + \ldots + w(u_s) \leq i + 1$.*

Proof. Suppose not, that is $w(u_2) + \ldots + w(u_s) \geq i + 2$. We may assume without loss of generality that $w(u_1) \geq w(u_2) \geq \ldots \geq w(u_s)$. Choose the largest index j such that $w(u_j) + \ldots + w(u_s) \geq i + 1$, and set $\alpha = i + 1 - (w(u_{j+1}) + \ldots + w(u_s))$. Since $w(u_1) + \ldots + w(u_s) \geq k-1$, we also have that $w(u_1) + \ldots + w(u_j) \geq (k-1) - (i+1-\alpha) = k - 2 - i - \alpha$, which is at least $i + 1 + \alpha$ as $i \leq (k-3)/2$.

By the choice of j and the ordering of the vertices, $0 < \alpha \leq w(u_j) \leq w(u_1)$. Define a multigraph graph H on the vertices $V(H) = \partial_G R$ with the following edges. Add α edges between u_1 and u_j, and $i + 1 - \alpha$ edges between the vertex sets $\{u_{j+1}, \ldots, u_s\}$ and $\{u_1, \ldots, u_j\}$ so that for each ℓ, the degree of u_ℓ in H is at most $w(u_\ell)$.

As G admits no $(i+1)$-edge-addition, there is a $(k-1)$-coloring φ of $G[R]+E(H)$, which is also a $(k-1)$-coloring of $G[R]$. For any color c and $M=\varphi^{-1}(c)$, we have that

$$\sum_{u\in(\partial_G)\setminus M} w(u) \geq \sum_{u\in(\partial_G R)\setminus M} d_H(u) \geq \frac{1}{2}\sum_{u\in\partial_G R} d_H(u) = i+1.$$

This contradicts the i-collapsibility of R. □

Now let $S=\{u_1u_j|2\leq j\leq s\}$, and note that $|S|\leq\sum_{2\leq j\leq s} w(u_j)\leq i+1$. Hence $G[R]+S$ has a $(k-1)$-coloring φ. We may assume that $\varphi(u_1)=1$ and hence $\varphi(u_j)\neq 1$ for all $j\geq 2$.

Note that $w(u_1)\geq k-1-(i+1)\geq i+1$. Thus for all c with $2\leq c\leq k-1$, we have that $|E(\varphi^{-1}(C\setminus\{c\})\cap R, V(G)\setminus R)|\geq i+1$. Since R is i-collapsible, it follows that $|E(\varphi^{-1}(C\setminus\{1\})\cap R, V(G)\setminus R)|\leq i$. Thus $w(u_2)+\ldots+w(u_s)\leq i$.

Let ψ be a $(k-1)$-coloring of $G[(V(G)\setminus R)\cup\{u_1\}]$ such that $\psi(u_1)=1$. Consider the *improper* coloring $\overline{\psi}$ induced on G by the union of φ and ψ. Now let us choose ψ so that the number of edges between $R, V(G)\setminus R$ with endpoints having the same color in $\overline{\psi}$ is minimized. Without loss of generality, assume that $\varphi(u_2)=2$ and that some neighbor z of u_2 in $V(G)\setminus R$ also has $\overline{\psi}(z)=2$.

To obtain a contradiction, we show that the colors on $V(G)\setminus R$ can be permuted so as to contradict the minimality of ψ. Let $\overline{C}$ be the set of colors

$$\{1\}\cup\{\varphi(u_j): 2\leq j\leq s\}\cup\{\varphi(v): v\in V(G)\setminus R, \exists u_jv\in E(G), \varphi(u_j)=2\}.$$

Now $\overline{C}$ contains at most $i+1$ colors, as there are at most i edges joining $u_2,\ldots,u_s$ to $V(G)\setminus R$. Hence, there exists a color $c'\in\{1,\ldots,k-1\}\setminus\overline{C}$. The coloring ψ' obtained by permuting the colors 2 and c' in ψ contradicts the minimality of ψ, and this completes the proof. □

A.2.8 Proof of Lemma 4.17

Proof. We will show by induction on i that for $0\leq i\leq(k-2)/2$, G does not admit an i-edge-addition, and there is no subset $R\subsetneq V(G)$, $G[R]$ not a clique, with $p_G(R)<p(G)+2i(k-1)+\Delta$. The base case $i=0$ is true since Lemma 4.11 implies that $p(G)\leq p_G(R)-2(k-1)$.

Now suppose it holds for $i-1$ and suppose to the contrary it does not hold for i.

Claim A.2. *If $R\subsetneq V(G)$, $G[R]$ not a clique, with $p_G(R)<p(G)+2i(k-1)+\Delta$, then every critical extension of R is spanning, at most $(i-1)$-incomplete, and has a core of size one.*

Proof. Let R' be a critical extension of R with extender W. By Lemma 4.11, if R' is ℓ-incomplete, then

$$p_G(R')\leq p_G(R)-2(\ell+1)(k-1)-\delta<p(G)+2(i-1-\ell)(k-1)+\Delta-\delta.$$

This is a contradiction unless $R' = V(G)$ and $\ell \leq i-1$. Hence R' is spanning and $(i-1)$-incomplete. Next suppose that R' has size greater than one. Then by Lemma 4.10,

$$\begin{aligned} p_G(R') &\leq p_G(R) + p(W) - (p(K_{k-1}) - (k-3)\delta) \\ &= p_G(R) + p(W) - (2(k-1)(k-2) - (k-1)(\delta - \varepsilon)). \end{aligned}$$

Since $p_G(R) < p(G) + 2i(k-1) + \Delta \leq p(G) + (k-1)(k-2) + \Delta$ and $p_G(R') \geq p(G)$, we have $p(W) > k(k-3) + 2 - (\Delta + (k-1)(\delta - \varepsilon)) \geq k(k-3)$. However, G is tight, so W satisfies Conjecture 1.4, in which case $p(W) \leq k(k-3)$, a contradiction. Hence R' has a core of size one as desired. □

Since the statement does not hold for i, there exists $R \subsetneq V(G)$, $G[R]$ not a clique, with $p_G(R) < p(G) + 2i(k-1) + \Delta$. Now let us assume that R is such a subset of minimal size. Note that by Claim A.2, every critical extension of R is spanning, at most $(i-1)$-incomplete and has a core of size one. By Lemma 4.14, R is $(i-1)$-collapsible.

Since $i - 1 \leq (k-3)/2$, Lemma 4.16 implies that G admits an i-edge-addition S consisting of non-edges of $G[R]$. By induction, G does not admit an $(i-1)$-edge-addition, so $|S| = i$. Let S be chosen so that the k-critical graph $H \subseteq G[R] + S$ has the minimum number of vertices, and let $R_0 = V(H)$. By the same calculation as in Lemma 4.15, $p_G(R_0) \leq p(H) + 2i(k-1) + i\delta$. As G is good and H is smaller than G, $p(H) \leq k(k-3)$. But then

$$\begin{aligned} p_G(R_0) &\leq k(k-3) + 2i(k-1) + i\delta \\ &< p(G) + 2i(k-1) + (P + Q + i\delta) \\ &< p(G) + 2i(k-1) + \Delta, \end{aligned}$$

where the second inequality follows since $p(G) \geq k(k-3) - P - Q$ and the third inequality follows since $P + Q + i\delta \leq \Delta$ by Assumption 1.4. Since R is a minimum subset with this property, we have that $R_0 = R$.

By assumption, G does not admits an $(i-1)$-edge-addition and hence by Lemma 4.16, R is not $(i-2)$-collapsible. It follows then from Claim A.2 that there exists a critical extension R' of R that is not $(i-2)$-incomplete.

Claim A.3. *H is k-Ore.*

Proof. Suppose not. As H is smaller than G, $p(H) \leq k(k-3) - P < p(G) + Q$. Since R' is spanning and has core of size one, Lemmas 4.10 and 4.15 yield

$$\begin{aligned} p(G) &\leq p_G(R) + p(W) - 2(i-1)(k-1) - (k^2 - k - 2 + \varepsilon - \delta) \\ &\leq (p(H) + 2i(k-1) + i\delta) + p(W) - 2(i-1)(k-1) - (k^2 - k - 2 + \varepsilon - \delta) \\ &< p(G) + p(W) + Q + 2(k-1) + i\delta - (k^2 - k + \varepsilon - \delta) \end{aligned}$$

and therefore $p(W) > k(k-3) - (Q + (i+1)\delta - \varepsilon)$. Since $Q + k\delta \leq P$, this implies that W is k-Ore.

Since $|X| = 1$, by Proposition 2.7, there exists a gem D in W with $x_1 \notin V(D)$. If $i = 1$, then as R' is complete, D is also a gem in G, a contradiction that G is ungemmed. Hence we may assume that $i > 1$. If D is a diamond, then G admits an edge-addition, which contradicts the induction hypothesis.

Thus we may assume that D is an emerald, with vertices $\{u_1, \ldots, u_{k-1}\}$. Since R' is $(i-1)$-incomplete, there are at most $i-1$ edges incident to D in G but not in W, so there are at least $k-i$ vertices in D of degree $k-1$ in G, which we may assume are $\{u_1, \ldots, u_\ell\}$, $\ell \leq k-i$. For each $1 \leq i \leq \ell$, let v_i denote the unique neighbor of u_i outside D. If $v_i \neq v_j$ for any $i \neq j$, then $v_i v_j$ is an edge-addition in G. To see this, suppose there exists a $(k-1)$-coloring φ of $G + v_i v_j - \{u_i, u_j\}$. Since $\varphi(v_i) \neq \varphi(v_j)$, φ extends to a $(k-1)$-coloring of G, a contradiction. Since $i > 1$, G does not admit an edge-addition, so $v_i = v_j$ for all i, j. But then adding the edges $v_1 u_j$ for $j = \ell+1, \ldots, k-1$ yields a K_k-subgraph; so G admits an $(i-1)$-edge-addition, a contradiction. Therefore W is not k-Ore, a contradiction. □

Claim A.4. *If $i = 1$, then $S = \{xy\}$ for a single edge xy and $H = G[R] + S$.*

Proof. Suppose not. Recall that $V(H) = R$. As $E(G[R]) \setminus E(H) \neq \emptyset$, then $p_G(R) \leq p(H) + \delta$. In which case

$$\begin{aligned} p(G) &\leq p_G(R) + p(W) - (k^2 - k - 2 + \varepsilon - \delta) \\ &\leq p(H) + \delta + (k(k-3) - P) - (k^2 - k - \varepsilon + \delta) \\ &\leq p(H) - 2(k-1) - P - \varepsilon + 2\delta. \end{aligned}$$

Rearranging yields $p(H) \geq p(G) + 2(k-1) + P + \varepsilon - 2\delta$. Since $p(G) > k(k-3) - P - Q$, we have that $p(H) > k(k-3)$, which is a contradiction, since we assumed H to be k-Ore. □

Claim A.5. *$H = K_k$.*

Proof. Suppose not. Since $H \neq K_k$ and H is k-Ore, we have that H is an Ore-composition of two k-Ore graphs H_1, H_2. We will work towards a contradiction showing that H cannot be an Ore-composition. Let $\{a, b\}$ denote the overlap vertices, and $\underline{ab}$ the split-vertex of H_2. The proof for $i = 1$ differs from the general case, so we first resolve $i = 1$.

For $i = 1$ and $S = \{xy\}$, we show that $\partial_G R$ intersects both $V(H_1) \setminus \{a, b\}, V(H_2) - \underline{ab}$.

Subclaim A.6. *There exists $u, v \in \partial_G R$ with $u \in V(H_1) \setminus V(H_2), v \in V(H_2) \setminus V(H_1)$.*

Proof. Suppose to the contrary that $\partial_G R$ is contained in $V(H_1) \setminus V(H_2)$. By Proposition 2.7, there exists a gem D in H_2 with $\underline{ab} \notin V(D)$. Since there are no edges between $V(H_2)$ and $G \setminus R$, D is a gem in G, a contradiction. Now suppose that

$\partial_G R$ was contained in $V(H_2) \setminus V(H_1)$. Notice that in any $(k-1)$-coloring of $G[R]$, a and b receive different colors, as otherwise there exists a $(k-1)$-coloring of H_2, a contradiction. Thus, at most one of a, b is in $\partial_G R$; we may assume without loss of generality that $b \notin \partial_G R$. By Proposition 2.7, there exists a gem D in H_1 with $a \notin V(D)$, in which case $ab \notin E(D)$. D does not contain the edge xy, or else one of the edges x, y would have degree strictly less than $k-1$ in G. Then D is a gem in G, a contradiction. $\square$

The remainder of the argument for $i = 1$ will appear again, so we label it for convenience.

Subclaim A.7. *If Subclaim A.6 holds, then for $i = 1$, we obtain the desired result that H cannot be an Ore-composition of two k-Ore graphs.*

Proof. From Subclaim A.6, there exists $u, v \in \partial_G R$ with $u \in V(H_1) \setminus V(H_2), v \in V(H_2) \setminus V(H_1)$. Let φ be a $(k-1)$-coloring of $G[R]$; since R is collapsible, $\varphi(u) = \varphi(v)$. Since H_2 is a split k-Ore subgraph of G, we have that $\varphi(a) \neq \varphi(b)$. So let us assume without loss of generality that $\varphi(a) = 1$ and $\varphi(b) = 2$.

Suppose $\varphi(u)$ is distinct from $\varphi(a), \varphi(b)$. Suppose without loss of generality that $\varphi(u) = 3$. Hence, by permuting the colors $3, 4$ on $V(H_1)$, we obtain a $(k-1)$-coloring of $G[R]$ in which u, v have distinct colors, a contradiction. So we may assume $\varphi(u) = \varphi(v) = 1$ in every $(k-1)$-coloring of $G[R]$. This implies that $H_2 + av$ is not $(k-1)$-colorable, but then $G[H_2]$ admits an edge-addition, which contradicts the minimality of H. This proves the subcase $i = 1$. $\square$

Now we consider $i > 1$. Let $\overline{S} = S \cap E(H_1)$ be the edges of S added on the vertex-side. If $\overline{S} \leq i-1$, then $\overline{S} \cup \{ab\}$ is an i-edge-addition contained in $G[V(H_1)]$, which contradicts the minimality of H. Hence $S \subseteq E(H_1)$, so $H_2 - \underline{ab}$ is a subgraph of G. By Proposition 2.7, there exists a gem D in H_2 with $\underline{ab} \notin V(D)$. If D is a diamond, then G admits an edge-addition, a contradiction. Thus D is an emerald, with vertices $\{u_1, \ldots, u_{k-1}\}$.

In a $(k-1)$-coloring of $G[R]$, the vertices of D receive different colors. Let $X = \{x_1\}$ be the core of R'. Since R' is $(i-1)$-incomplete, there are at most i vertices in $V(D) \cap \partial_G R$, as otherwise there are at least i edges from $\{x_2, \ldots, x_{k-1}\}$ to $V(G) \setminus R$ not included in W. Now let $\{u_1, \ldots, u_\ell\}$, be the vertices in $V(D) \setminus \partial_G R$. As before, the vertices $u_1, \ldots, u_\ell$ have a common neighbor v outside D, or else G would admit an edge-addition. Adding i edges between v and $u_{\ell+1}, \ldots, u_{k-1}$ yields a K_k, which contradicts the minimality of S as an i-edge-addition. $\square$

Now suppose that $i = 1$. By Claim A.4, $S = \{xy\}$ and $H = G[R] + S$. The vertices x, y are in $\partial_G R$, or else one of x or y has degree less than $k-1$ in G, a contradiction as G is k-critical. But R' has a core of size one and is complete, so $\partial_G R = \{x, y\}$. By Claim A.5, $H = K_k$. Hence $G[R]$ is a diamond in G, a contradiction.

So we may assume that $i > 1$. Let $V(H) = \{u_1, \ldots, u_k\}$, and note that $G[R]+S = H$ by the minimality of R. Without loss of generality, assume that $u_1u_k \in S$. Let $\varphi : G[R] \to \{1, \ldots, k-1\}$ be the improper coloring $\varphi(u_i) = i$ for $1 \leq i \leq k-1$ and $\varphi(u_k) = 1$. Since R is $(i-1)$-collapsible, we have that

$$\min_{c \in C} |E(\varphi^{-1}(C \setminus \{c\}) \cap R, V(G) \setminus R)| \leq i - 1.$$

Observe that as G is k-critical, $d_G(u_i) \geq k-1$, so each vertex u_i has at least as many neighbors in $V(G) \setminus R$ as the number of edges of S incident with u_i. Therefore, if $c \neq 1, k$, the vertex v_c is incident to at most $i-1$ edges of S, and the vertices $\{v_i : i \neq c\}$ are incident with at least $i-1$ edges with ends in $V(G) \setminus R$, with an additional 2 edges for the edge $u_1u_k \in S$. Hence for $c \neq 1$,

$$|E(\varphi^{-1}(C \setminus \{c\}) \cap R, V(G) \setminus R)| \geq i + 1.$$

Since R is $(i-1)$-collapsible, it follows that $|E(\varphi^{-1}(C \setminus \{1\}) \cap R, V(G) \setminus R)| \leq i-1$. By the pigeonhole principle, every edge in S is incident with either u_1 or u_k. If the edges u_1u_i and u_ku_j are both in S for some $u_i \neq u_j$, then let ψ be the improper coloring obtained from φ by setting $\psi(u_i) = 1, \psi(u_k) = \varphi(u_i)$ (that is, we swap the colors on u_i and u_k). Then ψ induces a proper coloring of $G[R]$ in which no color class covers all but $i-1$ endpoints of S, which contradicts the $(i-1)$-collapsibility of R. Hence the edges of S either form a star, or $i = 3$ and S forms a triangle.

Suppose first that S forms a star, with u_1 the center. In this case, for R to satisfy the $(i-1)$-collapsibility condition, every leaf of the star has exactly one edge going to $V(G) \setminus R$, and every $u_i \in R$ not incident with S has no edges to $V(G) \setminus R$. Then $d_G(u_i) = k-1$ for $2 \leq i \leq k$ and $G[R \setminus \{u_1\}]$ is an emerald in G, a contradiction.

Suppose instead that S forms a triangle on $\{u_1, u_2, u_3\}$. Then each of u_1, u_2, u_3 are incident with exactly 2 edges with ends in $V(G) \setminus R$ while for every $i > 3$, u_i is not incident with an edge with an end in $V(G) \setminus R$. However, $|S| = 3$ occurs only when $(k-2)/2 \geq 3$, or $k \geq 8$. Thus $|E(R, V(G) \setminus R)| \leq 6$. Yet G is at least 7-edge-connected as G is k-critical with $k \geq 8$, a contradiction.

We conclude that there is no subset $R \subsetneq V(G)$, $G[R]$ not a clique, with $p_G(R) < p(G)+2i(k-1)+\Delta$. By Lemma 4.15, if G admitted an i-edge-addition with $R = V(G)$, we would have $p_G(R) \leq p(G)+2i(k-1)+P+Q+i\delta$. Since $P+Q+i\delta < P+Q+\frac{k}{2}\delta \leq \Delta$, no such subset R exists, and thus no i-edge-addition exists. This completes the induction. □

A.3 Proofs for Section 5

A.3.1 Proof of Lemma 5.3

Proof. Suppose φ is a $(k-1)$-coloring of $G_{y \to x}$. Note $|\{\varphi(v) : v \in N(y) \setminus C\}| \geq k-2$. Thus there exists a color $c \in [k-1] \setminus \{\varphi(v) : v \in N(y) \setminus C\}$. If there does not exist a vertex $z \in C$ such that $\varphi(z) = c$, then φ extends to a $(k-1)$-coloring of G by letting $\varphi(y) = c$, a contradiction. So we may assume there exists a vertex $z \in C$ such that $\varphi(z) = c$; moreover, such a vertex is unique since $G[C]$ is a clique. Now define a coloring ψ on G by letting $\psi(y) = c$, $\psi(z) = \varphi(\widetilde{x})$, and $\psi = \varphi$ for all other vertices of G. But then ψ is a $(k-1)$-coloring of G, a contradiction. □

A.3.2 Proof of Lemma 5.4

Proof. Suppose not. By conditions 1 and 2 and Lemma 5.3, there exists a k-critical subgraph H of $G_{y\to x}$. By condition 3, H is smaller than G.

Let $R = V(H) \setminus \{\widetilde{x}\}$. If $G[R]$ is a clique, then $H = K_k$ and hence $G[R \cup \{y\}]$ is a K_k-clique in G, a contradiction. So we may assume that $G[R]$ is not a clique. The vertex $\widetilde{x}$ has degree $k-1$ in H. Thus the potential of R satisfies

$$p_G(R) \le p(H) - ((k-2)(k+1) + \varepsilon) + 2(k-1)(k-1) + \delta.$$

Let R' be a critical extension of R with extender W. Since $G[R]$ is not a clique, W is smaller than G. Let X be the core of R'.

Claim A.8. $|X| = 1$.

Proof. Suppose not. First suppose that $1 < |X| < k-1$. By Lemma 4.10,

$$\begin{aligned} p(G) \le p_G(R') &\le p_G(R) + p(W) - (p(K_2) - 2\delta) \\ &\le p(H) + p(W) - (k^2 - k - 6) + 3(\delta - \varepsilon). \end{aligned}$$

As G is good and H is not k-Ore, $p(H) \le k(k-3) - P$. Hence

$$p(W) \ge k(k-3) + 2k - 6 - (Q + 3(\delta - \varepsilon)) > k(k-3).$$

But W is smaller than G, so $p(W) \le k(k-3)$. This is a contradiction.

So we may assume that $|X| = k-1$. It follows from a similar calculation using Lemma 4.10 as above that

$$p(W) \ge k(k-3) - (Q + k(\delta - \varepsilon)) > k(k-3) - P,$$

and hence W is k-Ore. Furthermore, if R' is incomplete or not spanning, then by Lemma 4.11, $p_G(R') \ge p(G) + 2(k-1)$ and hence

$$p(W) \ge k(k-3) + 2k - 2 - (Q + k(\delta - \varepsilon)) > k(k-3),$$

a contradiction. So we may assume that R' is spanning and complete.

First suppose that $W = K_k$. Then as $|X| = k-1$ and R' is spanning and complete, $\partial_G(V(G) \setminus R) = \{y\}$. This implies that $V(G) = R \cup \{y\}$. Now $p(H) \le p(G) + Q$, $|V(H)| = |V(G)|$, and $|E(H)| \le |E(G)|$, and hence we have that $|E(H)| = |E(G)|$. Therefore $d_G(y) = k-1$ and $H = G_{y\to x}$, a contradiction.

So we may assume that $W \neq K_k$. Then Proposition 2.8 implies that there exists a gem D in W disjoint from X. If D is a diamond, then G admits an edge-addition, contradicting Lemma 4.17. So we may assume that D is an emerald. Since R' is complete, every vertex v of D satisfies $d_G(v) = d_W(v) = k-1$. But then D is an emerald of G, a contradiction. □

Claim A.9. *R' is spanning and i-incomplete for some $i \leq (k-4)/2$.*

Proof. Suppose not. Hence R' is j-incomplete for some $j \geq (k-3)/2$. By Lemma 4.10,

$$\begin{aligned} p_G(R') &\leq p(R) + p(W) - 2j(k-1) - (p(K_1) - \delta) \\ &\leq p(H) + p(W) - 2j(k-1) - (2k - 6 - 2(\delta - \varepsilon)). \end{aligned}$$

If $R' \subsetneq V(G)$ is proper, Lemma 4.17 implies that $p_G(R') \geq p(G) + (k-1)(k-2) + \Delta$. Then, as $p(H) \leq p(G) + Q$, we have that

$$p(W) \geq k(k-3) + 2j(k-1) + 4(k-2) + \Delta - (Q + 2(\delta - \varepsilon)) > k(k-3),$$

a contradiction. Thus, R' is spanning and so $p(G) = p_G(R')$. But then we have that

$$p(W) \geq 2j(k-1) + (2k - 6 - Q) - 2(\delta - \varepsilon).$$

Since $j \geq (k-3)/2$, we have that $p(W) \geq k(k-3) + (k - 3 - Q - 2\delta - \varepsilon) \geq k(k-3)$ by Assumption 1, a contradiction. □

By Claims A.8 and A.9, every critical extension of R is spanning, has a core of size one, and is at most $(k-4)/2$-incomplete. Thus by Lemma 4.14, R is $(k-4)/2$-collapsible. Hence by Lemma 4.16, G admits a $(k-2)/2$-edge-addition, contradicting Lemma 4.17. □

A.3.3 Proof of Lemma 5.5

Proof. Suppose not. Let $R \subsetneq V(G), |R| \geq k$ be a proper subset with $p_G(R) \leq p(G) + (k-1)(k-2) + (\Gamma + Q + \delta)$.

Claim A.10. *If R' is a critical extension of R, then R' is spanning and i-incomplete for some $i \leq (k-3)/2$.*

Proof. Suppose not. First suppose that R' is not spanning. Then Lemma 4.11 implies that

$$\begin{aligned} p_G(R') \leq p_G(R) - 2(k-1) - \delta &\leq p(G) + (k-1)(k-2) + (\Gamma + Q + \delta) - 2(k-1) - \delta \\ &= p(G) + (k-1)(k-2) - (2(k-1) - \Gamma - Q - \delta) \\ &< p(G) + (k-1)(k-2) + \Delta \end{aligned}$$

This contradicts Lemma 4.17 as $R' \neq V(G)$. So we may assume that R' is spanning.

So we may assume that R' is not i-incomplete for some $i \leq (k-4)/2$. For even k, if R' is at most $(k-2)/2$-incomplete, then Lemma 4.11 implies

$$p(G) = p_G(R') \leq p_G(R) - k(k-1) - \delta \leq p(G) - (2(k-1) - \Gamma - Q)$$

a contradiction. For odd k, if R' is at most $(k-3)/2$-incomplete, then the same calculation using Lemma 4.11 implies that $\Gamma + Q \geq k-1$, a contradiction. $\square$

It follows from Lemma 4.17 that R is not $(k-4)/2$-collapsible. But then by Lemma 4.14 and Claim A.10, there exists a critical extension R' of R that has a core of size greater than one. Let W be the extender of R and let X be the core of R.

First suppose that $2 \leq |X| < k-1$. Then by Lemma 4.10,

$$\begin{aligned} p(G) = p_G(R') &\leq p(R) + p(W) - (p(K_2) - 2\delta) \\ &\leq p(G) + p(W) - (k(k-3) + 4k - 6 - \Gamma - Q + 2\varepsilon - 3\delta). \end{aligned}$$

Thus $p(W) > k(k-3)$, a contradiction.

So we may assume that $|X| = k-1$. Repeating the previous calculation with $|X| = k-1$ in Lemma 4.10, we find that

$$\begin{aligned} p(G) = p_G(R') &\leq p(R) + p(W) - (p(K_{k-1}) - (k-3)\delta) \\ &\leq p(G) + p(W) - (k(k-3) - (\Gamma - 2 + Q + k\delta - (k-1)\varepsilon)). \end{aligned}$$

Since $(\Gamma - 2) + Q + k\delta - (k-1)\varepsilon < P$ by Assumption 1.3, we have that $p(W) > k(k-3) - P$. Hence W is k-Ore.

Now suppose that $W \neq K_k$. Then by Proposition 2.8, there exists a gem D with vertices in $V(W) \setminus X$. If D is a diamond, then G admits an edge-addition, contradicting Lemma 4.17. So we may assume that D is an emerald. As R' is spanning and at most $(k-4)/2$-incomplete by Claim A.10, there are at most $(k-4)/2$ vertices of D incident with an edge in $E(R') \setminus E(W)$. Hence at least $k-1-(k-4)/2 \geq k/2+1$ vertices of D have degree $k-1$ in G. Let $u_1, \ldots, u_\ell$ denote the vertices in D with $d_G(u_i) = k-1$. For each i with $1 \leq i \leq \ell$, let v_i denote the neighbor of u_i their neighbors not in D. By Corollary 4.6, $\overline{N}(u_i) = \overline{N}(u_j)$ for all i, j. Hence, there exists a vertex $v \in V(G) \setminus V(D)$ such that v is adjacent to all of $\{u_1, \ldots, u_\ell\}$. Adding at most $(k-4)/2$-edges between v and $u_{\ell+1}, \ldots, u_{k-1}$ yields a K_k-subgraph of G. Thus G admits a $(k-4/)2$-edge-addition, contradicting Lemma 4.17.

So we may assume that $W = K_k$. A R' is spanning and $|X| = k-1$, $|G - R| = 1$. Since R' is at most $(k-4)/2$-incomplete, the single vertex of $G - R$ has degree at most $k - 1 + (k-4)/2$, a contradiction. $\square$

For completeness, we include the statement of [2, Theorem 6].

Theorem A.11 (Kostochka and Yancey, Theorem 6 [2]). *Let $k \geq 4$ and G be a k-critical graph. Then G is k-extremal (i.e, G satisfies Theorem 1.1 with equality) if and only if it is a k-Ore graph. Moreover, if G is not a k-Ore graph, then*

$$|E(G)| \geq \frac{(k-2)(k+1)|V(G)| - y_k}{2(k-1)}$$

where $y_4 = 2, y_5 = 4$, and $y_k = k^2 - 5k + 2$ for $k \geq 6$.

A.3.4 Proof of Lemma 5.6

Proof. Suppose to the contrary that there exists a cluster C of size at least $k-3$. If $|C| \geq k-2$, then $G[C \cup N(C)]$ is a diamond of G, a contradiction. So we may assume that $|C| = k-3$. Let $C = \{u_1, \ldots, u_{k-3}\}$, and let v_1, v_2, v_3 be the common neighbors of vertices in C. Adding the edges v_1v_2, v_2v_3, v_1v_3 produces a K_k-subgraph, so G admits a 3-edge-addition, a contradiction to Lemma 4.17 when $k \geq 8$. Hence, we may assume that $k \in \{6, 7\}$. Furthermore, as G does not admit a 2-edge-addition by Lemma 4.17, we may assume that $v_1v_2, v_1v_3, v_2v_3 \notin E(G)$. For each $i \in \{1, 2, 3\}$, let G_i denote the graph obtained from $G - C - v_i$ by identifying the two vertices of $\{v_1, v_2, v_3\} \setminus \{v_i\}$.

Claim A.12. *For each $i \in \{1, 2, 3\}$, at least one of the following holds:*

1. *v_i has degree at most k, or*
2. *G_i has a k-Ore subgraph.*

Proof. Suppose not. We may assume without loss of generality that the claim does not hold for $i = 3$. Let w denote the identified vertex of G_3. Note that G_3 is not $(k-1)$-colorable, as a $(k-1)$-coloring φ of G_3 extends to a $(k-1)$-coloring of G by letting $\varphi(v_1) = \varphi(v_2) = \varphi(w)$ and coloring the vertices in C with the remaining $k-3$ colors of $[k-1] \setminus \{\varphi(w), \varphi(v_3)\}$. Let K be a k-critical subgraph of G_3 and note that $w \in V(K)$. Let $R = V(K) - w + \{v_1, v_2\} + C$. Observe that $|V(R)| = |V(K)| + k - 2$, $|E(G[R])| \geq |E(K)| + \binom{k-1}{2} - 1, T(G[R]) \geq T(K) - 1$. Thus,

$$\begin{aligned} p_G(R) &\leq p(K) + ((k-2)(k+1) + \varepsilon)(k-2) - 2(k-1)\left(\binom{k-1}{2} - 1\right) + \delta \\ &\leq p(K) + k^2 - 3k + 4 + 2\delta. \end{aligned}$$

Furthermore, if $v_3 \in V(K)$, then $|E(G[R])| \geq |E(K)| + \binom{k}{2} - 3$; in which case, $p_G(R) \leq p(K) + (-k^2 + 5k - 2) + 2\delta$.

For all $k \geq 5$, $-k^2 + 5k - 2 \leq -2$. As G is good, K satisfies Conjecture 1.4 and hence $p(K) \leq k(k-3)$. Since R has a non-edge v_1v_2, $G[R]$ is not a clique, and Lemma 4.11 implies that $p(G) \leq p_G(R)$. Thus, if $v_3 \in V(K)$, $p(G) \leq p_G(R) \leq$

$k(k-3)+(-k^2+5k-2) \leq k(k-3)-2$, which contradicts the fact that G is tight. We deduce that $v_3 \notin V(K)$.

Now K is not k-Ore as otherwise 2 holds. But then

$$p_G(R) \leq p(G)+(k-1)(k-2)+2+Q+2\delta \leq p(G)+(k-1)(k-2)+\Gamma+Q+\delta.$$

Now Lemma 5.5 implies that $G-K=\{v_3\}$ is a single vertex of degree at most $k-1+(k-4)/2$. Since $k \leq 7$, it follows that v_3 has degree at most k and 1 holds, a contradiction. □

Claim A.13. *At most one of the vertices v_1, v_2, v_3 has degree at most k.*

Proof. Suppose not. Suppose without loss of generality that both v_2 and v_3 have degree at most k. Since G is k-critical, v_2 and v_3 have degree at least $k-1$. We have two cases to consider.

First, suppose that both v_2 andv_3 have degree $k-1$. Let $\varphi : V(G)\setminus(C\cup\{v_2,v_3\}) \to \{1,\dots,k-1\}$ be a $(k-1)$-coloring of $G\setminus(C\cup\{v_2,v_3\})$. But then φ extends to G, a contradiction, as follows. For each $i \in \{2,3\}$, let $L_i = \{\varphi(v) : v \in N(v_i)\}$. As v_2 has degree $k-1$, v_2 has at most 2 neighbors not in $C\cup\{v_2,v_3\}$ and hence $|L_2| \leq 2$. Similarly $|L_3| \leq 2$.

If $\varphi(v_1) \notin L_2$, then let $\varphi(v_2)=\varphi(v_1)$, let $\varphi(v_3) \in [k-1]\setminus L_3$ and then color the vertices in C from $[k-1]\setminus\{\varphi(v_1),\varphi(v_3)\}$. So we may assume that $\varphi(v_1) \in L_2$ and similarly that $\varphi(v_1) \in L_3$. But then $|(L_2\cup L_3)\setminus\{\varphi(v_1)\}| \leq 2 < k-2$ since $k \geq 5$. But then we let $\varphi(v_2) \in [k-1]\setminus(L_2\cup L_3)$, let $\varphi(v_3)=\varphi(v_2)$, and color the vertices of C with colors from $[k-1]\setminus\{\varphi(v_1),\varphi(v_2)\}$.

Hence, we may assume without loss of generality that v_2 has degree k. Since $k \geq 6$, we have that $k \leq k-2+(k-3)$. By Lemma 5.4, $G_{v_2\to C}$ contains a k-Ore subgraph H. Since $d_G(v_2)=k$, we have that $|E(G)| > |E(H)|$. If $|V(H)|=|V(G)|$, then

$$p(G) \leq p(H)-2(k-1)+\delta \leq k(k-3)-2(k-1)+\delta < k(k-3)-P-Q,$$

which contradicts that G is tight.

So we may assume that $|V(H)| < |V(G)|$. Let $R=(V(H)\setminus\{\widetilde{v_2}\})\cup\{v_2\}$, which is a proper subset of $V(G)$. Observe that $H \subseteq G[R]+v_2v_1+v_2v_3$, and therefore G admits a 2-edge-addition, contradicting Lemma 4.17. □

By Claims A.12 and A.13, for at least two $i \in \{1,2,3\}$, G_i has a k-Ore subgraph. We may assume without loss of generality that both G_1 and G_2 contain a k-Ore subgraph. For each $i \in \{1,2\}$, let K_i be a k-Ore subgraph of G_i and let $R_i = V(K_i)\setminus\{w_i\}\cup\{v_i,v_3\}$. Notice that R_1 is disjoint from $C\cup\{v_2\}$ and R_2 is disjoint from $C\cup\{v_1\}$. Let $R=R_1\cup R_2\cup C$.

Claim A.14.

$$\overline{p}_G(R) \leq k(k-3)+2(k-1).$$

Proof. Suppose not. Let $H = R_1 \cap R_2$. For each $i \in \{1,2\}$, let

$$E_i = \{xy : x, y \in V(K_i) \cup \{v_3\}, xy \in E(G[R_i]), xy \notin E(K_i)\}.$$

That is, E_i is the set of edges which are present in G but not in K_i, and have both endpoints in $V(K_i) \cup \{v_3\}$. Finally, define

$$E_H = \{xy : x, y \in R_1 \cap R_2, xy \in E(G[R_1 \cap R_2]), xy \notin E(K_1)\}.$$

Note that $E_H \subseteq E_1$.

For each $i \in \{1,2\}$, $|R_i| \geq |V(K_i)| + 1$ and $|E(G[R_i])| \geq |E(K_i)| + |E_i|$, and hence we have that $\overline{p}_G(R_i) \leq \overline{p}(K_i) + (k-2)(k+1) - 2(k-1)|E_i|$.

Let $S = E(K_1[H])$. Observe that $(k-2)(k+1)|H| - 2(k-1)|S| = \overline{p}_{K_1}(H) \geq k(k-3) + 2(k-1)$, by Lemma 4.11. Then $\overline{p}_G(H) = \overline{p}_{K_1}(H) - 2(k-1)|E_H|$. We calculate that

$$\begin{aligned}
\overline{p}(R_1 \cup R_2) &\leq \overline{p}_G(R_1) + \overline{p}_G(R_2) - \overline{p}_G(H) \\
&\leq \overline{p}(K_1) + \overline{p}(K_2) - \overline{p}_{K_1}(H) + 2(k-2)(k+1) - 2(k-1)(|E_1| + |E_2|) + 2(k-1)|E_H| \\
&\leq k(k-3) + 2(k-2)(k+1) - 2(k-1)(|E_1| + |E_2| - |E_H| + 1) \\
&\leq k(k-3) + 2(k-2)(k+1) - 2(k-1).
\end{aligned}$$

Next, adding the vertices and edges of the cluster C, we find that

$$\begin{aligned}
\overline{p}_G(R) &\leq k(k-3) + 2(k-2)(k+1) - 2(k-1) + (k-2)(k+1)(k-3) - (k-1)(k-3)(k+2) \\
&= k(k-3) + 2(k-1).
\end{aligned}$$

□

By Lemma 2.5, for each $i \in \{1,2\}$, $T(K_i) \geq 2 + \frac{|V(K_i)|-1}{k-1}$. Note that $T(G[R]) \geq \max\{T(K_1), T(K_2)\} - 1$. Since $|V(K_1)| + |V(K_2)| \geq |R| - (k-2)$, we may assume without loss of generality that $|V(K_1)| \geq \frac{1}{2}(|R| - (k-2))$. Therefore $T(G[R]) \geq 1 + \frac{|R|-(k-1)}{2(k-1)}$. But then

$$\begin{aligned}
p_G(R) &\leq \overline{p}_G(R) + \varepsilon|R| - \delta\left(1 + \frac{|R| - (k-1)}{2(k-1)}\right) \\
&\leq k(k-3) + 2(k-1) + \varepsilon|R| - \delta\left(1 + \frac{|R| - (k-1)}{2(k-1)}\right) \\
&= k(k-3) + 2(k-1) + (\varepsilon - \frac{1}{2(k-1)}\delta)|R| - \frac{1}{2}\delta.
\end{aligned}$$

It follows that $R = V(G)$ by Lemma 4.17. As G is not k-Ore, Theorem A.11 implies that $\overline{p}(G) \leq k(k-3) - 2(k-1)$. Thus, substituting $R = V(G)$ above, we have that

$$\begin{aligned}
p(G) &\leq \overline{p}(G) + \varepsilon|V(G)| - \delta\left(1 + \frac{|V(G)| - (k-1)}{2(k-1)}\right) \\
&\leq k(k-3) - 2(k-1) \\
&< k(k-3) - P - Q,
\end{aligned}$$

which contradicts the fact that G is tight. $\square$

A.3.5 Proof of Lemma 5.7

Proof. Suppose not. Let $\widetilde{x}$ denote the new vertex in $G_{y\to x}$. First, suppose that $G_{y\to x}$ contains a diamond D. If $\widetilde{x} \notin V(D)$, then $G[V(D)]$ is a $K_k - e$ subgraph of G, then G admits an edge-addition, contradicting Lemma 4.17. Hence $\widetilde{x} \in V(D)$, and therefore $x \in V(D)$ as well. Let C be the set of vertices in $D - \widetilde{x}$ of degree $k-1$ in $G_{y\to x}$. Note that $|C| \geq k-3$. Every vertex $v \in C$ is adjacent to y in G, as otherwise v has degree strictly less than $k-1$ in G. Hence C is a cluster in G of size $k-3$, contradicting Lemma 5.6.

So we may assume that $G_{y\to x}$ contains an emerald D. Further suppose that $\widetilde{x} \in V(D)$. If $x \notin V(D)$, then as every vertex of D is a neighbor of x, $G[(V(D) \setminus \{\widetilde{x}\}) \cup \{x\}] = K_k$. Note that every vertex $v \in D$ is adjacent to y in G since v has degree $k-1$ in $G_{y\to x}$. Then it follows that $G[(V(D) \setminus \{\widetilde{x}\}) \cup \{x, y\}] = K_k$, a contradiction. So we may assume that both $x, \widetilde{x} \in V(D)$. But then $G[(V(D) \setminus \{\widetilde{x}\}) \cup \{y\}] = K_{k-1}$, and Corollary 4.6 implies that the $k-2$ vertices of degree $k-1$ in $D - \widetilde{x}$ are in the same cluster in G. This contradicts Lemma 5.6.

So we may assume that $x, \widetilde{x} \notin V(D)$. As $G[V(D)] = K_{k-1}$, Corollary 4.6 implies that the vertices of degree $k-1$ in D are in the same cluster C. The only edges present in G that are not present in $G_{y\to x}$ are those incident with y. Hence if a vertex v in D has degree greater than $k-1$ in G, then v is adjacent to y. Therefore the vertices of $D - C$ have degree k in G. If $C = \emptyset$, then $G[V(D) \cup \{y\}] = K_k$, a contradiction. Hence there exists a vertex $z \in V(D)$ with $d_G(z) = k-1$. Let w be the unique neighbor of z not in D. Corollary 4.7 implies that every vertex in $D - C$ is also adjacent to w. But then $G[V(D) \cup \{w\}] = K_k$, a contradiction. $\square$

A.3.6 Proof of Lemma 5.8

Proof. Observe that $|V(G_{y\to x})| = |V(G)|$, $|E(G_{y\to x})| \leq |E(G)|$, and $T(G_{y\to x}) \leq T(G) + 1$. The result follows by evaluating $p(G_{y\to x})$. $\square$

A.3.7 Proof of Lemma 5.9

Proof. We claim that any frame $V^*(H)$ must be the closed neighborhood of x in $G_{y\to x}$. Suppose to the contrary that x is inside a replacement edge with endpoints a, b. Note that $\widetilde{x}$ is then also within the replacement edge a, b. Then $G[V(H) \setminus \{\widetilde{x}\}] + ab$ is a k-Ore subgraph, as we have added a real edge over the replacement edge in ab, so G admits an edge-addition, contradicting Lemma 4.17. Therefore $x, \widetilde{x} \in V^*(H)$, and the other vertices of $V^*(H)$ are neighbors of x. $\square$

A.3.8 Proof of Lemma 5.11

Proof. We argue by induction on $|V(H)|$. If $|V(H)| = k+1$, then $H - \{a,b\} = K_{k-1}$, which is a proto-gadget containing x as desired. Note that $V^*(H)$ is unique for $H = K_{k-1}$.

Proceeding inductively, let $H^\bullet$ be the k-Ore graph obtained by identifying a, b to a single vertex $\underline{ab}$, and let $V^*(H^\bullet)$ be a frame of $H^\bullet$. First suppose that $\underline{ab} \notin V^*(H^\bullet)$. But then $\underline{ab}$ is contained within a replacement edge with endpoints u, v. Then $G + uv$ contains a k-Ore graph and thus G admits an edge-addition, a contradiction to Lemma 4.17.

Now, if x is not in a replacement edge, then delete $\underline{ab}$ and its incident replacement edges to obtain a proto-gadget H' with $x \in V(H')$ and $H' \subseteq H - \{a,b\}$ as desired.

So we may assume that x is in a replacement edge of $H^\bullet$ with endpoints u, v. Let $H^{\bullet\bullet}$ be the split k-Ore graph contained in the replacement edge. As $\underline{ab} \in V^*(H)$, either $\underline{ab} \notin V(H^{\bullet\bullet})$, or $\underline{ab} \in \{u, v\}$. Clearly $|V(H^{\bullet\bullet})| < |V(H)|$, so the induction hypothesis implies that there exists a proto-gadget $H' \subseteq H^{\bullet\bullet} - \{u, v\}$ with $x \in V^*(H')$. Since $\underline{ab} \notin V(H')$, H' is a proto-gadget of G and $a, b \notin V(H')$ as desired. □

A.3.9 Proof of Lemma 5.13

Proof. Suppose to the contrary that there exists an induced path $P = v_1v_2v_3v_4$ such that each vertex of P has degree $k-1$. Let $\{z_1, \ldots, z_{k-3}\}$ be the neighbors of v_2 not in P. Consider $G_{v_2 \to v_3}$, that is cloning v_2 with v_3. As v_3 has degree $k-2$ in $G_{u \to y}$, $G_{v_2 \to v_3}$ is not k-critical. Hence, Lemma 5.4 implies that $G_{v_2 \to v_3}$ contains a k-Ore subgraph H, and Lemma 5.9 implies that $V^*(H) = \{v_1, \widetilde{v_1}, v_2, z_1, \ldots, z_{k-3}\}$.

We claim that $v_1z_i \in E(G)$ for all $1 \leq i \leq k-3$. Suppose not. We may assume without loss of generality that $v_1z_1 \notin E(G)$ and hence v_1z_1 is a replacement edge in H instead of a real edge. Let T be the set of neighbors of v_1 within the replacement edge v_1z_1, and let S be the set of edges $\{wz_1 : w \in T\}$.

Observe that $G+S$ contains a k-Ore subgraph, as z_1 now has the same neighbors as z_1 identified with v_1. Thus $|T| \geq 3$, as G does not admit 2-edge-additions by Lemma 4.17. But v_1 has at least one real edge for each of the vertices $v_1, z_2, \ldots, z_{k-3}$, so $d_G(x) \geq k-3+3 = k$, a contradiction. It follows that v_1z_i is an edge of G for all i with $1 \leq i \leq k-3$. But then v_1, v_2 have $k-3$ common neighbors. Hence Corollary 4.6 implies that v_1, v_2 are in the same cluster. So v_1 is adjacent to v_3, which contradicts the assumption that P is induced. □

A.3.10 Proof of Lemma 5.14

Proof. Suppose to the contrary that G has a 2-cut $\{x, y\}$, separating G into two edge-disjoint subgraphs G_1, G_2. As G is k-critical, x and y as otherwise there exists a $(k-1)$-coloring of G by taking the union of a $(k-1)$-coloring of G_1 and a $(k-1)$-coloring of G_2 as necessary (cf. Theorem 3 of [1]). As noted in [1], either $\varphi_1(x) =$

$\varphi_1(y), \varphi_2(x) \neq \varphi_2(y)$ for all possible $(k-1)$-colorings φ_1, φ_2 of G_1, G_2 respectively, or vice versa, as otherwise, by permuting colors if necessary, the union of φ_1, φ_2 is a $(k-1)$-coloring of G, a contradiction. So without loss of generality, assume that $\varphi_1(x) = \varphi_1(y)$ for all $(k-1)$-colorings φ_1 of G_1.

We claim that G is an Ore-composition of $H_1 = G_1$ and $H_2 = G_2 - \{x, y\} + \underline{xy}$, with H_1 being the edge-side. Suppose not. Then there exists a vertex $z \in V(G_2)$ such that z is adjacent to both x and y. Let φ be a $(k-1)$-coloring of $G - xz$. Since φ induces a $(k-1)$-coloring of G_1, it follows that $\varphi(x) = \varphi(y)$. Since $yz \in E(G - xz)$, we have that $\varphi(x) \neq \varphi(z)$. But then φ is a $(k-1)$-coloring of G, a contradiction. This proves the claim.

Since G is not k-Ore, at least one of H_1 or H_2 is not k-Ore. Note that both H_1 and H_2 are smaller than G and so satisfy Conjecture 1.4. Observe that $|V(G)| = |V(G_1)| + |V(G_2)| - 1, |E(G)| = |E(H_1)| + |E(H_2)| - 1$, and by Lemma 2.4, $T(G) \geq T(H_1) + T(H_2) - 2$. Thus

$$p(G) \leq p(H_1) + p(H_2) - (k-2)(k+1) - \varepsilon + 2(k-1) + 2\delta.$$

As G is a minimal counterexample, $p(G) > k(k-3) - P$.

First suppose that H_2 is not k-Ore. Then $p(H_2) \leq k(k-3) - P$, and

$$k(k-3) - P < p(G) \leq p(H_1) - P - \varepsilon + 2\delta,$$

whence $p(H_1) > k(k-3) - 2\delta + \varepsilon$. It then follows that $H_1 = K_k$, as $\delta \geq 2(k-1)\varepsilon$ implies

$$k(k-3) - 2\delta + \varepsilon > k(k-3) + \varepsilon|V(G)| - \left(2 + \frac{|V(G)| - 1}{k-1}\right)\delta.$$

But then the sharper form of Lemma 2.4 when $H_1 = K_k$ implies that $T(G) \geq T(H_1) + T(H_2) - 1$, and so

$$p(G) \leq k(k-3) - P - (\delta - (k-1)\varepsilon) < k(k-3) - P,$$

which contradicts the assumption that G is a counterexample. So we may assume that H_2 is k-Ore and H_1 is not. But then the same argument applies using again the sharper form of Lemma 2.4. □

A.3.11 Proof of Lemma 5.15

Proof. The proof is the same as $i = 1$ in Lemma 4.17, with a small modification to account for the fact that we are no longer assuming G to be ungemmed. Let $R \subsetneq V(G)$ be a subset of minimum size such that $G[R] + xy$ contains a k-critical subgraph H. The same argument as Claim A.2 in Lemma 4.17 shows that R is collapsible. Since H is proper, by Lemma 4.11 we have that $k(k-3) - P < p(G) < p(H)$ and hence H

is k-Ore. Thus we have established the analogue of Claim A.3 in Lemma 4.17. The same argument as in Claim A.4 holds here. Together, these facts imply that H is k-Ore and $H = G[R] + xy$, from which we deduce that H cannot be K_k, since then $\{x, y\}$ would form a 2-cut in G, contradicting Lemma 5.14.

Having shown that $H = K_k$ is impossible, it only remains to show that H cannot be an Ore-composition of two k-Ore graphs H_1, H_2. Clearly there must exist $u, v \in \partial_G R$ with $u \in V(H_1) \setminus V(H_2), v \in V(H_2) \setminus V(H_1)$, as otherwise the overlap vertices of H_1, H_2 would be a 2-cut in G. Thus, the conditions for Subclaim A.7 hold, and Subclaim A.7 implies that H cannot be an Ore-composition of H_1, H_2, which completes the argument. □

A.4 Proofs from Section 6

A.4.1 Proof of Lemma 6.7

Proof. Suppose to the contrary that v_1, v_2, v_3, v_4 have degree 5 and $G[\{v_1, v_2, v_3, v_4\}] = K_4$. Let u_1, u_2 be the other neighbors of v_1. By Lemma 5.4, $G_{v_2 \to v_1}$ is 6-critical or $G_{v_2 \to v_1}$ contains a 6-Ore subgraph.

First suppose $G_{v_2 \to v_1}$ has a 6-Ore subgraph H. Then by Lemma 5.9, $V^*(H) = \{v_1, \widetilde{v_1}, v_3, v_4\}$. By Corollary 6.4 as v_3, v_4 have degree 5, we find that v_3, v_4 are not incident with a replacement edges of H. Hence v_3, v_4 are each adjacent to both of u_1, u_2. But then Corollary 4.6 implies that v_1, v_3, v_4 are in the same cluster, contradicting Lemma 5.6.

So we may assume that $G_{v_2 \to v_1}$ is 6-critical. By Lemmas 5.7 and 5.8, $G_{v_2 \to v_1}$ is tight and ungemmed. But $G_{v_2 \to v_1}$ contains a cluster of size 2 with two neighbors having degree 5, which contradicts Lemma 6.6. □

A.4.2 Proof of Lemma 6.8

Proof. Suppose not. First suppose that $G_{v_3 \to v_2}$ is not 6-critical. By Lemma 5.4 and Lemma 5.9, $G_{v_3 \to v_2}$ contains a 6-Ore subgraph H with $V^*(H) = (N(v_2) \setminus \{v_3\}) \cup \{\widetilde{v_2}\}$. The vertex v_1 has degree 6 in $G_{v_3 \to v_2}$. By Corollary 6.4, v_1 is not incident with a replacement edge of H. Hence v_1 is adjacent to the other 3 neighbors of v_2. But then Corollary 4.6 implies that v_1, v_2 are in the same cluster and hence v_1 is adjacent to v_3, a contradiction since P is induced.

So we may assume that $G_{v_3 \to v_2}$ is 6-critical. By Lemma 5.8 $p(G_{v_3 \to v_2}) \geq p(G) - \delta > p(G) - P - Q$ and hence $G_{v_3 \to v_2}$ is tight. By Lemma 5.7, $G_{v_3 \to v_2}$ is ungemmed. By symmetry $G_{v_1 \to v_3}$ is also tight and ungemmed, a contradiction. □

A.4.3 Proof of Lemma 6.9

Proof. Suppose not. Thus we may assume there exists a triangle $T = v_1 v_2 v_3$ in $D_5(G)$ and $u \in D_5(G)$ adjacent to v_1. Note that in $G_{v_2 \to v_1}$, v_3 has degree 5 and u has degree at most 6. Hence we deduce from Lemma 6.6 that $G_{v_2\to v_1}$ is not 6-critical. Thus Lemma 5.4 and Lemma 5.9 imply that $G_{v_2\to v_1}$ contains a 6-Ore subgraph H with $v_3, u \in V^*(H)$. Since v_3 has degree 5 in G, it follows from Corollary 6.4 that v_3 is not incident to a replacement edge of H. Hence $v_3u \in E(G)$. Therefore, both v_2 and u have degree 5 in $G_{v_3\to v_1}$.

Similarly by Lemma 6.6, $G_{v_3\to v_1}$ is not 6-critical. Hence $G_{v_3\to v_1}$ has a 6-Ore subgraph H with $v_2, u \in V^*(H)$. Since both v_2 and u have degree 5 in G, it follows from Corollary 6.4 that neither v_2 nor u is incident to a replacement edge of H. Hence $v_2u \in E(G)$. But then $G[\{v_1, v_2, v_3, u\}] = K_4$, contradicting Lemma 6.7. □

A.4.4 Proof of Lemma 6.10

Proof. Suppose to the contrary that $v_1v_2v_3v_4v_1$ is a 4-cycle in $D_5(G)$. By Lemma 6.9, $v_1v_3, v_2v_4 \notin E(G)$. By Lemma 6.8, $G_{v_3\to v_2}$ is 1-tight and ungemmed. But v_4 has degree 4 in $G_{v_3\to v_2}$, a contradiction. □

A.4.5 Proof of Lemma 6.11

Proof. Suppose to the contrary that $u \in D_5(G)$ is a vertex with neighbors $v_1, v_2, v_3 \in D_5(G)$. By Lemma 6.9, we find that v_1 is not adjacent to v_2 and hence v_1uv_2 is an induced path in G. Then by Lemma 6.8, $G_{v_1\to u}$ is a 1-tight and ungemmed graph with a cluster $\{u, \widetilde{u}\}$ that has two neighbors v_2, v_3 having degree at most 6 in $G_{v_1\to u}$, contradicting Lemma 6.6. □

A.5 Proofs from Section 7

A.5.1 Proof of Lemma 7.3

Proof. Note that $\frac{\partial\psi(d,r)}{\partial r} = \frac{28+\varepsilon-5d}{r^2} \le 0$ as $\varepsilon \le 7$; so $\psi(d,r)$ is decreasing in r and (1) holds. Note that $\frac{\partial\psi(d,r)}{\partial d} = \frac{5}{r} \ge 0$ as $r \ge 1$; so $\psi(d,r)$ is increasing in d and (2) holds. For $0 \le i \le 5$, we have that $\frac{\partial\psi(d,d-i)}{\partial d} = \frac{28+\varepsilon-5i}{(d-i)^2} \ge 0$; so $\psi(d, d-i)$ is increasing in d and (3) holds.

A vertex of degree $d \ge 8$ sends $\psi(d,r)$ charge in STAGE 1 each neighbor of degree 5. By (1), $\psi(d,r) \ge \psi(d,d)$. By (3) with $i = 0$, $\psi(d,d) \ge \psi(8,8)$. Since $\psi(8,8) = \frac{3}{2} - \frac{1}{8}\varepsilon$, (4) holds as desired.

So suppose $r < d$. By (1), $\psi(d,r) \ge \psi(d,d-1)$. By (3) with $i = 1$, $\psi(d,d-1) \ge \psi(8,7)$. Since $\psi(8,7) = \frac{12}{7} - \frac{1}{7}\varepsilon$, (5) holds as desired.

By RULE 2A, a vertex of degree 7 discharges if it has $r \le 5$ unhappy upward neighbors. The amount of charge sent to each such neighbor is $\psi(7,r)$. By (1) and since $r \le 5$, $\psi(7,r) \ge \psi(7,5)$. Since $\psi(7,5) = \frac{7}{5} - \frac{1}{5}\varepsilon$, (6) holds as desired. □

A.5.2 Proof of Lemma 7.4

Proof. Suppose not. Let H and C be a counterexample such that $|V(H)|$ is minimized. First suppose that $C \setminus V^*(H) \neq \emptyset$. Let $w \in C \setminus V^*(H)$. Hence w is inside a replacement edge $e = uv$ of H. Since e is a split 6-Ore graph with overlap vertices $\{u, v\}$, it follows that $d_H(w) = 5 = d_G(w)$ and hence $N(w) \cap V(H) = N(w) \cap V(G)$. Since u and v each have a neighbor outside e, it follows that $u, v \notin C$ and hence C is entirely inside e. By Lemma 5.11, there exists a proto-gadget $H' \subseteq H - \{a, b\}$ with $C \subseteq V(H')$. Since $|V(H')| < |V(H)|$, H' and C contradict the minimality of H and C.

So we may assume that $C \subseteq V^*(H)$. Next, note that Corollary 6.5 implies that $V^*(H) \cap D_5(G) = C$. Furthermore, Corollary 6.17 implies that C is upward of each neighbor of C having degree 6 or 7. Let $\{w\} = N(C) \setminus V^*(H)$. By Lemma 4.17 and Lemma 6.1, G does not admit two edge-additions. This implies that $|N(w) \cap V(H)| \leq 2$ and hence $|N(w) \cap V(H) \setminus C| \leq 2 - |C|$. Let D denote the set of vertices of $V^*(H) \cap D_6(G)$ that are incident with a replacement edge of H. Note that if $v \in D$, then since $d(v) = 6$, it follows that v is incident with at most one replacement edge of H.

Claim A.15. *If $v \in V^*(H) \cap D_6(G) \setminus D$, then v is adjacent to w; hence $|V^*(H) \cap D_6(G) \setminus D| \leq 2 - |C|$.*

Proof. Follows from Corollary 4.7. □

Claim A.16. *$H \neq K_5$.*

Proof. Suppose not. Since $H = K_5$, $D = \emptyset$. Then by Claim A.15, it follows that $|V^*(H) \cap D_6(G)| \leq 2 - |C|$. Let $S = V(H) \setminus (D_6(G) \cup D_5(G))$. Since $V(H) \cap D_5(G) = C$, it follows that $|S| \geq 5 - (2 - |C|) - |C| \geq 3$. Moreover, each vertex in S has degree at least 7. Since S is a clique, it follows that for each vertex $s \in S$, $|N(s) \cap D_5(G)| \leq d(s) - 2$ and hence s either discharges in STAGE 1 or triggers RULE 2A.

First suppose $|C| = 2$. Thus each vertex in C receives at least

$$\sum_{s \in S} \psi(d(s), d(s) - 2) \geq 3\psi(7, 5) = \frac{21}{5} - \frac{3}{5}\varepsilon$$

total charge from vertices of S. This discharge leaves C with $ch_2(C) \geq -6 - 2\varepsilon + 2\left(\frac{21}{5} - \frac{3}{5}\varepsilon\right) \geq 2 + \frac{2}{5} - \frac{16}{5}\varepsilon$. Since $\varepsilon \leq \frac{1}{13}$, we find that $ch_2(C) \geq 2 + 2\varepsilon$ and hence C is happy after STAGE 2, a contradiction.

So we may assume that $|C| = 1$. Let $C = \{x\}$. Note that now for each vertex $s \in S$, $|N(s) \cap D_5(G)| \leq d(s) - 3$. Thus x receives at least

$$\sum_{s \in S}^{3} \psi(d(s), d(s) - 3) \geq 3\psi(7, 4)$$

charge from vertices in S. This discharge then leaves x with at least $ch_2(x) \geq -3 - \varepsilon + \frac{21}{4} - \frac{3}{4}\varepsilon \geq 2 + \frac{1}{4} - \frac{7}{4}\varepsilon$. Since $\varepsilon \leq \frac{1}{15}$, we find that $ch_2(x) \geq 2 + 2\varepsilon$ and hence x is happy after STAGE 2, a contradiction. □

Claim A.17. $|C| = 1$.

Proof. Suppose not. Thus $|C| = 2$. Let $C = \{x, y\}$. Let $\{v_1, v_2, v_3\} = V^*(H) \setminus C$. Lemma 6.6 implies that $V^*(H)$ has at most 1 vertex of degree 6.

Now further suppose that $V^*(H)$ has no vertices of degree 6. Then $\deg(v_j) \geq 7$ for each $j \in \{1, 2, 3\}$. By Corollary 6.16, it follows that $d(v_j) \geq 8$ or v_j has at most 5 upward neighbors. Hence each v_j then discharges either in STAGE 1 or RULE 2A, sending at least $\min\{\psi(8, 8), \psi(7, 5)\} = \frac{7}{5} - \frac{1}{5}\varepsilon$ charge to both of x, y. Thus $ch_2(C) \geq -6-2\varepsilon+2(\frac{21}{5}-\frac{3}{5}\varepsilon) \geq 2+\frac{2}{5}-\frac{16}{5}\varepsilon$. Since $\varepsilon \leq \frac{1}{13}$, we find that $ch_2(C) \geq 2+2\varepsilon$ and hence C is happy after STAGE 2, a contradiction.

So we may assume that $V^*(H)$ has a single vertex of degree 6, say v_1 without loss of generality. Since $|C| = 2$, it follows from Claim A.15 that $v_1 \in D$ and hence v_1 is incident with a unique replacement edge e of H. Suppose that $e = v_1v_2$ without loss of generality. Corollary 6.16 implies that any neighbors of v_1 inside e are not upward of v_1. Hence v_1 has no upward neighbors aside from C and therefore triggers RULE 2B, sending $2 - \varepsilon$ charge to C.

Note that $d(v_2), d(v_3) \geq 7$ as $V^*(H)$ has a single vertex of degree 6. Also note that v_3 is incident with v_1 and hence v_3 has at most $d(v_3) - 1$ neighbors of degree 5.

Subclaim A.18. *v_2 and v_3 together send at least $3 + \frac{4}{35} - \frac{12}{35}\varepsilon$ charge to each vertex in C.*

Proof. First suppose that $d(v_3) \geq 8$. Then v_3 sends at least $\psi(8, 7) = \frac{12}{7} - \frac{1}{7}\varepsilon$ charge to C, and v_2 sends at least $\psi(7, 5)$. Then together v_2, v_3 send to each vertex in C at least $\psi(8, 7) + \psi(7, 5) = \frac{12}{7} - \frac{1}{7}\varepsilon + \frac{7}{5} - \frac{1}{5}\varepsilon = 3 + \frac{4}{35} - \frac{12}{35}\varepsilon$ charge to each vertex in C as desired.

So we may assume that $d(v_3) = 7$. Suppose that v_2v_3 is a replacement edge of H. Since v_2v_3 is a replacement edge of H, we have that $d(v_2) \geq 8$ and v_3 has at most 3 upward neighbors. Then together v_2, v_3 send to each vertex in C at least $\psi(8, 8) + \psi(7, 3) = 3 + \frac{5}{6} - \frac{11}{24}\varepsilon$ charge, which is at least $3 + \frac{4}{35} - \frac{12}{35}\varepsilon$ charge as desired since $\varepsilon \leq 1 \leq \frac{604}{97}$.

So we may assume that v_2v_3 is a real edge of H. Since v_3 is adjacent to v_1 and v_2, we find that v_3 has at most 5 upward neighbors. Similarly since v_2 is adjacent to v_3, we find that v_2 has at most $d(v_2) - 1$ neighbors of degree 5; furthermore v_2 has at most $d(v_2) - 4$ upward neighbors since by Corollary 6.16 v_2 is upward of its neighbors inside the replacement edge $e = v_1v_2$. Then together v_2, v_3 send at least $\psi(7, 5)+\min\{\psi(8, 7), \psi(7, 3)\} = 3+\frac{4}{35}-\frac{12}{35}\varepsilon$ charge to each vertex in C as desired. □

By Subclaim A.18, the total charge sent to C by v_2, v_3 is at least $2\left(3 + \frac{4}{35} - \frac{12}{35}\varepsilon\right) = 6 + \frac{8}{35} - \frac{24}{35}\varepsilon$. Since v_1 sends at least $2 - \varepsilon$ charge to C, it follows that the charge of C

is then $ch_2(C) \geq -6 - 2\varepsilon + (2-\varepsilon) + \left(6 + \frac{8}{35} - \frac{24}{35}\varepsilon\right) \geq 2 + \frac{8}{35} - \frac{129}{35}\varepsilon$. Since $\varepsilon \leq \frac{8}{199}$, we find that $ch_2(C) \geq 2 + 2\varepsilon$ and hence C is happy after STAGE 2, a contradiction. □

By Claim A.17, $|C| = 1$. Let $C = \{x\}$. Recall that $V^*(H) \cap D_5(G) = C$.

Claim A.19. *If $v \in V^*(H)$ such that $d(v) \geq 7$, then v sends at least $\psi(8,7)$ charge to x.*

Proof. First suppose $d(v) = 7$. Then either v is incident with only real edges of H, in which case v has at most 4 neighbors of degree 5, or v is incident with exactly one replacement edge of H, in which case v has at most 2 upward neighbors. In either case, v triggers RULE 2A and sends at least $\psi(7,4)$ charge to x, which is at least $\psi(8,7)$ charge as desired. Next suppose $d(v) = 8$. Then v is incident with at most 2 replacement edges and hence v has at most 7 neighbors of degree 5. Thus v sends at least $\psi(8,7)$ charge to x as desired. Finally suppose $d(v) \geq 9$. Then v sends at least $\psi(9,9)$ charge to x, which is at least $\psi(8,7)$ charge as desired. □

Claim A.20. *If $v \in D$, then v triggers* RULE 2B *and hence sends $2 - \varepsilon$ charge to x.*

Proof. Suppose not. Since $v \in D$, v is incident to a replacement edge e of H. Since $d(v) = 6$, this is the only replacement edge of H incident with v. Note then that $N(v) \cap D_5(G) \setminus \{x\}$ is inside e. By Corollary 6.16, the neighbors of v inside e are downward neighbors, so x is the only upward neighbor of v. Therefore v triggers RULE 2B and sends $2 - \varepsilon \geq \psi(8,7)$ to x as desired. This proves the second statement. □

By Claim A.15, $|V^*(H) \cap D_6(G) \setminus D| \leq 2 - |C| \leq 1$. By Claim A.20, every vertex in D sends at least $2 - \varepsilon$ charge to x. By Claim A.19, every vertex in $V^*(H) \setminus (\{x\} \cup D_6(G))$ sends at least $\psi(8,7) = \frac{12}{7} - \frac{1}{7}\varepsilon$ charge to x. Since $2 - \varepsilon \geq \psi(8,7)$ as $\varepsilon \leq \frac{1}{3}$, it follows that at least 3 of the neighbors of x will each send at least $\psi(8,7)$ charge to x. Hence x receives at least $3\psi(8,7) = 5 + \frac{1}{7} - \frac{3}{7}\varepsilon$ charge. Thus $ch_2(x) \geq -3 - \varepsilon + 5 + \frac{1}{7} - \frac{3}{7}\varepsilon = 2 + \frac{1}{7} - \frac{10}{7}\varepsilon$; since $\varepsilon \leq \frac{1}{24}$, this is at least $2 + 2\varepsilon$ and hence x is happy after STAGE 2, a contradiction. □

A.5.3 Proof of Lemma 7.18

Proof. We consider x. The same argument applies symmetrically to y. Assume that x is unhappy after STAGE 3 and hence after STAGE 2 as well. By Corollary 7.5, x is upward of its neighbors and therefore receives charge from them in STAGE 2.

Let $w \neq y$ be a neighbor of x. If $d(w) \geq 7$, then w sends at least $\psi(7,7) \geq 1 - \frac{1}{2}\varepsilon$ charge to x as w discharges during either STAGE 2 or STAGE 3. Hence, we may assume $d(w) = 6$.

If w has at most one neighbor of degree 5 other than x, then w sends at least $1 - \frac{1}{2}\varepsilon$ charge to x by RULE 3A and (2) holds. Thus, we may assume that w has at least two other neighbors $z_1, z_2 \neq x$ of degree 5.

First suppose z_1, z_2 are adjacent. Then $\{w, z_1, z_2\}$ is a triangle in $D_5(G_{x\to y})$. By Lemma 6.9, it follows that w has exactly three neighbors $x, z_1, z_2 \in D_5(G)$ and furthemore that $z_1z_2 \in E(G)$. Therefore w triggers RULE 3B and sends $1 - \frac{1}{2}\varepsilon$ charge to x.

So we may assume that z_1 and z_2 are not adjacent. Let $G' = G_{x\to y}$. By assumption, G' is 2-tight and ungemmed. Now (z_1, w, z_2) is an induced path of $D_5(G')$. Yet $G'_{w\to z_1}, G'_{w\to z_2}$ have vertices of degree 4, and therefore are not 6-critical. By Lemma 5.4, it follows that there exist gadgets H_1, H_2 of G' containing z_1, z_2 respectively.

Suppose that H_1 is not a gadget in G. Then $y, \widetilde{y} \in V(H_1)$, as $\widetilde{y}$ is the only vertex in $V(G') \setminus V(G)$. Clearly $y \notin V^*(H_1)$, as otherwise y is adjacent to z_1. Applying Lemma 5.11 to the replacement edge containing y in H_1, there exists a proto-gadget $H_3 \subseteq H_1$ with $y \in V^*(H_3)$, so $\widetilde{y} \in V^*(H_3)$ as well. But $H_3 - \widetilde{y}$ is then a kite in G containing y, with $x \notin V(H_3) \setminus \{\tilde{y}\}$ (since $x \notin V(G')$), and hence y dangles from x and (3) holds.

So we may assume that H_1 is a gadget of G and by symmetry that H_2 is also a gadget of G. Then w has at most three neighbors z_1, z_2, z_3 having degree 5, all of which are pairwise not adjacent, and such that there exist gadgets H_1, H_2, H_3 in G containing z_1, z_2, z_3 respectively. None of z_1, z_2, z_3 is in a cluster of size 2. So by Lemma 7.4, z_1, z_2, z_3 are happy after STAGE 2. Thus w sends $2 - \varepsilon$ charge to x in RULE 3B and (2) holds. □

A.5.4 Proof of Lemma 7.19

Proof. By assumption, $G' = G_{x\to y}$ is 2-tight and ungemmed. Let $N(x) \cap N(y) \cap D_6(G) = \{w\}$. Since w is adjacent to y, w has degree 6 in G'. Let C be the cluster $\{y, \widetilde{y}\}$ in G', and consider $G'' = G'_{w\to C}$. Since G"$'$ contains a cluster of size 3, Lemma 5.6 and Lemma 5.4 imply that G'' contains a 6-Ore subgraph H. Lemma 5.9 implies that $V^*(H) = (N(y) \setminus \{x, w\}) \cup \{\widetilde{y}, \widetilde{w}\}$. Deleting the cloned vertices $\{\widetilde{y}, \widetilde{w}\}$ from H yields a kite H_2 of G. Since w is the only common neighbor of x, y having degree 6 and $w \notin V^*(H_2)$, $x \notin V(H_2)$, H_2 is a kite which verifies that y dangles from x and vice versa. □

A.5.5 Proof of Proposition 7.22

Proof. If $C = \{x, y\}$ is contained in a proto-gadget of G, then Lemma 7.4 implies that $ch_3(C) \geq ch_2(C) \geq 0$. Thus, if $ch_2(C) < 0$, then C is not contained in a proto-gadget of G. By Lemma 6.6, the neighbors of C all have degree at least 7, and Corollary 7.5 implies that x and y are both upward of the neighbors of C. Thus x and y each receive at least $\psi(7, 7) = 1 - \frac{1}{7}\varepsilon$ charge from four vertices. So $ch_3(C) \geq -6 - 2\varepsilon + 2(4 - \frac{4}{7}\varepsilon) \geq 0$. □

A.5.6 Proof of Proposition 7.23

Proof. Note that $G_{z \to C}$ has a cluster of size 3. So C is contained in a gadget H. By Lemma 7.4, $ch_2(C) \geq 2+2\varepsilon$, as the neighbor z of C not in H has degree 5. Therefore $ch_3(S) = ch_3(C) + ch_3(z) \geq -1 + \varepsilon$. If any of the neighbors of z is not a reserved degree-6 neighbor, then z receives at least another $1 - \frac{1}{2}\varepsilon$ charge, whence $ch_3(S) \geq 0$ as desired. □

A.5.7 Proof of Proposition 7.24

Proof. By Corollary 7.7, we have $ch_2(x), ch_2(z) \geq 2+2\varepsilon$. Therefore $ch_3(S) \geq ch_2(x) + ch_2(z) - 3 - \varepsilon \geq 1 + 3\varepsilon > 0$. □

References

[1] G. Dirac, *The structure of k-chromatic graphs*, Fund. Math., 40 (1953), pp. 42–55.

[2] A. Kostochka and M. Yancey, *A Brooks-type result for sparse critical graphs*, Combinatorica, 38 (2018), pp. 887—-934.